MATH1050 Assignment 12

1. *You are not required to justify your answer in this question.*

   Denote by A the set of all non-zero polynomial functions on $\mathbb{R}$.

   Define $G = \left\{ (f,g) \;\middle|\; f,g \in A \text{ and } \lim_{x\to+\infty} \frac{f(x)}{g(x)} \text{ exists in } \mathbb{R} \right\}$. Define $R = (A, A, G)$.

   Note that R is a relation in A with graph G.

   For each statement below, decide whether it is true or false:

   (a) *R is reflexive.*

   (b) *R is symmetric.*

   (c) *R is anti-symmetric.*

   (d) *R is transitive.*

2. Define $J = \left\{ ((s,t),(u,v)) \;\middle|\; \begin{array}{l} s,t,u,v \in \mathbb{N}, \text{ and} \\ [s < u \text{ or } (s = u \text{ and } t \leq v)] \end{array} \right\}$, and $T = (\mathbb{N}^2, \mathbb{N}^2, J)$.

   Note that $J \subset \mathbb{N}^2 \times \mathbb{N}^2$. Hence T is a relation in $\mathbb{N}^2$ with graph J.

   Fill in the blanks in the blocks below, all labelled by capital-letter Roman numerals, with appropriate words so that it gives a proof for the statement $(A1)$, a proof for the statement $(A2)$, a proof for the statement $(A3)$, a proof for the statement $(A4)$, and a proof for the statement $(A5)$. (*The 'underline' for each blank bears no definite relation with the length of the answer for that blank.*)

   (a) Here we prove the statement $(A1)$:

   $(A1)$ *T is reflexive.*

   > We verify the statement '___(I)___ $p \in \mathbb{N}^2$, ___(II)___'.
   > ___(III)___ $p \in \mathbb{N}^2$. ___(IV)___ $s,t \in \mathbb{N}$ ___(V)___.
   > We have $s = s$ ___(VI)___.
   > Then by the definition of J, we have ___(VI)___.
   > It follows that T is reflexive.

   (b) Here we prove the statement $(A2)$:

   $(A2)$ *T is anti-symmetric.*

   > We verify the statement '___(I)___ *then* ___(II)___'.
   > Pick any $p,q \in \mathbb{N}^2$. ___(III)___.
   > There exist some $s,t,u,v \in \mathbb{N}$ such that $p = (s,t)$ and $q = (u,v)$.
   > Since $(p,q) \in J$, we have $(s < u$ or $(s = u$ and $t \leq v))$.
   > Then $s \leq u$ ___(IV)___ $(s < u$ ___(V)___ $t \leq v)$, by the Law of Distribution for conjunction and disjunction.
   > In particular, $s \leq u$.
   > Since ___(VI)___, we also deduce ___(VII)___ by modifying the argument above.
   > Then $s \leq u$ and ___(VIII)___. Therefore $s = u$.
   > Now recall that $(s < u$ or $(s = u$ and $t \leq v))$.
   > Then $s = u$ and ___(IX)___. In particular, $t \leq v$.
   > Similarly, we deduce that ___(X)___.
   > Then $t \leq v$ ___(XI)___ $v \leq t$. Therefore ___(XII)___.
   > Then $s = u$ and $t = v$. Hence ___(XIII)___
   > It follows that T is anti-symmetric.

   (c) Here we prove the statement $(A3)$:

($A3$) *T is transitive.*

We verify the statement '______(I)______ *then* ____(II)____'.

Pick any $p, q, r \in \mathbb{N}^2$. __________(III)__________

- (Case 1.) ______(IV)______ . Then $(p, r) = (q, r) \in J$.
- (Case 2.) Suppose $q = r$. Then ______(V)______ .
- (Case 3.) Suppose ______(VI)______ .

  __________(VII)__________ .

  Since ____(VIII)____ , we have $s < u$ or ($s = u$ and $t \leq v$).

  Since $p \neq q$, '$s = u$ and $t = v$' is ____(IX)____ .

  Then we have $s < u$ ____(X)____ ($s = u$ ____(XI)____ $t < v$).

  Since ____(XII)____ , we can also deduce that ______(XIII)______ , by modifying the argument above.

  Now we have [$s < u$ or ($s = u$ and $t < v$)] and [$u < w$ or ($u = w$ and $v < x$)].

  - (Case 3a.) Suppose $s < u$ and $u < w$. Then ____(XIV)____ .
    Therefore $s < w$ or ($s = w$ and $t < x$).
  - (Case 3b.) ______(XV)______ . Then $s < u = w$.
    Therefore $s < w$ or ($s = w$ and $t < x$).
  - (Case 3c.) ______(XVI)______ . Then $s = u < w$.
    Therefore $s < w$ or ($s = w$ and $t < x$).
  - (Case 3d.) ______(XVII)______ . Then $s = u = w$ and $t < v < x$.
    Therefore $s < w$ or ($s = w$ and $t < x$).

  Then in any case $s < w$ or ($s = w$ and $t < x$). Hence ____(XVIII)____ .

Hence in any case, $(p, r) \in J$.
It follows that T is transitive.

(d) Here we prove the statement ($A4$):

($A4$) *T is strongly connected.*

We verify the statement '*For any $p, q \in \mathbb{N}^2$,* ______(I)______ '.

______(II)______ .

There exist some $s, t, u, v \in \mathbb{N}$ such that $p = (s, t)$ and $q = (u, v)$.

We have $s < u$ or $s = u$ or $s > u$.

- (Case 1.) Suppose $s < u$. Then ______(III)______ . Therefore $(p, q) \in J$.
- (Case 2.) ____(IV)____ . We have $t \leq v$ ____(V)____ $v \leq t$.
  - (Case 2a.) ____(VI)____ . Then $s = u$ and $t \leq v$.
    Therefore $s < u$ or ($s = u$ and $t \leq v$). Hence $(p, q) \in J$.
  - (Case 2b.) ____(VII)____ . Then $u = s$ and $v \leq t$.
    Therefore $u < s$ or ($u = s$ and $v \leq t$). Hence ____(VIII)____ .
- (Case 3.) ____(IX)____ . Then ______(X)______ . Therefore ____(XI)____ .

Hence in any case, ________(XII)________ .
It follows that T is strongly connected.

(e) Here we prove the statement ($A5$):

($A5$) *T is a well-ordered relation in $\mathbb{N}^2$.*

> According to the previous parts, T is a total ordering.
> From now on, we write $p \leq_{\text{lex}} q$ exactly when $(p, q) \in J$.
> We verify the statement '___(I)___ *subset B of* ___(II)___ *, if* ___(III)___ *, then* ___(IV)___ '.
> ___(V)___ . Take some $p \in B$. ___(VI)___ such that $p = (u, v)$.
>
> - Define the set $B' = \{x \in \mathbb{N} : (x, y) \in B$ ___(VII)___ $y \in \mathbb{N}\}$.
>
> B' is a ___(VIII)___ according to its definition.
>
>   By definition of B', ___(IX)___ . Then B' is non-empty.
>
>   By the Well-ordering Principle for Integers , B' has a ___(X)___ . Denote it by s.
>
> - Define $B'' = \{y \in \mathbb{N} : (s, y) \in B\}$.
>
> B'' is a subset of $\mathbb{N}$ according to its definition.
>
>   Recall that $s \in B'$. Then by definition of B', ___(XI)___ $w \in \mathbb{N}$ ___(XII)___ .
>
>   By definition of B'', $w \in B''$.
>
>   Therefore ___(XIII)___ .
>
>   By ___(XIV)___ , B'' has a least element.
>
>   Denote it by t.
>
> - We verify that (s, t) is the least element of B with respect to T.
>
>   According to the definition for the notion of least element, we verify the statement '___(XV)___ '
>
>   Pick any $q \in B$. By definition, $q \in \mathbb{N}^2$.
>   Then there exist some $x, y \in \mathbb{N}$ such that $q = (x, y)$.
>   By definition of B', ___(XVI)___ , we have $x \in B'$.
>   Then, since ___(XVII)___ , we have $s \leq x$.
>   Note that $s < x$ ___(XVIII)___ $s = x$.
>
>   - (Case 1.) Suppose $s < x$. Then $s < x$ ___(XIX)___ ($s = x$ ___(XX)___ $t \leq y$).
>   Therefore $(s, t) \leq_{\text{lex}} (x, y) = q$.
>   - (Case 2.) ___(XXI)___ . Then, by definition of B'', since $(s, y) = (x, y) \in B$, we have ___(XXII)___ .
>   Now, since t is a least element of B'', we have ___(XXIII)___
>   Then $s = x$ and $t \leq y$. Therefore $s < x$ or ($s = x$ and $t \leq y$).
>   Then ___(XXIV)___ .
>
>   Hence, in any case, $(s, t) \leq_{\text{lex}} q$.
>
> It follows that T is a well-order relation in $\mathbb{N}^2$.

3. Define the relation $R = (\mathbb{R}, \mathbb{R}, P)$ in $\mathbb{R}$ by $P = \{(x, y) \in \mathbb{R}^2 :$ There exists some $n \in \mathbb{N}$ such that $y = 2^n x\}$.
   (a) Verify that R is a partial ordering in $\mathbb{R}$.
   (b) Is R a total ordering in $\mathbb{R}$? Why?
   (c) Is R an equivalence relation in $\mathbb{R}$? Why?

4. Let $R = (\mathbb{R}, \mathbb{R}, G)$ be the relation in $\mathbb{R}$ defined by $G = \{(x, y) \mid x \in \mathbb{R}$ and $y \in \mathbb{R}$ and $|y - x| \leq 1\}$.
   (a) Verify that R is reflexive.

(b) Is R symmetric? Justify your answer.

(c) Is R anti-symmetric? Justify your answer.

(d) Is R transitive? Justify your answer.

(e) Is R an equivalence relation in $\mathbb{R}$? Is R a partial ordering in $\mathbb{R}$? Or neither? Why?

5. Let $A = \{\varphi \mid \varphi : \mathbb{N} \longrightarrow \mathbb{R} \text{ is a function and } \varphi(0) = 0.\}$. Define the relation $R = (A, A, H)$ in A by $H = \{(\varphi, \psi) \in A^2 : \varphi(n+1) - \varphi(n) \leq \psi(n+1) - \psi(n) \text{ for any } n \in \mathbb{N}\}$.

(a) Verify that R is reflexive.

(b) Verify that R is transitive.

(c) Is R a partial ordering in A? Justify your answer.

(d) Is R an equivalence relation in A? Justify your answer.

6. Define the relation $R = (\mathbb{Z}, \mathbb{Z}, G)$ in $\mathbb{Z}$ by $G = \{(x, y) \in \mathbb{Z}^2 : \text{There exist some } m, n \in \mathbb{N} \text{ such that } y = mx + n.\}$.

(a) Verify that R is reflexive.

(b) Verify that R is transitive.

(c) Verify that R is not a partial ordering in $\mathbb{Z}$.

(d) Is R is an equivalence relation in $\mathbb{Z}$? Justify your answer.

7. Let $I = (0, +\infty)$, $J = [-1, 1]$.

(a) Prove that $\dfrac{1}{a+1} \in J$ for any $a \in I$.

(b) Define the function $g : I \longrightarrow J$ by $g(x) = \dfrac{1}{x+1}$ for any $x \in I$. Is g injective? Justify your answer.

(c) Apply the Schröder-Bernstein Theorem to prove that $I \sim J$.

8. Let $A = [-1, 1]$, $B = (-4, -2] \cup [2, 4)$.

(a) Name one injective function from A to B, if there is any at all, and verify that it is indeed an injective function from A to B.

(b) Apply the Schröder-Bernstein Theorem, or otherwise, to prove that A is of cardinality equal to B.

9.$^\diamond$ Let $A = [1010, 1050] \backslash \{1030\}$ and $B = (2040, 2050) \cup ([2060, +\infty) \cap \mathbb{Q})$.

(a) Name one injective function from A to B, if there is any at all, and verify that it is indeed an injective function from A to B.

(b) Apply the Schröder-Bernstein Theorem to prove that $A \sim B$.

10. Let $D = \{\zeta \in \mathbb{C} : |\zeta| \leq 1\}$. Define $F = \left\{ (z, w) \mid z \in \mathbb{C} \text{ and } w \in D \text{ and } w = \dfrac{iz|z|}{1 + |z| + |z|^2} \right\}$. Note that $F \subset \mathbb{C} \times D$. Define $f = (\mathbb{C}, D, F)$.

(a)$^\diamond$ Verify that f is a function.

(b)$^\diamond$ Is f injective? Justify your answer.

(c) Is f surjective? Justify your answer.

(d) Is it true that $D \sim \mathbb{C}$? Justify your answer.

Remark. Where appropriate and relevant, you may apply the Schröder-Bernstein Theorem in your argument.

11. *You are not required to justify your answer in this question.*

Consider the sets below. Classify them according to whether such a set is of cardinality equal to $\mathbb{N}$, or whether it is of cardinality equal to $\mathfrak{P}(\mathbb{N})$, or whether it is of cardinality equal to $\mathfrak{P}(\mathfrak{P}(\mathbb{N}))$.

$$\begin{array}{llll} \mathbb{N}, & \mathbb{Q}, & \mathbb{R}\backslash\mathbb{Q}, & \mathsf{Map}(\{0,1\}, \mathbb{N}), \\ \mathbb{N}^3, & [0,1], & \mathfrak{P}(\mathbb{N}), & \mathsf{Map}(\mathbb{N}, \{0,1\}), \\ \mathbb{N} \times \mathbb{R}, & \mathbb{C}, & \mathfrak{P}(\mathbb{R}), & \mathsf{Map}(\mathbb{R}, \{0,1\}). \end{array}$$