MATH1050 Assignment 10 (Answers and selected solutions)

1. **Solution.**

   (a) Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be the function defined by $f(x) = x^4 - 4x^2$ for any $x \in \mathbb{R}$.

   i. We verify that $f$ is not injective:

   - Take $x_0 = 0$, $w_0 = 2$. We have $x_0, w_0 \in \mathbb{R}$ and $x_0 \neq w_0$. Also, $f(x_0) = 0 = f(w_0)$.

   ii. We verify that $f$ is not surjective:

   - Take $y_0 = -5$.
     Pick any $x \in \mathbb{R}$. We have $f(x) = x^4 - 4x^2 = (x^2 - 2)^2 - 4 \geq -4 > -5$. Then $f(x) \neq -5$.
     Hence, for any $x \in \mathbb{R}$, $f(x) \neq y_0$.

   (b) Let $x \in (\sqrt{2}, +\infty)$.
   $x^4 - 4x^2 = (x^2 - 2)^2 - 4 > 0 - 4 = -4$.

   (c) Let $g : (\sqrt{2}, +\infty) \longrightarrow (-4, +\infty)$ be the function defined by $g(x) = x^4 - 4x^2$ for any $x \in (\sqrt{2}, +\infty)$.

   i. Pick any $x, w \in (\sqrt{2}, +\infty)$. Suppose $g(x) = g(w)$. Then $x^4 - 4x^2 = w^4 - 4w^2$.
   Therefore $(x - w)(x + w)(x^2 + w^2) = (x^2 - w^2)(x^2 + w^2) = 4(x^2 - w^2) = 4(x - w)(x + w)$.
   Then $(x - w)(x + w)(x^2 + w^2 - 4) = 0$.
   Note that $x \geq \sqrt{2} > 0$ and $w \geq \sqrt{2} > 0$. Then $x + w > 0$ and $x^2 + w^2 - 4 > 0$.
   Then $x = w$.
   It follows that $g$ is injective.

   ii. Pick any $y \in (-4, +\infty)$. Note that $y + 4 > 0$. Then $\sqrt{y + 4}$ is well-defined and $2 + \sqrt{4 + y} > 2$. Therefore $\sqrt{2 + \sqrt{4 + y}}$ is well-defined and $\sqrt{2 + \sqrt{4 + y}} > \sqrt{2}$.
   Take $x = \sqrt{2 + \sqrt{4 + y}}$. Note that $x \in (\sqrt{2}, +\infty)$.
   We have $g(x) = x^4 - 4x^2 = (\sqrt{2 + \sqrt{4 + y}})^4 - 4(\sqrt{2 + \sqrt{4 + y}})^2 == (2 + \sqrt{4 + y})^2 - 4(2 + \sqrt{4 + y}) + 4 - 4 = [(2 + \sqrt{4 + y}) - 2]^2 - 4 = (4 + y) - 4 = y$.
   It follows that $g$ is surjective.

   iii. Since $g$ is both injective and surjective, $g$ is bijective. Its inverse function $g^{-1} : (-4, +\infty) \longrightarrow (\sqrt{2}, +\infty)$ is given by $g^{-1}(y) = \sqrt{2 + \sqrt{4 + y}}$ for any $y \in (-4, +\infty)$.

   **Remark.** Although $f$ and $g$ have the same 'formula of definition', one is bijective and the other is not. So when talking about a function, be aware of its domain and its range, and don't just look at its 'formula of definition'.

2. **Answer.**

   (a) $J = (1, +\infty)$.

   (b) $f^{-1}(y) = \dfrac{1}{4}\left(\ln\left(\dfrac{y+1}{y-1}\right)\right)^2$ for any $y \in J$.

3. (a) **Solution.**
   Let $f : \mathbb{C} \longrightarrow \mathbb{C}$ be the function defined by $f(z) = \bar{z}$ for any $z \in \mathbb{C}$.

   - Let $z, w \in \mathbb{C}$. Suppose $f(z) = f(w)$. Then $\bar{z} = \bar{w}$. Therefore $z = \bar{\bar{z}} = \bar{\bar{w}} = w$.
     It follows that $f$ is injective.
   - Let $\zeta \in \mathbb{C}$. Take $z = \bar{\zeta}$. By definition, $z \in \mathbb{C}$. $f(z) = \bar{z} = \bar{\bar{\zeta}} = \zeta$.
     It follows that $f$ is surjective.

   Hence $f$ is bijective.

   (b) **Answer.**
   The inverse function $f^{-1} : \mathbb{C} \longrightarrow \mathbb{C}$ of the function $f$ is given by $f^{-1}(z) = \bar{z}$ for any $z \in \mathbb{C}$.
   *Comment.* The conjugate of the conjugate of a complex number is the complex number itself.

4. **Solution.**
   Let $a, b, c, d \in \mathbb{C}$. Suppose $c \neq 0$ and $ad - bc \neq 0$.

   (a) Let $z \in \mathbb{C}$.
   $\dfrac{az + b}{cz + d} - \dfrac{a}{c} = \dfrac{c(az + b) - a(cz + d)}{c(cz + d)} = -\dfrac{ad - bc}{c(cz + d)} \neq 0$ because $ad - bc \neq 0$. Then $\dfrac{az + b}{cz + d} \neq \dfrac{a}{c}$.

(b) Define the function $f : \mathbb{C}\backslash\{-d/c\} \longrightarrow \mathbb{C}\backslash\{a/c\}$ by $f(z) = \dfrac{az+b}{cz+d}$ for any $z \in \mathbb{C}\backslash\{-d/c\}$.

i. Pick any $z, w \in \mathbb{C}\backslash\{-d/c\}$. Suppose $f(z) = f(w)$. Then $\dfrac{az+b}{cz+d} = \dfrac{aw+b}{cw+d}$. Therefore

$$aczw + bd + adz + bcw = (az+b)(cw+d) = (aw+b)(cz+d) = aczw + bd + adw + bcz.$$

Hence $(ad-bc)z = (ad-bc)w$. Since $ad - bc \neq 0$, we have $z = w$.
It follows that $f$ is injective.

ii. Pick any $\zeta \in \mathbb{C}\backslash\{a/c\}$. Since $\zeta \neq \dfrac{a}{c}$, we have $-c\zeta + a \neq 0$. Take $z = \dfrac{d\zeta - b}{-c\zeta + a}$. By definition, $z \in \mathbb{C}$.
Moreover, $z - \dfrac{-d}{c} = \dfrac{d\zeta - b}{-c\zeta + a} + \dfrac{d}{c} = \dfrac{c(d\zeta - b) + d(-c\zeta + a)}{c(-c\zeta + a)} = \dfrac{ad - bc}{c(-c\zeta + a)} \neq 0$. Then $z \neq -\dfrac{d}{c}$. Hence $z \in \mathbb{C}\backslash\{-d/c\}$.
$f(z) = \dfrac{a[(d\zeta - b)/(-c\zeta + a)] + b}{c[(d\zeta - b)/(-c\zeta + a)] + d} = \dfrac{a(d\zeta - b) + b(-c\zeta + a)}{c(d\zeta - b) + d(-c\zeta + a)} = \dfrac{(ad-bc)\zeta + 0}{0 \cdot \zeta + (ad - bc)} = \zeta.$
It follows that $f$ is surjective.

iii. The inverse function of $f$, which is $f^{-1} : \mathbb{C}\backslash\{a/c\} \longrightarrow \mathbb{C}\backslash\{-d/c\}$, is given by $f^{-1}(\zeta) = \dfrac{d\zeta - b}{-c\zeta + a}$ for any $\zeta \in \mathbb{C}\backslash\{a/c\}$.

5. **Answer.**

(a) ——

(b) i. ——

ii. ——

iii. Yes.

6. **Answer.**

(a) (I) $L$ is a subset of $H \times K$

(II) a relation

(III) For any $x \in D$

(IV) $y \in R$

(V) $(x, y) \in G$

(VI) For any $x \in D$, for any $y, z \in R$

(VII) $(x, y) \in G$ and $(x, z) \in G$

(VIII) $y = z$

(IX) $D$

(X) $R$

(XI) $G$

(b) i. $(p, q) = (0, 0)$ or $(p, q) = (0, 1)$. $(s, t) = (1, \tau)$, provided that $-2 \leq \tau < 2$.

ii. $(p, q) = (1, 1)$ and $(s, t) = (2, 1)$. *Alternative answer*: $(p, q) = (1, 2)$ and $(s, t) = (2, 2)$.

iii. $(m, n) = (0, 0)$. $(p, q) = (1, 1)$ or $(p, q) = (1, 2)$.

7. **Answer.**

(I) $F$ is a subset of $A \times B$

(II) there exists some $y \in B$

(III) Pick any $x \in A$.

(IV) 0

(V) 4

(VI) Define $y = 4 + \sqrt{\dfrac{16 - (x-2)^4}{4}}$

(VII) $y \in B$

(VIII) $(x-2)^4 + 4 \cdot \left[4 + \sqrt{\frac{16-(x-2)^4}{4}} - 4\right]^2 = (x-2)^4 + 4 \cdot \frac{16-(x-2)^4}{4} = 16$

(IX) $(x, y) \in F$

(X) if $(x, y) \in F$ and $(x, z) \in F$ then $y = z$

(XI) Pick any $x \in A$. Pick any $y, z \in B$. Suppose $(x, y) \in F$ and $(x, z) \in F$.

(XII) $(x-2)^4 + 4(y-4)^2 = 16$

(XIII) since $(x, z) \in F$, we have $(x-2)^4 + 4(z-4)^2 = 16$

(XIV) $\frac{16-(x-2)^4}{4}$

(XV) $\sqrt{(y-4)^2} = \sqrt{(z-4)^2} = z - 4$

(XVI) $y = z$

8. **Answer.**

(a) (I) Suppose $y \in f(S)$

(II) there exists some $x \in S$ such that $y = f(x)$

(III) $y = f(x) = 2x^4 - 4 \geq 2 \cdot 1^4 - 4 = -2$

(IV) Since $x \leq 2$

(V) Take $x = \sqrt[4]{\frac{y+4}{2}}$

(VI) $x = \sqrt[4]{\frac{y+4}{2}} \geq 1$

(VII) Since $y \leq 28$, we have $\frac{y+4}{2} \leq 16$

(VIII) $x = \sqrt[4]{\frac{y+4}{2}} \leq 2$

(IX) $f(x) = 2x^4 - 4 = 2\left(\sqrt[4]{\frac{y+4}{2}}\right)^4 - 4 = 2 \cdot \frac{y+4}{2} - 4 = y + 4 - 4 = y$

(X) $y \in f(S)$

(b) (I) Suppose $x \in f^{-1}(U)$

(II) there exists some $y \in U$ such that $y = f(x)$

(III) $y \in U$

(IV) $2x^4 - 4 = f(x) = y \leq 4$

(V) Suppose $x \in [-\sqrt{2}, \sqrt{2}]$

(VI) Define $y = f(x)$

(VII) $y = f(x) = 2x^4 - 4 \leq 4$

(VIII) $y = f(x) = 2x^4 - 4 \geq -6$

(IX) and

(X) $x \in f^{-1}(U)$

9. **Answer.**

(a) (I) Pick any subset $U$ of $B$

(II) For any $y$, if $y \in\in f(S \cap f^{-1}(U))$ then $y \in f(S) \cap U$.

(III) Pick any object $y$

(IV) $y \in f(S \cap f^{-1}(U))$

(V) $x \in S \cap f^{-1}(U)$

(VI) $y = f(x)$

(VII) $x \in S$ and $x \in f^{-1}(U)$

(VIII) $y = f(x)$

(IX) $y \in f(S)$

(X) there exists some $z \in U$ such that $z = f(x)$

(XI) $f(x)$

(XII) $U$

(XIII) $y \in f(S)$ and

(XIV) $y \in f(S) \cap U$

(XV) Suppose $y \in f(S) \cap U$

(XVI) $y \in f(S)$ and

(XVII) $y \in f(S)$

(XVIII) there exists some $x \in S$ such that $y = f(x)$

(XIX) $y = f(x)$

(XX) $x \in f^{-1}(U)$

(XXI) $x \in S \cap f^{-1}(U)$

(XXII) and $y = f(x)$

(XXIII) $y \in f(S \cap f^{-1}(U))$

(XXIV) $f(S \cap f^{-1}(U)) = f(S) \cap U$

(b) (I) $A = \{0, 1\}$

(II) the function $f : A \longrightarrow B$

(III) $f(0) = 2$

(IV) 0

(V) $\{2\}$

(VI) $\{2\}$

(VII) $\{0, 1\}$

(VIII) 1

(IX) $1 \notin f^{-1}(U) \cap S$

(X) $f^{-1}(U \cap f(S)) \not\subset f^{-1}(U) \cap S$