MATH1050 Proof-writing Exercise 2

Advice.

- All the questions are concerned with the handling of '*there exists*' and/or with proof-by-contradiction argument.
- When doing proofs, remember to adhere to definition, always.

  Study the handouts *Basic results on divisibility*, and *Rationals and irrationals*.
- Besides the handout mentioned above, Question (6), Question (7), Question (8), Question (9) in Assignment 2 are also suggestive on what it takes to give the types of argument meant to be written here, and on the level of rigour required.

1. Prove the statements below:
   (a) *Let x, y be real numbers. Suppose x, y are rational. Then $x - y$ is rational.*
   (b) *Let x, y be real numbers. Suppose x, y are rational and $y \neq 0$. Then $\frac{x}{y}$ is rational.*
2. Prove the statement ($\sharp$):
   ($\sharp$) *Let $x, y \in \mathbb{Z}$. Suppose x is divisible by y and y is divisible by x. Then $|x| = |y|$.*
3. Apply proof-by-contradiction to justify the statements below:
   (a) *Let a, b be complex numbers. Suppose $a^4 + a^3b + a^2b^2 + ab^3 + b^4 \neq 0$. Then at least one of a, b is non-zero.*
   (b) *Let a, b be real numbers. Suppose $ab > 1$. Then $a^2 + 4b^2 > 4$.*
   (c) *Let ζ be a complex number. Suppose that $|\zeta| \leq \varepsilon$ for any positive real number ε. Then $\zeta = 0$.*
4. In this question, take for granted that $\sqrt{2}, \sqrt{3}$ are irrational numbers.

   Apply proof-by-contradiction to justify the statements below:
   (a) *$\sqrt{2} + \sqrt{3}$ is an irrational number.*

   Remark. *Hint.* Write $r = \sqrt{2} + \sqrt{3}$. Can you re-express one of $\sqrt{2}, \sqrt{3}$ as a fractional expression whose numerator and denominator involve only integers and the non-negative integral powers of r?

   (b) *$\sqrt{3} - \sqrt{2}$ is an irrational number.*

   Remark. See if you can generalize the argument to prove the statement ($\sharp$):

   ($\sharp$) *Suppose a, b are non-zero rational numbers. Then $a\sqrt{2} + b\sqrt{3}$ is an irrational number.*
5. Take for granted the validity of Euclid's Lemma where appropriate and necessary. You may also take for granted that $2, 3, 5$ are prime numbers.

   Apply proof-by-contradiction to justify the statements below:
   (a) *$\sqrt{3}$ is an irrational number.*
   (b) *$\sqrt[3]{5}$ is an irrational number.*
   (c) *$\sqrt[3]{4}$ is an irrational number.*
6. Apply proof-by-contradiction to justify the statements below:
   (a) *2 is not divisible by 3.*

   Remark. Apply the definition for the notion of divisibility to obtain an equality with 2 on one side and an expression involving 3 and some intger on the other side. Then obtain a contradiction by considering the magnitudes of the integers involved.

   (b)$^\diamondsuit$ *3 is not divisible by 2.*

   (c)$^\clubsuit$ *$\sqrt{6}$ is irrational.*

   Remark. Take for granted the validity of Euclid's Lemma where appropriate and necessary. You may also need the results described in the previous parts.

7. We introduce the definitions for the notions of *algebraicity* and *transcendence* for complex numbers:

- *Let α be a complex number. We say that α is* **algebraic** *if there exists some non-constant polynomial $f(x)$ whose coefficients are rational numbers such that $f(\alpha) = 0$.*
- *Let τ be a complex number. We say that τ is* **transcendental** *if τ is not algebraic.*

(a) Verify that the numbers below are algebraic:

i. 0.
ii. 1.
iii. i.
iv. $\sqrt{2}$.
v. $\sqrt{2}i$.
vi. $\sqrt{2}+\sqrt{3}$.
vii. $\sqrt{2}+i$.
viii. $\sqrt{5+\sqrt[3]{2}}$.

(b) Prove the statements below:

i. *Let α be a non-zero complex number. Suppose α is algebraic. Then $\dfrac{1}{\alpha}$ is algebraic.*
ii. *Let α be a positive real number. Suppose α is algebraic. Then $\sqrt{\alpha}$ is algebraic.*
iii.♣ *Let α be a complex number. Suppose α is algebraic. Then α^2 is algebraic.*

(c) Prove the statements below:

i. *Let τ be a non-zero complex number. Suppose τ is transcendental. Then $\dfrac{1}{\tau}$ is transcendental.*
ii. *Let τ be a positive real number. Suppose τ is transcendental. Then τ^2 is transcendental.*
iii. *Let τ be a positive real number. Suppose τ is transcendental. Then $\sqrt{\tau}$ is transcendental.*

8. For each $n \in \mathbb{N}\backslash\{0\}$, define $A_n = \sum_{j=1}^{n} \dfrac{1}{j}$, $B_n = \sum_{k=1}^{n} \dfrac{1}{2k}$, $C_n = \sum_{k=1}^{n} \dfrac{1}{2k-1}$.

(a) i. Prove that $B_n = \dfrac{1}{2}A_n$ and $C_n = A_{2n} - \dfrac{1}{2}A_n$ for any $n \in \mathbb{N}\backslash\{0\}$.

ii. Prove that $C_n - B_n \geq \dfrac{1}{2}$ for any $n \in \mathbb{N}\backslash\{0,1\}$.

(b) By applying proof-by-contradiction, or otherwise, prove that $\{A_n\}_{n=1}^{\infty}$ does not converge in $\mathbb{R}$.

Remark. Take for granted the result about inequality for limits of infinite sequences:

Let $\{x_n\}_{n=0}^{\infty}$ be an infinite sequence of real number, and t be a real number. Suppose $x_n \geq t$ for any $n \in \mathbb{N}$. Also suppose $\{x_n\}_{n=0}^{\infty}$ converges in $\mathbb{R}$. Then $\lim_{n\to\infty} x_n \geq t$.