MATH1050 Proof-writing Exercise 1 (Answers and selected solutions)

1. **Solution.**

   (a) Let x, y be real numbers. Suppose $x < y < 1$.

   Note that $\dfrac{y}{1-y} - \dfrac{x}{1-x} = \dfrac{y(1-x)-x(1-y)}{(1-x)(1-y)} = \dfrac{y-x}{(1-x)(1-y)}$. —— $(\star)$

   Since $x < 1$, we have $1 - x > 0$. Since $y < 1$, we have $1 - y > 0$.
   Since $x < y$, we have $y - x > 0$.

   Since $1-x > 0$ and $1-y > 0$ and $y - x > 0$, we have $\dfrac{y-x}{(1-x)(1-y)} > 0$.

   Then by $(\star)$, we have $\dfrac{y}{1-y} - \dfrac{x}{1-x} > 0$.

   Therefore $\dfrac{x}{1-x} < \dfrac{y}{1-y}$.

   (b) *Argument with the help of the previous part.*
   Let x, y be real numbers. Suppose $0 < x < y < 1$.
   By $(\sharp)$, we have $\dfrac{x}{1-x} < \dfrac{y}{1-y}$. —— $(\dagger)$

   By assumption, we have $x > 0$. Then by $(\dagger)$ also, we have $\dfrac{x^2}{1-x} = x \cdot \dfrac{x}{1-x} < x \cdot \dfrac{y}{1-y}$. —— $(\ddagger)$

   By assumption, we have $0 < y < 1$. Then $\dfrac{y}{1-y} > 0$.

   Therefore by $(\ddagger)$ also, we have $x \cdot \dfrac{y}{1-y} < y \cdot \dfrac{y}{1-y} = \dfrac{y^2}{1-y}$.

   We now have $\dfrac{x^2}{1-x} < x \cdot \dfrac{y}{1-y}$ and $x \cdot \dfrac{y}{1-y} < \dfrac{y^2}{1-y}$.

   Then $\dfrac{x^2}{1-x} < \dfrac{y^2}{1-y}$.

   Direct argument.
   Let x, y be real numbers. Suppose $0 < x < y < 1$.

   Note that $\dfrac{y^2}{1-y} - \dfrac{x^2}{1-x} = \dfrac{y^2(1-x)-x^2(1-y)}{(1-x)(1-y)} = \cdots\cdots\cdots = \dfrac{(y-x)[1-(1-x)(1-y)]}{(1-x)(1-y)}$. —— $(\star)$

   Since $0 < x < 1$, we have $0 < 1 - x < 1$. Since $0 < y < 1$, we have $0 < 1 - y < 1$.
   Now we have $0 < 1-x < 1$ and $0 < 1-y < 1$. Then $0 < (1-x)(1-y) < 1$. Therefore $1-(1-x)(1-y) > 0$.
   Since $x < y$, we have $y - x > 0$.

   Since $1-x > 0$ and $1-y > 0$ and $y-x > 0$ and $1-(1-x)(1-y) > 0$, we have $\dfrac{(y-x)[1-(1-x)(1-y)]}{(1-x)(1-y)} > 0$.

   Then by $(\star)$, we have $\dfrac{y^2}{1-y} - \dfrac{x^2}{1-x} > 0$.

   Therefore $\dfrac{x^2}{1-x} < \dfrac{y^2}{1-y}$.

2. (a) *Hint.*

   Make use of the 'rationalization formula' $\sqrt{u} - \sqrt{v} = \dfrac{u-v}{\sqrt{u}+\sqrt{v}}$. (But be careful on whether the formula is indeed valid with what you 'substitute' into u, v.)

   (b) *Hint.*
   Make use of the 'telescopic formula'

$$(u_1 - u_0) + (u_2 - u_1) + (u_3 - u_2) + \cdots + (u_p - u_{p-1}) = u_p - u_0.$$

3. *Hint.*

Start by verifying the equality $\left(x^m + \frac{1}{x^m}\right) - \left(x^n + \frac{1}{x^n}\right) = \frac{(x^m - x^n)(x^{m+n} - 1)}{x^{m+n}}$. Study the factors in the right-hand side.

4. (a) *Hint.*

Start by re-expressing $(u^2 + v^2)(x^2 + y^2) - (ux + vy)^2$ as a 'whole square' of real quantities.

(b) ——

5. (a) **Solution.**

Suppose u, v are positive real numbers.

$\sqrt{u}, \sqrt{v}, \sqrt{uv}$ are well-defined, and $u = (\sqrt{u})^2$, $v = (\sqrt{v})^2$, $\sqrt{uv} = \sqrt{u}\sqrt{v}$.

$\sqrt{u} - \sqrt{v}$ is well-defined as a real number. Then $(\sqrt{u} - \sqrt{v})^2 \geq 0$.

Therefore $u + v = (\sqrt{u})^2 + (\sqrt{v})^2 = (\sqrt{u} - \sqrt{v})^2 + 2\sqrt{u}\sqrt{v} \geq 2\sqrt{u}\sqrt{v} = 2\sqrt{uv}$.

Hence $\frac{u+v}{2} \geq \sqrt{uv}$.

Remark. Much of the work done prior to presenting the calculations should be about explaining why the expressions involving the 'square root' in the calculations are *well-defined*, and clarifying the behaviour of such expressions.

(b) *Hint.*

Make use of the chain of inequalities $\frac{1}{2}\left(\frac{a+b}{2} + \frac{c+d}{2}\right) \geq \frac{\sqrt{ab} + \sqrt{cd}}{2} \geq \sqrt{(\sqrt{ab}) \cdot (\sqrt{cd})}$.

(c) *Hint.*

With the positive real numbers r, s, t, define the positive real number $u = \frac{r+s+t}{3}$. Now r, s, t, u are four positive real numbers, on which the result in the previous part may be applied.

6. ——

7. (a) Let a, b be real numbers, and $h : \mathbb{R} \longrightarrow \mathbb{R}$ be the function defined by $h(x) = x^3 - 3a^2x + b$ for any $x \in \mathbb{R}$. Suppose $a > 0$.

Pick any $s, t \in [a, +\infty)$. Suppose $s < t$.

Note that

$$\begin{aligned} h(t) - h(s) &= (t^3 - 3a^2t + b) - (s^3 - 3a^2s + b) \\ &= (t^3 - s^3) - 3a^2(t - s) \\ &= (t - s)(t^2 + st + s^2 - 3a^2) \end{aligned}$$

Since $s < t$, we have $t - s > 0$.

Since $0 < a \leq s < t$, we have $0 < a^2 \leq s^2 < st < t^2$.

Then $t^2 + st + s^2 > 3s^2 \geq 3a^2$.

Therefore $t^2 + st + s^2 - 3a^2 > 0$.

Hence $h(t) - h(s) = (t - s)(t^2 + st + s^2 - 3a^2) > 0$. Then $h(s) < h(t)$.

It follows that h is strictly increasing on the interval $[a, +\infty)$.

(b) Let a be a real number, and $h : (0, +\infty) \longrightarrow \mathbb{R}$ be the real-valued function of one real variable defined by $h(x) = x + \frac{a^2}{x}$ for any $x \in (0, +\infty)$. Suppose $a > 0$.

Pick any $s, t \in (0, a]$. Suppose $s < t$.

Note that

$$h(t) - h(s) = (t + \frac{a^2}{t}) - (s + \frac{a^2}{s}) = (t - s) - a^2(\frac{1}{s} - \frac{1}{t}) = (t - s) - \frac{a^2}{st}(t - s) = \frac{t - s}{st} \cdot (st - a^2)$$

Since $0 < s < t$, we have $t - s > 0$ and $st > 0$. Then $\frac{t-s}{st} > 0$.

Since $0 < s < t \leq a$, we have $0 < st < t^2 \leq a^2$. Then $st - a^2 < 0$.

Therefore $h(t) - h(s) = \dfrac{t-s}{st} \cdot (st - a^2) < 0$. Then $h(t) < h(s)$.

It follows that h is strictly decreasing on $(0, a]$.

8. (a) *Hint.*

Start in this way:

Let a, b be real numbers, and $h : \mathbb{R} \longrightarrow \mathbb{R}$ be the function defined by $h(x) = x^2 + ax + b$ for any $x \in \mathbb{R}$.

Remember that we want to verify the statement (for such a function h)

For any $s, t \in \mathbb{R}$, for any $\theta \in (0,1)$, if $s < t$ then $h((1-\theta)s + \theta t) < (1-\theta)h(s) + \theta h(t)$.

So we continue with these words:

Pick any $s, t \in \mathbb{R}$. Pick any $\theta \in (0,1)$. Suppose $s < t$.

Then patiently verify $(1-\theta)h(s)+\theta h(t)-h((1-\theta)s+\theta t) = \theta(1-\theta)(s-t)^2$, and argue that the 'right-hand-side' of this equality is a positive number.

(b) ——