MATH1050 Proof-writing Exercise 1

Advice.

- All the questions are concerned with 'direct proofs' for inequalities.
- The algebraic methods that you have learnt at school will suffice for the construction of the arguments.

  There is no need to use anything that you are taught in the calculus of one real variable. (In fact, an argument done here using calculus may well be wrong because it is likely a 'circular argument'.)
- Study the handout *From simple inequalities to basic properties of the reals*, and Questions (2), (3) in Assignment 1, if you have not done so.

  The material will give you an idea on the type of argument meant to be written here, and on the level of rigour required.

1. (a) Prove the statement ($\sharp$):

   ($\sharp$) *Let x, y be real numbers. Suppose $x < y < 1$. Then* $\dfrac{x}{1-x} < \dfrac{y}{1-y}$.

   (b) Prove the statement ($\flat$):

   ($\flat$) *Let x, y be real numbers. Suppose $0 < x < y < 1$. Then* $\dfrac{x^2}{1-x} < \dfrac{y^2}{1-y}$.

   Remark. There are two possible approaches. One is to make a clever and careful use of the previous part. The other is a direct argument (which involves a careful analysis of the algebra).

2. In this question you may take for granted the validity of the statement ($\star$):

   ($\star$) *Let u, v be positive real numbers. Suppose $u > v$. Then $\sqrt{u} > \sqrt{v}$.*

   (a) Prove the statement ($\sharp$):

   ($\sharp$) *Let x be a real number. Suppose $x > 0$. Then* $\sqrt{x+2} - \sqrt{x+1} < \dfrac{1}{2\sqrt{x+1}} < \sqrt{x+1} - \sqrt{x}$.

   (b) Hence prove that $193 < \displaystyle\sum_{k=10}^{10000} \frac{1}{\sqrt{k}} < 194$.

   Remark. You may need the inequality $3.3^2 > 10$.

3.$^\diamond$ Prove the statement ($\sharp$):

   ($\sharp$) *Let $m, n \in \mathbb{N}\backslash\{0\}$. Let x be a positive real number. Suppose $m > n$. Then* $x^m + \dfrac{1}{x^m} \geq x^n + \dfrac{1}{x^n}$. *Moreover, equality holds iff $x = 1$.*

4. (a) Prove the statement ($\sharp$) below:

   ($\sharp$) *Suppose u, v, x, y are real numbers. Then $(ux + vy)^2 \leq (u^2 + v^2)(x^2 + y^2)$.*

   Remark. This is a 'baby version' of the Cauchy-Schwarz Inequality.

   (b) Hence, or otherwise, prove the statement ($\flat$) below:

   ($\flat$) *Suppose s, t be positive real numbers. Then* $(s+t)\left(\dfrac{1}{s} + \dfrac{1}{t}\right) \geq 4$.

5. (a) By considering the non-negativity of squares, or otherwise, prove the statement ($\natural$) below:

   ($\natural$) *Suppose u, v are positive real numbers. Then* $\dfrac{u+v}{2} \geq \sqrt{uv}$.

   (b)$^\diamond$ Hence, or otherwise, prove the statement ($\sharp$) below:

   ($\sharp$) *Suppose a, b, c, d are positive real numbers. Then* $\dfrac{a+b+c+d}{4} \geq \sqrt[4]{abcd}$.

(c)$^\clubsuit$ Hence, or otherwise, prove the statement ($\flat$) below:

($\flat$) *Suppose* r, s, t *be positive real numbers. Then* $\dfrac{r+s+t}{3} \geq \sqrt[3]{rst}$.

Remark. These are 'baby versions' of the Arithmetico-Geometrical Inequality.

6. Let c, ε be positive real numbers.

(a)$^\diamondsuit$ Define $\delta = \sqrt{c^2+\varepsilon} - c$.

i. Prove that $\delta > 0$.

ii. Let x be a real number. Suppose $|x-c| < \delta$.

A. Prove that $|x+c| \leq \sqrt{c^2+\varepsilon} + c$.

B. Hence, or otherwise, deduce that $|x^2-c^2| < \varepsilon$.

(b)$^\clubsuit$ Define $\delta = \min\left\{1, \dfrac{\varepsilon}{1+2c}\right\}$.

i. Prove that $\delta > 0$.

ii. Let x be a real number. Suppose $|x-c| < \delta$.

A. Prove that $|x+c| \leq 1+2c$.

B. Hence, or otherwise, deduce that $|x^2-c^2| < \varepsilon$.

Remark. Overall we have given two independent proofs for the statement below:

For any $c > 0$, *for any* $\varepsilon > 0$, *there exists some* $\delta > 0$ *such that for any* $x \in \mathbb{R}$, *if* $|x-c| < \delta$ *then* $|x^2-c^2| < \varepsilon$.

Hence we have argued for the continuity of the 'square function' t^2 at every positive value of t.

7. *Recall the definitions for the notion of* strict monotonicity *for real-valued functions of one real variable from Assignment 1.*

Prove the statements below, with direct reference to the definitions on strict monotonicity.

(a) *Let* a, b *be real numbers, and* $h : \mathbb{R} \longrightarrow \mathbb{R}$ *be the function defined by* $h(x) = x^3 - 3a^2x + b$ *for any* $x \in \mathbb{R}$. *Suppose* $a > 0$. *Then* h *is strictly increasing on the interval* $[a, +\infty)$.

(b) *Let* a *be a real number, and* $h : (0, +\infty) \longrightarrow \mathbb{R}$ *be the real-valued function of one real variable defined by* $h(x) = x + \dfrac{a^2}{x}$ *for any* $x \in (0, +\infty)$. *Suppose* $a > 0$. *Then* h *is strictly decreasing on* $(0, a]$.

8. We introduce/recall the definitions on *strict convexity* for real-valued functions of one real variable. Do not use any results from the calculus of one real variable.

Let I *be an interval, and* $h : D \longrightarrow \mathbb{R}$ *be a real-valued function of one real variable with domain* D *which contains* I *as a subset entirely.*

h *is said to be* **strictly convex** *on* I *if the statement* (StrConv) *holds:*

(StrConv) *For any* $s, t \in I$, *for any* $\theta \in (0,1)$, *if* $s < t$ *then* $h((1-\theta)s + \theta t) < (1-\theta)h(s) + \theta h(t)$.

(We can define 'strict concavity' in an analogous manner.)

Prove the statements below, with direct reference to the definitions on strict convexity:

(a) *Let* a, b *be real numbers, and* $h : \mathbb{R} \longrightarrow \mathbb{R}$ *be the function defined by* $h(x) = x^2 + ax + b$ *for any* $x \in \mathbb{R}$. *The function* h *is strictly convex on* $\mathbb{R}$.

(b) Let $h : (0, +\infty) \longrightarrow \mathbb{R}$ be the function defined by $h(x) = \dfrac{1}{x}$ for any $x \in (0, +\infty)$. The function h is strictly convex on $(0, +\infty)$.