1. (a) Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be the function defined by $f(x) = x^4 - 4x^2$ for any $x \in \mathbb{R}$.
   i. Is f injective? Justify your answer.
   ii. Is f surjective? Justify your answer.

   (b) Verify that for any $x \in (\sqrt{2}, +\infty)$, $x^4 - 4x^2 > -4$.

   (c) Let $g : (\sqrt{2}, +\infty) \longrightarrow (-4, +\infty)$ be the function defined by $g(x) = x^4 - 4x^2$ for any $x \in (\sqrt{2}, +\infty)$.
   i. Is g injective? Justify your answer.
   ii. Is g surjective? Justify your answer.
   iii. Is g bijective? If *yes*, also write down the 'formula of definition' for its inverse function.

2. *You are not required to prove your answers in this question.*

   The function $f : (0, +\infty) \longrightarrow J$, given by $f(x) = \dfrac{e^{\sqrt{x}} + e^{-\sqrt{x}}}{e^{\sqrt{x}} - e^{-\sqrt{x}}}$ for any $x \in (0, +\infty)$ is known to be a bijective function from $(0, +\infty)$ to the set J.

   (a) Express the set J explicitly as an interval.

   (b) Write down the explicit 'formula of definition' for the inverse function f^{-1} of the function f.

3. Let $f : \mathbb{C} \longrightarrow \mathbb{C}$ be the function defined by $f(z) = \bar{z}$ for any $z \in \mathbb{C}$.

   (a) Verify that f is bijective.

   (b) Write down the 'formula of definition' of the inverse function of f.

4. Let $a, b, c, d \in \mathbb{C}$. Suppose $c \neq 0$ and $ad - bc \neq 0$.

   (a) Prove that for any $z \in \mathbb{C}$, $\dfrac{az + b}{cz + d} \neq \dfrac{a}{c}$.

   (b) Define the function $f : \mathbb{C}\backslash\{-d/c\} \longrightarrow \mathbb{C}\backslash\{a/c\}$ by $f(z) = \dfrac{az + b}{cz + d}$ for any $z \in \mathbb{C}\backslash\{-d/c\}$.
   i. Verify that f is injective.
   ii. Verify that f is surjective.
   iii. Write down the 'formula of definition' of the inverse function of f.

5. (a)$^\diamond$ Let $n \in \mathbb{N}\backslash\{0\}$, and $a \in \mathbb{C}\backslash\{0\}$. Define the function $\mu : \mathbb{C} \longrightarrow \mathbb{C}$ by $\mu(z) = az^n$ for any $z \in \mathbb{C}$. Prove that μ is bijective iff $n = 1$.

   (b) Let $h : \mathbb{C} \longrightarrow \mathbb{C}$ be the function defined by

$$h(z) = \begin{cases} iz & \text{if} \quad |z| \in \mathbb{Q} \\ \dfrac{3i}{2\bar{z}} & \text{if} \quad |z| \in \mathbb{R}\backslash\mathbb{Q} \end{cases}.$$

   i. Prove the statement (♯):
   (♯) *For any $\zeta \in \mathbb{C}$, if $|\zeta|$ is irrational then $|h(\zeta)|$ is irrational.*
   ii. Prove that $(h \circ h)(z) = -z$ for any $z \in \mathbb{C}$.
   iii.$^\diamond$ Is h bijective? Justify your answer. (*Hint.* Make good use of the result in the previous part.)

6. (a) Fill in the blanks in the passage below so as to give the respective definitions for the notions of *relation* and *function*:
   - *Let H, K, L be sets. We say that (H, K, L) is a* **relation** *if* ____(I)____ .
   - *Let D, R, G be sets. We say that (D, R, G) is a* **function** *if (D, R, G) is* ____(II)____ *and the statements (E), (U) below hold:*
     (E) ____(III)____ *, there exists some* ____(IV)____ *such that* ____(V)____ .
     (U) ____(VI)____ *, if* ____(VII)____ *then* ____(VIII)____ .

     For such a function, we say that ____(IX)____ *is its domain,* ____(X)____ *is its range, and* ____(XI)____ *is its graph.*

(b) *You are not required to justify your answers in this question. In each part, you are only required to give one correct answer, although there are different correct answers.*

i. Let $A = (-1, 1]$, $B = [-2, 2)$, $G = \{(x, x) \mid x \leq 0\}$, $H = \{(x, x+1) \mid x \geq 0\}$ and $F = (A \times B) \cap (G \cup H)$.
Name some appropriate $(p, q), (s, t) \in A \times B$, if such exist, for which the ordered triple $(A, B, (F \backslash \{(p, q)\}) \cup \{(s, t)\})$ is a function from A to B.

ii. Let $A = [0, 2]$, $G = \{(x, x^2) \mid 0 \leq x \leq 1\}$, $H = \{(x, 3 - x) \mid 1 \leq x < 2\}$ and $F = A^2 \cap (G \cup H)$.
Name some appropriate $(p, q), (s, t) \in A^2$, if such exist, for which the ordered triple $(A, A, (F \backslash \{(p, q)\}) \cup \{(s, t)\})$ is an injective function from A to A.

iii. Let $A = [0, +\infty)$ and E, F be the subsets of $\mathbb{R}^2$ defined respectively by $E = \{(x, x^{-1}) \mid 0 < x \leq 1\}$, $F = \{(x, 2x^{-2}) \mid x \geq 1\}$.
Name some appropriate $(m, n), (p, q) \in A^2$, if such exist, for which the ordered triple $(A, A, (E \cup F \cup \{(m, n)\}) \backslash \{(p, q)\})$ is a surjective function from A to A.

7. Let $A = [0, 4]$, $B = [4, 6]$, and $F = \{(x, y) \mid x \in A \text{ and } y \in B \text{ and } (x-2)^4 + 4(y-4)^2 = 16\}$. Define $f = (A, B, F)$.

Fill in the blanks in the block below, all labelled by capital-letter Roman numerals, with appropriate words so that it gives a proof for the statement (A). (*The 'underline' for each blank bears no definite relation with the length of the answer for that blank.*)

Here we prove the statement (A):

(A) *f is a function from A to B with graph F.*

> By definition, _____(I)_____ . Then f is a relation from A to B with graph F.
> We verify the statement '*for any $x \in A$,* _____(II)_____ *such that $(x, y) \in F$*':
>
> - _____(III)_____
>
>   By definition, $0 \leq x \leq 4$. Then $-2 \leq x - 2 \leq 2$. Therefore $0 \leq (x-2)^4 \leq 16$.
>
>   Hence __(IV)__ $\leq \dfrac{16 - (x-2)^4}{4} \leq$ __(V)__ .
>
>   _____(VI)_____ . By definition, $4 \leq y \leq 6$. Then _____(VII)_____ .
>
>   Also by definition, $(x-2)^4 + 4(y-4)^2 =$ _____(VIII)_____ .
>
>   Hence _____(IX)_____ .
>
> We verify the statement '*for any $x \in A$, for any $y, z \in B$,* _____(X)_____ ':
>
> - _____(XI)_____
>
>   Since $(x, y) \in F$, we have _____(XII)_____ .
>
>   Also, _____(XIII)_____ .
>
>   Then $(y-4)^2 =$ _____(XIV)_____ $= (z-4)^2$.
>
>   Since $y, z \in B$, we have $y - 4 \geq 0$ and $z - 4 \geq 0$.
>
>   Then $y - 4 =$ _____(XV)_____ .
>
>   Therefore _____(XVI)_____ .
>
> It follows that f is a function.

8. Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be the function defined by $f(x) = 2x^4 - 4$ for any $x \in \mathbb{R}$.

Fill in the blanks in the blocks below, all labelled by capital-letter Roman numerals, with appropriate words so that they give respectively a proof for the statement (B) and a proof for the statement (C). (*The 'underline' for each blank bears no definite relation with the length of the answer for that blank.*)

(a) Here we prove the statement (B):

(B) $f([1, 2]) = [-2, 28]$.

Write $S = [1, 2]$.

- [We want to verify the statement (†): '*for any* y, *if* $y \in f(S)$ *then* $y \in [-2, 28]$.']

  Pick any y. ____(I)____ . Then by the definition of $f(S)$, ________(II)________ .

  For the same x, since $x \in S$, we have $1 \leq x \leq 2$.

  Since $x \geq 1$, we have ______(III)______ .

  ________(IV)________ we have $y = f(x) = 2x^4 - 4 \leq 2 \cdot 2^4 - 4 = 28$.

  Therefore $-2 \leq y \leq 28$. Hence $y \in [-2, 28]$.

- [We want to verify the statement (‡): '*for any* y, *if* $y \in [-2, 28]$ *then* $y \in f(S)$.']

  Pick any y. Suppose $y \in [-2, 28]$. Then $-2 \leq y \leq 28$.

  [We want to verify that for this y, there exists some $x \in S$ such that $y = f(x)$.]

  ____(V)____ . We verify that $x \in S$:

  * Since $y \geq -2$, we have $\dfrac{y+4}{2} \geq 1$. Then ______(VI)______ .

    __________(VII)__________ . Then __________(VIII)__________ .

    Therefore $1 \leq x \leq 2$. Hence $x \in [1, 2] = S$.

  For the same x, we have __________(IX)__________ .

  Then, for the same x, y, we have $x \in S$ and $y = f(x)$. Hence by the definition of $f(S)$, ____(X)____ .

It follows that $f(S) = [-2, 28]$.

(b) Here we prove the statement (C):

(C) $f^{-1}([-6, 4]) = [-\sqrt{2}, \sqrt{2}]$.

Write $U = [-6, 4]$.

- [We want to verify the statement (†): '*for any* x, *if* $x \in f^{-1}(U)$ *then* $x \in [-\sqrt{2}, \sqrt{2}]$.']

  Pick any x. ____(I)____ . Then by the definition of $f^{-1}(U)$, ______(II)______

  For the same y, since ____(III)____ , we have $-6 \leq y \leq 4$.

  Since $y \geq -6$, we have $2x^4 - 4 = f(x) = y \geq -6$. Then $x^4 \geq -1$. (This provides no information other than re-iterating '$x \in \mathbb{R}$'.)

  Since $y \leq 4$, we have ______(IV)______ . Then $x^4 \leq 4$. Since $x \in \mathbb{R}$, we have $-\sqrt{2} \leq x \leq \sqrt{2}$. Then $x \in [-\sqrt{2}, \sqrt{2}]$.

- [We want to verify the statement (‡): '*for any* x, *if* $x \in [-\sqrt{2}, \sqrt{2}]$ *then* $x \in f^{-1}(U)$.']

  Pick any x. ____(V)____ . Then $-\sqrt{2} \leq x \leq \sqrt{2}$.

  [We want to verify that for this x, there exists some $y \in U$ such that $y = f(x)$.]

  ____(VI)____ . We verify that $y \in U$:

  * Since $-\sqrt{2} \leq x \leq \sqrt{2}$, we have $x^4 \leq 4$. Then ________(VII)________ .

    Since $x \in \mathbb{R}$, we have $x^4 \geq 0 \geq -1$. Then ______(VIII)______ .

    Therefore $-6 \leq y \leq 4$. Hence $y \in [-6, 4] = U$.

  Then, for the same x, y, we have $y = f(x)$ ____(IX)____ $y \in U$. Hence by the definition of $f^{-1}(U)$, ____(X)____.

It follows that $f^{-1}(U) = [-\sqrt{2}, \sqrt{2}]$.

9. Fill in the blanks in the blocks below, all labelled by capital-letter Roman numerals, with appropriate words so that they give respectively a proof for the statement (K) and a dis-proof against the statement (L). (*The 'underline' for each blank*

bears no definite relation with the length of the answer for that blank.)

(a) Consider the statement (K):

(K) *Suppose A, B are sets, and $f : A \longrightarrow B$ is a function. Then for any subset S of A, for any subset U of B, $f(S \cap f^{-1}(U)) = f(S) \cap U$.*

We now give a proof for the statement (K).

> Suppose A, B are sets, and $f : A \longrightarrow B$ is a function. Pick any subset S of A. ___(I)___ .
>
> - [We verify the statement (†): '___(II)___']
>   ___(III)___ . Suppose ___(IV)___ . Then by the definition of image set, there exists some ___(V)___ such that ___(VI)___ .
>
>   We have $x \in S \cap f^{-1}(U)$. Then by the definition of intersection, we have ___(VII)___ .——$(*)$
>
>   In particular, $x \in S$. We have $x \in S$ and ___(VIII)___ . Then, by the definition of image set, we have ___(IX)___ .——$(**)$
>
>   By $(*)$, we also have $x \in f^{-1}(U)$. Then, by the definition of pre-image set, ___(X)___ . Now $y =$ ___(XI)___ $= z \in$ ___(XII)___ .——$(***)$
>
>   Now by $(**)$, $(***)$, we have ___(XIII)___ $y \in U$. Hence, by the definition of intersection, we have ___(XIV)___ .
>
> - [We verify the statement (†): '*For any y, if $y \in f(S) \cap U$ then $y \in\in f(S \cap f^{-1}(U))$*'.]
>
>   Pick any object y. ___(XV)___ . Then, by the definition of intersection, ___(XVI)___ $y \in U$.
>
>   In particular, ___(XVII)___ . Then, by the definition of image set, ___(XVIII)___ . ——$(\star)$
>
>   For the same y, we have ___(XIX)___ and $y \in U$. Then, by the definition of pre-image set, we have ___(XX)___ .——$(\star\star)$
>
>   By $(\star)$, $(\star\star)$, we have $x \in S$ and $x \in f^{-1}(U)$. Hence, by the definition of intersection, we have ___(XXI)___ .
>
>   Now we have $x \in S \cap f^{-1}(U)$ ___(XXII)___ . Therefore, by the definition of image set, we have ___(XXIII)___ .
>
> It follows that ___(XXIV)___ .

(b) Consider the statement (L):

(L) *Suppose A, B are sets, and $f : A \longrightarrow B$ is a function. Then for any subset S of A, for any subset U of B, $f^{-1}(U \cap f(S)) \subset f^{-1}(U) \cap S$.*

We now give a dis-proof against the statement (L).

> Regard $0, 1, 2, 3$ as pairwise distinct objects. Take ___(I)___ , $B = \{2, 3\}$.
> Define ___(II)___ by ___(III)___ , $f(1) = 2$.
> Take $S = \{0\}$, $U = B$.
> Note that $f^{-1}(U) = \{0, 1\}$. Then $f^{-1}(U) \cap S = \{$ ___(IV)___ $\}$.
> Note that $f(S) =$ ___(V)___ . Then $U \cap f(S) =$ ___(VI)___ . Therefore $f^{-1}(U \cap f(S)) =$ ___(VII)___ .
> Now we have ___(VIII)___ $\in f^{-1}(U \cap f(S))$ and ___(IX)___ . Then by the definition of subset relation, ___(X)___ .