MATH1050 Assignment 11

1. *You are not required to justify your answer in this question. You are only required to give one correct answer for each ordered pair, although there may be different correct answers.*

   Let $A = [0, +\infty)$ and G, H be the subsets of $\mathbb{R}^2$ defined respectively by

   $$G = \{(x, x) \mid x > 0\},$$
   $$H = \{(x, y) \mid x \geq 0 \text{ and } y > 0 \text{ and } x^2 + y^2 = 1\}.$$

   Name some appropriate ordered pairs $(s, t), (u, v) \in A^2$, if such exist, for which the ordered triple $(A, A, (G \cup H \cup \{(s, t), (u, v)\}))$ is a reflexive and symmetric relation in A.

2. *You are not required to justify your answer in this question.*

   Let $\alpha, \beta, \gamma, \delta \in \mathbb{R}$, and

   $$A = [0, +\infty), \qquad B = \{(x, y) \mid y = \alpha x\},$$
   $$C = \{(x, y) \mid y = 2x + 1\}, \qquad D = \{(x, y) \mid \beta y = x + \gamma\},$$
   $$E = \{(x, y) \mid (x - 1)^2 + (y - \delta)^2 = \delta^2\}, \qquad G = A^2 \cap (B \cup C \cup D \cup E).$$

   Suppose (A, A, G) is a reflexive and symmetric relation.

   What are the values of $\alpha, \beta, \gamma, \delta$?

3. Fill in the blanks in the block below, all labelled by capital-letter Roman numerals, with appropriate words so that it gives a proof for the statement (A). (*The 'underline' for each blank bears no definite relation with the length of the answer for that blank.*)

   We prove the statement (A):

   (A) *Suppose $E = \{(x, y) \in \mathbb{R}^2 : x - y = a$ for some $a \in \mathbb{Q}\}$, and $S = (\mathbb{R}, \mathbb{R}, E)$. Then S is an equivalence relation in $\mathbb{R}$.*

   Suppose $E = \{(x, y) \in \mathbb{R}^2 : x - y = a$ for some $a \in \mathbb{Q}\}$, and $S = (\mathbb{R}, \mathbb{R}, E)$. By definition, E is a subset of $\mathbb{R}^2$. Then S us a relation in $\mathbb{R}$.

   - [We want to verify the reflexivity of S.

     We verify the statement '_____ (I) _____'.]

     _____ (II) _____ .

     Note that _____ (III) _____ , and $0 \in \mathbb{Q}$. Then _____ (IV) _____ , by the definition of S.

     It follows that S is reflexive.

   - [We want to verify the symmetry of S.

     We verify the statement '_____ (V) _____'.]

     Pick any $x, y \in \mathbb{R}$. _____ (VI) _____ .

     Then _____ (VII) _____ such that $x - y = a$.

     Note that _____ (VIII) _____ .

     Since _____ (IX) _____ , we have $-a \in \mathbb{Q}$.

     Now we have $y - x = -a$ and _____ (X) _____ . Then _____ (XI) _____ , by the definition of S.

     It follows that S is symmetric.

   - [We want to verify the transitivity of S.

     We verify the statement '_____ (XII) _____'.]

     Pick any _____ (XIII) _____ . Suppose _____ (XIV) _____ .

     Since $(x, y) \in E$, there exists some $a \in \mathbb{Q}$ such that _____ (XV) _____ .

     _____ (XVI) _____ .

     Note that $x - z = (x - y) + (y - z) = a + b$.

     _____ (XVII) _____ .

     Now we have $x - z = a + b$, and $a + b \in \mathbb{Q}$. Then _____ (XVIII) _____ , by the definition of S.

     It follows that S is transitive.

   Since S is reflexive, symmetric and transitive, S is an equivalence relation.

4. Define $G = \{(x, y) \in \mathbb{R}^2 : x - y > xy\}$, and $S = (\mathbb{R}, \mathbb{R}, G)$.

Note that G is a subset of $\mathbb{R}^2$, and hence S is a relation in $\mathbb{R}$.

Fill in the blanks in the blocks below, all labelled by capital-letter Roman numerals, with appropriate words so that it gives a proof for the statement $(B1)$, a proof for the statement $(B2)$, and a proof for the statement $(B3)$. (*The 'underline' for each blank bears no definite relation with the length of the answer for that blank.*)

(a) Here we prove the statement $(B1)$:

$(B1)$ *S is not reflexive.*

> We verify the statement '___(I)___ *such that* ___(II)___':
>
> ___(III)___. Note that $x_0 \in \mathbb{R}$.
>
> Also note that ___(IV)___ $= 0 \leq 0 =$ ___(V)___. Therefore '$x_0 - x_0 > x_0 \cdot x_0$' is false.
>
> Hence, by the definition of G, ___(VI)___.
>
> It follows that S is not reflexive.

(b) Here we prove the statement $(B2)$:

$(B2)$ *S is not symmetric.*

> We verify the statement '___(I)___ such that ___(II)___':
>
> Take ___(III)___. Note that $x_0, y_0 \in \mathbb{R}$.
>
> Note that $x_0 - y_0 =$ ___(IV)___ $= x_0y_0$. Then by the definition of G, ___(V)___.
>
> Also note that $y_0 - x_0 = -1 \leq 0 = y_0x_0$. Then '___(VI)___' is false.
>
> Therefore by the definition of G, ___(VII)___.
>
> Hence for the same $x_0, y_0 \in \mathbb{R}$, ___(VIII)___ simultaneously.
>
> It follows that S is not symmetric.

(c) Here we prove the statement $(B3)$:

$(B3)$ *S is not transitive.*

> We verify the statement '___(I)___ *such that* ___(II)___':
>
> Take $x_0 = -2$, $y_0 = 3$, ___(III)___. Note that $x_0, y_0, z_0 \in \mathbb{R}$.
>
> Note that ___(IV)___ Then ___(V)___, $(x_0, y_0) \in G$.
>
> Also note that $y_0 - z_0 = 5 > -6 = y_0z_0$. Then by the definition of G, ___(VI)___.
>
> Finally, note that ___(VII)___. Then '$x_0 - z_0 > x_0z_0$' is false.
>
> Therefore by the definition of G, ___(VIII)___.
>
> Hence for the same $x_0, y_0, z_0 \in \mathbb{R}$, ___(IX)___ simultaneously.
>
> It follows that S is not transitive.

5. Write $\mathbb{C}^* = \mathbb{C}\backslash\{0\}$, $\mathbb{R}^* = \mathbb{R}\backslash\{0\}$. Define the relation $R = (\mathbb{C}^*, \mathbb{C}^*, E)$ in $\mathbb{C}^*$ by $E = \left\{(\zeta, \eta) \in (\mathbb{C}^*)^2 : \frac{\eta}{\zeta} \in \mathbb{R}^*\right\}$.

Note that E is a subset of $(\mathbb{C}^*)^2$, and hence $(\mathbb{C}^*, \mathbb{C}^*, E)$ is a relation in $\mathbb{C}^*$ with graph E.

Fill in the blanks in the blocks below, all labelled by capital-letter Roman numerals, with appropriate words so that it gives a proof for the statement $(C1)$, a proof for the statement $(C2)$, and a proof for the statement $(C3)$. (*The 'underline' for each blank bears no definite relation with the length of the answer for that blank.*)

(a) Here we prove the statement $(C1)$:

$(C1)$ *R is reflexive.*

> We verify the statement '___(I)___, $(\zeta, \zeta) \in E$':
>
> ___(II)___ ζ ___(III)___. We have ___(IV)___ $= 1 \in$ ___(V)___. Then (ζ, ζ) ___(VI)___.
>
> It follows that R is reflexive.

(b) Here we prove the statement $(C2)$:

$(C2)$ *R is transitive.*

> We verify the statement '*For any* $\zeta, \eta, \xi \in \mathbb{C}^*$, ______(I)______ ':
>
> Pick any $\zeta, \eta, \xi \in \mathbb{C}^*$. ______(II)______
>
> Since $(\zeta, \eta) \in E$, we have ___(III)___ . Since ___(IV)___ , we have $\frac{\xi}{\eta} \in \mathbb{R}^*$.
>
> Note that $\frac{\xi}{\zeta} = \frac{\xi}{\eta} \cdot \frac{\eta}{\zeta}$. Since $\frac{\eta}{\zeta} \in \mathbb{R}$ and $\frac{\xi}{\eta} \in \mathbb{R}$, we have ___(V)___ . Since ______(VI)______ , we have $\frac{\xi}{\zeta} \neq 0$.
>
> Now we have $\frac{\xi}{\zeta} \in \mathbb{R}$ and $\frac{\xi}{\zeta} \neq 0$. Then ______(VII)______ . Hence ___(VIII)___ $\in E$.
>
> It follows that R is transitive.

(c) Here we prove the statement $(C3)$:

$(C3)$ *R is an equivalence relation in $\mathbb{C}^*$.*

> We want to verify that R is ___(I)___ . We verify the statement ' ______(II)______ ':
>
> ______(III)______ . Suppose ___(IV)___ . Then $\frac{\eta}{\zeta} \in \mathbb{R}^*$.
>
> Since ___(V)___ , $\frac{1}{\eta/\zeta}$ is well-defined as a non-zero number, and $\frac{\zeta}{\eta} = \frac{1}{\eta/\zeta}$.
>
> Since $\frac{\eta}{\zeta} \in \mathbb{R}$, we have ___(VI)___ .
>
> We have $\frac{\zeta}{\eta} \in \mathbb{R}$ and $\frac{\zeta}{\eta} \neq 0$. Then $\frac{\zeta}{\eta} \in \mathbb{R}^*$. Therefore ___(VII)___ .
>
> It follows that R is symmetric.
>
> Since R is ______(VIII)______ , R is an equivalence relation.

6. *You are not required to justify your answer in this question.*

   Let $A = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$.

   Which of the sets below are partitions of A? Which not?

   $\Omega = \{\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}\}$,
   $\Sigma = \{\{0, 1\}, \{2, 3, 4, 5, 5\}, \{7, 8, 9\}\}$,
   $\Pi = \{\{1, 3, 5, 7, 9\}, \{0, 2, 4, 6, 8\}\}$,
   $\Gamma = \{\emptyset, \{0, 2\}, \{1, 3, 5\}, \{4, 6, 8\}, \{7, 9\}\}$,
   $\Xi = \{\{0\}, \{1\}, \{2\}, \{3\}, \{4\}, \{5\}, \{6\}, \{7\}, \{8\}, \{9\}\}$,
   $\mathrm{T} = \{\{0, 1\}, \{2, 3, 4\}, \{4, 5, 6\}, \{6\}, \{7, 8, 9\}\}$,
   $\Upsilon = \{\{0, 1\}, \{2, 3, 4\}, \{5, 6, 7, 8, 9\}, \{4, 3, 2\}\}$,
   $\Delta = \{\{0, 1\}, \{2, 3, 4\}, \{5, 6, 7, 8, 9, 10\}\}$.

7. *You are not required to justify your answer in this question.*

   Let $A = \{0, 1, 2, 3, 4, 5\}$, and Ω be the partition of A given by $\Omega = \{\{0\}, \{1, 3, 5\}, \{2, 4\}\}$.

   Write down all the elements of the graph E_Ω of the equivalence relation R_Ω in A induced by the partition Ω.

8. *You are not required to justify your answer in this question.*

   Let $A = \mathbb{C} \backslash \{0\}$, and $f : A \longrightarrow \mathbb{R}$ be the function defined $f(z) = \left| z + \frac{1}{z} \right|$ for any $z \in A$.

   Consider each of the ordered pairs. Decide whether it belongs to the graph E_f of the equivalence relation R_f in A induced by the function f.

   (a) $(1, 1)$
   (b) $(1, -1)$
   (c) $(1, i)$
   (d) $(2, \frac{1}{2})$
   (e) $(2, \frac{1}{2}i)$
   (f) $(2i, \frac{1}{2}i)$
   (g) $(2i, -\frac{1}{2}i)$
   (h) $(2, 3)$
   (i) $(1 + i, 1 - i)$
   (j) $(1 - i, 1 + i)$
   (k) $(4+3i, 4-3i)$
   (l) $(4+3i, 3+4i)$

9. Let $J = [0, 1)$, $K = (0, 1]$.

(a) Fill in the blanks in the block below, all labelled by capital-letter Roman numerals, with appropriate words so that it gives a proof for the statement (D). (*The 'underline' for each blank bears no definite relation with the length of the answer for that blank.*)

We prove the statement (D):

(D) *J is of cardinality equal to K.*

> Suppose $H = \{(x, y) \mid x \in J \text{ and } y \in K \text{ and } x + y = 1\}$, and $h = (J, K, H)$.
> Note that ____(I)____. Hence h is a relation from J to K with graph H.
> We verify the statement (E): '*for any $x \in J$,* ____(II)____'.
>
> Pick any ____(III)____. Take ____(IV)____. By definition, we have $x + y = x + (1 - x) = 1$.
> Since ____(V)____, we have $0 \leq x < 1$.
> Since $x \geq 0$, we have ____(VI)____.
> Since $x < 1$, we have ____(VII)____.
> Then $0 < y \leq 1$. Therefore ____(VIII)____.
> Hence ____(IX)____.
>
> We verify the statement (U): '*for any $x \in J$, for any $y, z \in K$,* ____*(X)*____'.
>
> ____(XI)____. Suppose ____(XII)____.
> ____(XIII)____, we have $x + y = 1$. Then $y = 1 - x$.
> Since $(x, z) \in H$, we have ____(XIV)____. Then ____(XV)____.
> Therefore ____(XVI)____.
>
> We verify the statement (S): '____(XVII)____ *such that* ____(XVIII)____'.
>
> ____(XIX)____. Take $x = 1 - y$. By definition, we have $x + y = (1 - y) + y = 1$.
> Since $y \in K$, we have ____(XX)____.
> Since ____(XXI)____, we have $x = 1 - y < 1 - 0 = 1$.
> Since $y \leq 1$, we have ____(XXII)____.
> Then $0 \leq x < 1$. Therefore $x \in J$.
> Hence ____(XXIII)____.
>
> We verify the statement (I): '____(XXIV)____ *if $(x, y) \in H$ and $(w, y) \in H$* ____(XXV)____'.
>
> Pick any $x, w \in J$, $y \in K$. ____(XXVI)____.
> ____(XXVII)____, we have $x + y = 1$. Then $x = 1 - y$.
> Since $(w, y) \in H$, we have ____(XXVIII)____. Then $w = 1 - y$.
> Therefore ____(XXIX)____.
>
> By $(E), (U)$, ____(XXX)____. By (S), ____(XXXI)____. By (I), ____(XXXII)____.
> Hence h is a bijective function from J to K with graph H.
> It follows that ____(XXXIII)____.

(b) Let $L = (0, 1)$.

By applying the Glueing Lemma to 'glue together' the identity function id_L and some appropriate bijective function, prove that J is of cardinality equal to K.

10. Let $J = [0, 1)$, $L = (0, 1)$, $M = [0, +\infty)$, $N = (0, +\infty)$.

(a) By writing down appropriate bijective functions, verify that $J \sim M$ and $L \sim N$. Justify your answer.

(b) By writing down an appropriate bijective function, verify that $L \sim \mathbb{R}$. Justify your answer.