MATH1050 Assignment 12 (Answers and selected solutions)

1. **Answer.**

   (a) Yes.

   (b) No.

   (c) No.

   (d) Yes.

2. **Answer.**

   (a) (I) For any
   (II) $(p,p) \in J$
   (III) Pick any
   (IV) There exist some
   (V) such that $p = (s,t)$
   (VI) and $t \leq t$
   (VII) $(p,p) \in J$

   (b) (I) For any $p, q \in \mathbb{N}^2$, if $(p,q) \in J$ and $(q,p) \in J$
   (II) $p = q$
   (III) Suppose $(p,q) \in J$ and $(q,p) \in J$
   (IV) and
   (V) or
   (VI) $(q,p) \in J$
   (VII) $u \leq s$
   (VIII) $u \leq s$
   (IX) $t \leq v$
   (X) $v \leq t$
   (XI) and
   (XII) $t = v$
   (XIII) $p = q$

   (c) (I) For any $p, q, r \in \mathbb{N}^2$, if $(p,q) \in J$ and $(q,r) \in J$
   (II) $(p,r) \in J$
   (III) Suppose $(p,q) \in J$ and $(q,r) \in J$.
   (IV) Suppose $p = q$
   (V) $(p,r) = (p,q) \in J$
   (VI) $p \neq q$ and $q \neq r$
   (VII) There exist some $s, t, u, v, w, x \in \mathbb{N}$ such that $p = (s,t)$, $q = (u,v)$ and $r = (w,x)$
   (VIII) $(p,q) \in J$
   (IX) false
   (X) or
   (XI) and
   (XII) $(q,r) \in J$
   (XIII) $u < w$ or ($u = w$ and $v < x$)
   (XIV) $s < u < w$
   (XV) Suppose $s < u$ and ($u = w$ and $v < x$)
   (XVI) Suppose ($s = u$ and $t < v$) and $u < w$
   (XVII) Suppose ($s = u$ and $t < v$) and ($u = w$ and $v < x$)
   (XVIII) $(p,r) \in J$

(d) (I) $(p, q) \in J$ or $(q, p) \in J$

(II) Pick any $p, q \in \mathbb{N}^2$

(III) $s < u$ or ($s = u$ and $t \leq v$)

(IV) Suppose $s = u$

(V) or

(VI) Suppose $t \leq v$

(VII) Suppose $v \leq t$

(VIII) $(q, p) \in J$

(IX) Suppose $s > u$

(X) $u < s$ or ($u = s$ and $v \leq t$)

(XI) $(q, p) \in J$

(XII) $(p, q) \in J$ or $(q, p) \in J$

(e) (I) for any

(II) $\mathbb{N}^2$

(III) B is non-empty

(IV) B has a least element with respect to T

(V) Let B be a non-empty subset of $\mathbb{N}^2$

(VI) There exists some $u, v \in \mathbb{N}$

(VII) for some

(VIII) subset of $\mathbb{N}$

(IX) $u \in B'$

(X) least element

(XI) there exists some

(XII) such that $(s, w) \in B$

(XIII) B'' is non-empty

(XIV) the Well-ordering Principle for Integers

(XV) for any $q \in B$, $(s, t) \leq_{\text{lex}} q$

(XVI) $q \in B$

(XVII) s is a least element of B'

(XVIII) or

(XIX) or

(XX) and

(XXI) Suppose $s = x$

(XXII) $y \in B''$

(XXIII) $t \leq y$

(XXIV) $(s, t) \leq_{\text{lex}} (x, y) = q$

3. **Answer.**

(a) ——

(b) No.

(c) No.

4. **Answer.**

(a) ——

(b) Yes.

(c) No.

(d) No.

(e) Neither.

5. **Answer.**
   (a) ——
   (b) ——
   (c) Yes.
   (d) No.

6. **Answer.**
   (a) ——
   (b) ——
   (c) ——
   (d) No.

7. **Answer.**
   (a) ——
   (b) Yes.
   (c) ——

8. **Answer.**
   (a) An injective function from A to B is $f : A \longrightarrow B$ defined by $f(x) = \frac{x}{2} + \frac{5}{2}$ for any $x \in A$.
   (b) An injective function from B to A is $g : B \longrightarrow A$ defined by $g(y) = \frac{y}{4}$ for any $y \in B$.

9. **Solution.**
   Let $A = [1010, 1050] \backslash \{1030\}$ and $B = (2040, 2050) \cup ([2060, +\infty) \cap \mathbb{Q})$.
   (a) Define the function $f : A \longrightarrow B$ by $f(x) = \frac{x}{1050} + 2040$ for any $x \in A$.
   Note that $\frac{x}{1050} + 2040 \in B$ for any $x \in A$. Then f is well-defined as a function.
   We verify that f is injective:
   * Pick any $x, w \in A$. Suppose $f(x) = f(w)$. Then $\frac{x}{1050} + 2040 = \frac{w}{1050} + 2040$. Therefore $\frac{x}{1050} = \frac{w}{1050}$. Hence $x = w$.

   Therefore $A \lesssim B$.
   (b)
   - Define the function $g : B \longrightarrow A$ by $g(y) = \frac{1}{y} + 1010$ for any $y \in B$.
     Note that $\frac{1}{y} + 1010 \in A_6$ for any $y \in B_6$. Then g is well-defined as a function.
     We verify that g is injective:
     * Pick any $y, z \in B$. Suppose $g(y) = g(z)$. Then $\frac{1}{y} + 1010 = \frac{1}{z} + 1010$. Therefore $\frac{1}{y} = \frac{1}{z}$. Hence $y = z$.

     Therefore $B \lesssim A$.
   - By Schröder-Bernstein Theorem, since $A \lesssim B$ and $B \lesssim A$, we have $A \sim B$.

10. **Answer.**
    (a) ——
    (b) Yes.
    (c) No.
    (d) Yes.

11. **Answer.**
    These sets are of cardinality equal to $\mathbb{N}$:

$\mathbb{N}$, $\mathbb{Q}$, $\mathbb{N}^3$, $\mathsf{Map}(\{0,1\},\mathbb{N})$.

These sets are of cardinality equal to $\mathfrak{P}(\mathbb{N})$:

$\mathbb{R}\backslash\mathbb{Q}$, $[0,1]$, $\mathfrak{P}(\mathbb{N})$, $\mathsf{Map}(\mathbb{N},\{0,1\})$, $\mathbb{C}$, $\mathbb{N}\times\mathbb{R}$.

These sets are of cardinality equal to $\mathfrak{P}(\mathfrak{P}(\mathbb{N}))$:

$\mathfrak{P}(\mathbb{R})$, $\mathsf{Map}(\mathbb{R},\{0,1\})$.