MATH1050 Assignment 9 (Answers and suggested solutions)

1. (a) **Answer.**

   There exist some set A, some functions $f, g : A \longrightarrow A$ such that $g \circ f \neq f \circ g$ as functions.

   (b) **Answer.**

   i. For any $x \in \mathbb{R}$, $(g \circ f)(x) = -\dfrac{1}{1+x^2}$, $(f \circ g)(x) = \dfrac{(x-1)^2}{1+(x-1)^2}$.

   ii. One of the correct choices of $x_0$is 0.

   Justification: Note that $(g \circ f)(x_0) = -1$, $(f \circ g)(x_0) = \dfrac{1}{2}$. Then $(g \circ f)(x_0) \neq (f \circ g)(x_0)$.

   iii. There exists some $x_0 \in \mathbb{R}$, namely, $x_0 = 0$, such that $(g \circ f)(x_0) \neq (f \circ g)(x_0)$. Hence it is not true that $g \circ f = f \circ g$ as functions.

   (c) **Solution.**

   Let $A = \{0, 1\}$.

   Define $f, g : A \longrightarrow A$ by $f(0) = f(1) = 0$, $g(0) = g(1) = 1$.

   By definition, $g \circ f : A \longrightarrow A$ is given by $(g \circ f)(0) = g(f(0)) = g(0) = 1$, $(g \circ f)(1) = g(f(1)) = g(0) = 1$.

   $f \circ g : A \longrightarrow A$ is given by $(f \circ g)(0) = f(g(0)) = f(1) = 0$, $(f \circ g)(1) = f(g(1)) = f(1) = 0$.

   Take $x_0 = 0$. We have $(g \circ f)(x_0) = 1$ and $(f \circ g)(x_0) = 0$. Then $(g \circ f)(x_0) \neq (f \circ g)(x_0)$.

   Therefore $g \circ f \neq f \circ g$ as functions.

   Remark. Let A be a set. (This set is fixed in our subsequent discussion.) Suppose $f, g : A \longrightarrow A$ are two functions from the set A to A itself.

   - When we want to verify that $g \circ f$, $f \circ g$ are the same function from A to A, we have to verify that for any $x \in A$, $(g \circ f)(x) = (f \circ g)(x)$.
   - To verify that $g \circ f$, $f \circ g$ are not the same function from A to A, we check that there exists some $x_0 \in A$ such that $(g \circ f)(x_0) \neq (f \circ g)(x_0)$. Hence we have to name an appropriate x_0 and show that $(g \circ f)(x_0) \neq (f \circ g)(x_0)$.

2. **Answer.**

   (a) (I) Pick any
   Alternative answer. Let
   Alternative answer. Suppose
   Alternative answer. Assume
   Alternative answer. Take any

   (II) $x = (y+1)^{\frac{3}{5}}$

   (III) $f(x) = x^{\frac{5}{3}} - 1 = \left[(y+1)^{\frac{3}{5}}\right]^{\frac{5}{3}} - 1 = (y+1) - 1 = y$

   (IV) f is surjective.

   (b) (I) $x, w \in \mathbb{R}$

   (II) Suppose
   Alternative answer. Assume

   (III) $f(x) + 1 = f(w) + 1$

   (IV) $(w^{\frac{5}{3}})^{\frac{3}{5}} = w$

   (V) f is injective

3. **Answer.**

   (a) (I) Take
   Alternative answer. Let
   Alternative answer. Define
   Alternative answer. Pick
   Alternative answer. Suppose
   Alternative answer. Assume

(II) for any $x \in \mathbb{R}$

(III) Suppose

Alternative answer. Assume

(IV) there existed some $x_0 \in \mathbb{R}$ such that $f(x_0) = y_0$.

(V) 0

(VI) $\left(x_0 - \dfrac{1}{2}\right)^2 + \dfrac{3}{4} \geq 0 + \dfrac{3}{4}$

(VII) f is not surjective.

(b) (I) $w_0 = 2$.

(II) $x_0 \neq w_0$.

(III) $f(w_0) = \dfrac{2}{2^2+1} = \dfrac{2}{5}$

(IV) $f(w_0)$

(V) f is not injective

4. **Answer.**

(a) (I) for any $\zeta \in \mathbb{C}$, there exists some $z \in \mathbb{C}$ such that $\zeta = f(z)$

(II) Pick any $\zeta \in \mathbb{C}$.

(III) there exists some $\theta \in \mathbb{R}$

(IV) Take $z = \sqrt[5]{|\zeta|} \cdot \left(\cos\left(\dfrac{\theta}{5}\right) + i\sin\left(\dfrac{\theta}{5}\right)\right)$.

(V) $\left[\sqrt[5]{|\zeta|} \cdot \left(\cos\left(\dfrac{\theta}{5}\right) + i\sin\left(\dfrac{\theta}{5}\right)\right)\right]^5 = \left(\sqrt[5]{|\zeta|}\right)^5 \cdot \left(\cos\left(5 \cdot \dfrac{\theta}{5}\right) + i\sin\left(5 \cdot \dfrac{\theta}{5}\right)\right) = |\zeta|(\cos(\theta) + i\sin(\theta)) = \zeta$

(VI) f is surjective

(b) (I) there exist some $z_0, w_0 \in \mathbb{C}$ such that $f(z_0) = f(w_0)$ and $z_0 \neq w_0$

(II) Take $z_0 = 1$, $w_0 = \cos\left(\dfrac{2\pi}{5}\right) + i\sin\left(\dfrac{2\pi}{5}\right)$.

Remark. There are many alternative answers.

(III) $z_0 \neq w_0$

(IV) $f(z_0) = {z_0}^5 = 1^5 = 1$

(V) $f(w_0) = {w_0}^5 = \cos\left(5 \cdot \dfrac{2\pi}{5}\right) + i\sin\left(5 \cdot \dfrac{2\pi}{5}\right) = \cos(2\pi) + i\sin(2\pi) = 1$

(VI) f is not injective

5. (a) **Answer.**

(I) Let C be a set

(II) $C \longrightarrow C$

(III) $\mathsf{id}_C(x) = x$ for any $x \in C$

(b) **Answer.**

(I) Suppose f is injective or f is surjective.

(II) $x \in A$

(III) $f(x)$

(IV) $(f \circ f)(x)$

(V) the definition of injectivity

(VI) $f(x) = x = \mathsf{id}_A(x)$

(VII) Suppose f is surjective.

(VIII) Pick any

(IX) there exists some $u \in A$ such that $x = f(u)$

(X) $(f \circ f)(u)$

(XI) $(f \circ f)(u) = f(u)$

(XII) $f = \mathsf{id}_A$

(c) ——

6. (a) i. *Hint.* Obtain the inequality $\frac{1}{2} \geq \frac{|p|}{b} + \frac{|q|}{b^2} + \frac{|r|}{b^3}$ —— $(\star)$.

Then apply the Triangle Inequality and the inequality $(\star)$ to deduce $f(b) \geq \frac{b^3}{2}$.

ii. ——

(b) *Hint.* For each $\gamma \in \mathbb{R}$, apply the result in the previous to the function $f_\gamma : \mathbb{R} \longrightarrow \mathbb{R}$ defined by $f_\gamma(x) = \frac{g(x) - \gamma}{A}$ for any $x \in \mathbb{R}$.

(c) ——

7. ——

8. **Answer.**

(a) (I) Pick any $x \in A$

(II) Pick any $y \in B$

(III) $f(t) = y$ for any $t \in A$

There are many correct answers: as long as the 'formula of f' is 'complete' and $f(x) = y$ according to the 'formula of f'.

(IV) $\mathsf{Map}(A, B)$

(V) $E_x(f) = f(x) = y$

(b) (I) some distinct

(II) there exists some $u \in A$ such that E_u is injective

(III) $x \in A \backslash \{u\}$

(IV) Define $f : A \longrightarrow B$ by

(V) for any

(VI) $t = u$

(VII) $f(u) = y = g(u) = E_u(g)$

(VIII) $f = g$

(IX) $y \neq z$

(X) $f \neq g$