MATH1050 Proof-writing Exercise 2 (Answers and selected solutions)

1. *Comment.*

   Whenever you want to write a fractional expression $\dfrac{\textit{blah-blah-blah}}{\textit{bleh-bleh-bleh}}$, you must make sure the denominator *bleh-bleh-bleh* has been known (whether by being assumed or by being verified) to be non-zero.

2. *Comment.*

   Be aware that the assumption '$x, y \in \mathbb{Z}$ *and* x *is divisible by* y *and* y *is divisible by* x' does not preclude the possibilities '$x = 0$', '$y = 0$'.

   Be careful when you are tempted to write an expression like $\dfrac{y}{x}$, or 'cancelling' x from both sides of an equality in which the expressions contain the 'factor' x.

3. (a) **Solution.**

   Let a, b be complex numbers. Suppose $a^4 + a^3b + a^2b^2 + ab^3 + b^4 \neq 0$. Further suppose it were true that both of a, b were zero.

   Then we would have $a^4 + a^3b + a^2b^2 + ab^3 + b^4 = 0^4 + 0^3 \cdot 0 + 0^2 \cdot 0^2 + 0 \cdot 0^3 + 0^4 = 0$.

   However, by assumption, $a^4 + a^3b + a^2b^2 + ab^3 + b^4 \neq 0$. Then $0 \neq 0$. Contradiction arises.

   Hence at least one of a, b is non-zero in the first place.

   (b) **Solution.**

   Let a, b be real numbers. Suppose $ab > 1$. Further suppose it were true that $a^2 + 4b^2 \leq 4$.

   Since $ab > 1$, we have $-4ab < -4$.

   Therefore $(a - 2b)^2 = a^2 + 4b^2 - 4ab < 4 - 4 = 0$.

   Since a, b are real numbers, $(a - 2b)^2 \geq 0$.

   Then $0 \leq (a - 2b)^2 < 0$. Contradiction arises.

   Hence $a^2 + 4b^2 > 4$ in the first place.

   (c) **Solution.**

   Let ζ be a complex number. Suppose that $|\zeta| \leq \varepsilon$ for any positive real number ε. Further suppose it were true that $\zeta \neq 0$.

   Since $\zeta \neq 0$, we would have $|\zeta| \neq 0$. Then $|\zeta| > 0$.

   Define $\varepsilon = \dfrac{|\zeta|}{2}$. Since $|\zeta| > 0$ and $\dfrac{1}{2} > 0$, we would have $\varepsilon > 0$.

   Then by assumption, $|\zeta| \leq \varepsilon = \dfrac{|\zeta|}{2}$.

   Therefore $\dfrac{|\zeta|}{2} = |\zeta| - \dfrac{|\zeta|}{2} \leq 0$. Hence $|\zeta| = 2 \cdot \dfrac{|\zeta|}{2} \leq 0$ (because $2 > 0$).

   Now we have $0 < |\zeta| \leq 0$. Contradiction arises.

   Hence $\zeta = 0$ in the first place.

4. ——

5. ——

6. ——

7. (a) ——

   (b) i. **Solution.**

   Let α be a non-zero complex number. Suppose α is algebraic.

   By definition, there exists some non-constant polynomial $f(x)$ whose coefficients are rational numbers such that $f(\alpha) = 0$.

   For the same $f(x)$, there exist some positive integer k and some rational numbers $a_0, a_1, a_2, \cdots, a_k$ such that one of a_k is non-zero and $f(x) = a_0 + a_1x + a_2x^2 + \cdots + a_kx^k$ as polynomials.

   Define the polynomial $g(x)$ by $g(x) = a_0x^k + a_1x^{k-1} + a_2x^{k-2} + \cdots + a_k$.

   By definition, $g(x)$ is a polynomial whose coefficients are rational numbers.

   We verify that at least one of $a_0, a_1, \cdots, a_{k-1}$ is non-zero, with the proof-by-contradiction argument:

- Suppose it were true that $a_0 = a_1 = \cdots = a_{k-1} = 0$. Then $f(x) = a_k x^k$ as a polynomial. Since $f(\alpha) = 0$, we would have $a_k \alpha^k = 0$. Since $a_k \neq 0$, we would have $\alpha^k = 0$. Then $\alpha = 0$. Contradiction arises.

Now it follows that $g(x)$ is is non-constant.

We have $g(\frac{1}{\alpha}) = a_0 \left(\frac{1}{\alpha}\right)^k + a_1 \left(\frac{1}{\alpha}\right)^{k-1} + a_2 \left(\frac{1}{\alpha}\right)^{k-2} + \cdots + a_k = \frac{1}{\alpha^k} \cdot (a_0 + a_1\alpha + a_2\alpha^2 + \cdots + a_k\alpha^k) = \frac{1}{\alpha^k} \cdot f(\alpha) = 0$.

Therefore, by definition, $\frac{1}{\alpha}$ is algebraic.

ii. ——

iii. ——

(c) *Comment.* Apply the proof-by-contradiction argument, and make use of the results established in the previous part.

8. ——