MATH1050 Assignment 9

1. (a) Denote the statement below by (M):

   (M): *Suppose A is a set, and $f, g : A \longrightarrow A$ are functions. Then the equality $g \circ f = f \circ g$ as functions holds.*

   Write down the negation $(\sim M)$ of the statement (M).

   (b) Let $f, g : \mathbb{R} \longrightarrow \mathbb{R}$ be functions defined by $f(x) = \dfrac{x^2}{1+x^2}$, $g(x) = x - 1$ for any $x \in \mathbb{R}$.

   i. Write down the respective 'formulae of definition' of the functions $g \circ f$, $f \circ g$ explicitly.

   ii. Name an appropriate real number x_0 for which $(g \circ f)(x_0) \neq (f \circ g)(x_0)$. Justify your answer.

   iii. Is it true that $g \circ f = f \circ g$ as functions? Justify your answer.

   Remark. Hence we have dis-proved the statement (M) by giving a counter-example. (Why?)

   (c) Define $A = \{0, 1\}$.

   Name a pair of functions $f, g : A \longrightarrow A$ for which $g \circ f \neq f \circ g$ as functions. Justify your answer.

   Remark. Hence we have dis-proved the statement (M) with another counter-example. (Why?)

2. Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be the function defined by $f(x) = x^{\frac{5}{3}} - 1$ for any $x \in \mathbb{R}$.

   Fill in the blanks in the blocks below, all labelled by capital-letter Roman numerals, with appropriate words so that they give respectively a proof for the statement (A) and a proof for the statement (B). (*The 'underline' for each blank bears no definite relation with the length of the answer for that blank.*)

   (a) Here we prove the statement (A):

   (A) *The function f is surjective.*

   > [We want to verify that f is surjective. This amounts to verifying the statement '__(I)__ $y \in \mathbb{R}$, __(II)__ $x \in \mathbb{R}$ *such that* __(III)__.']
   >
   > __(IV)__ $y \in \mathbb{R}$.
   >
   > Take __(V)__. Note that $x \in \mathbb{R}$.
   >
   > Also note that __(VI)__.
   >
   > It follows that __(VII)__.

   (b) Here we prove the statement (B):

   (B) *The function f is injective.*

   > [We want to verify that f is injective. This amounts to verifying the statement '__(I)__ $x, w \in \mathbb{R}$, __(II)__ *then* $x = w$.']
   >
   > Pick any __(III)__.
   >
   > __(IV)__ $f(x) = f(w)$.
   >
   > Then $x^{\frac{5}{3}} =$ __(V)__ $= w^{\frac{5}{3}}$.
   >
   > Since $x, w \in \mathbb{R}$, we have $x = (x^{\frac{5}{3}})^{\frac{3}{5}} =$ __(VI)__.
   >
   > It follows that __(VII)__.

3. Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be the function defined by $f(x) = \dfrac{x}{x^2+1}$ for any $x \in \mathbb{R}$.

   Fill in the blanks in the blocks below, all labelled by capital-letter Roman numerals, with appropriate words so that they give respectively a proof for the statement (C) and a proof for the statement (D). (*The 'underline' for each blank bears no definite relation with the length of the answer for that blank.*)

   (a) Here we prove the statement (C):

(C) *The function f is not surjective.*

> [We want to verify that f is not surjective. This amounts to verifying the statement '__(I)__ $y_0 \in \mathbb{R}$ *such that* __(II)__ $x \in \mathbb{R}$ *such that* __(III)__.']
>
> __(IV)__ $y_0 = 1$.
>
> We verify, using the method of proof-by-contradiction, that __(V)__, $f(x) \neq y_0$:
>
> * __(VI)__ it were true that __(VII)__.
>
>   Then $\dfrac{x_0}{{x_0}^2+1} = f(x_0) = y_0 = 1$.
>
>   Therefore __(VIII)__ $= {x_0}^2 - x_0 + 1 =$ __(IX)__ > 0. Contradiction arises.
>
> It follows that __(X)__.

(b) Here we prove the statement (D):

(D) *The function f is not injective.*

> [We want to verify that f is not injective. This amounts to verifying the statement '__(I)__ $x_0, w_0 \in \mathbb{R}$ *such that* $f(x_0)$__(II)__ *and* __(III)__.']
>
> Take $x_0 = \dfrac{1}{2}$, __(IV)__. Note that $x_0, w_0 \in \mathbb{R}$.
>
> Also note that __(V)__.
>
> We have $f(x_0) = \dfrac{1/2}{(1/2)^2+1} = \dfrac{2}{5}$ and __(VI)__. Then $f(x_0) =$ __(VII)__.
>
> It follows that __(VIII)__.

4. Let $f : \mathbb{C} \longrightarrow \mathbb{C}$ be the function defined by $f(z) = z^5$ for any $z \in \mathbb{C}$.
   Fill in the blanks in the blocks below, all labelled by capital-letter Roman numerals, with appropriate words so that they give respectively a proof for the statement (E) and a proof for the statement (F). (*The 'underline' for each blank bears no definite relation with the length of the answer for that blank.*)

(a) Here we prove the statement (E):

(E) *The function f is surjective.*

> [We want to verify the statement '__(I)__'.]
>
> __(II)__ For this ζ, __(III)__ such that $\zeta = |\zeta|(\cos(\theta) + i\sin(\theta))$.
>
> __(IV)__
>
> By definition, $z \in \mathbb{C}$.
>
> Note that $f(z) = z^5 =$ __(V)__.
>
> It follows that __(VI)__.

(b) Here we prove the statement (F):

(F) *The function f is not injective.*

> [We want to verify the statement '__(I)__']
>
> __(II)__
>
> Note that $z_0, w_0 \in \mathbb{C}$. Also note that __(III)__.
>
> We have __(IV)__ and __(V)__.
>
> Then $f(z_0) = f(w_0)$.
>
> It follows that __(VI)__.

5. (a) Fill in the blanks in the passage below so as to give the definition for the notion of *identity function on a set*:

$\underline{\quad\quad(\text{I})\quad\quad}$. *The* **identity function on** C *is the function* $\mathsf{id}_C : \underline{\;(\text{II})\;}$ defined by $\underline{\quad\quad(\text{III})\quad\quad}$.

(b) Consider the statement (T):

(T) *Let A be a set, and $f : A \longrightarrow A$ be a function. Suppose $f \circ f = f$. Further suppose (f is injective or f is surjective). Then $f = \mathsf{id}_A$.*

Fill in the blanks in the blocks below, all labelled by capital-letter Roman numerals, with appropriate words so that they give respectively a proof for the statement (T). (*The 'underline' for each blank bears no definite relation with the length of the answer for that blank.*)

> Let A be a set, and $f : A \longrightarrow A$ be a function.
> Suppose $f \circ f = f$. —— $(\star)$
> $\underline{\quad\quad\quad\quad(\text{I})\quad\quad\quad\quad}$
>
> [We want to verify that $f = \mathsf{id}_A$. This amounts to verifying '*for any* $x \in A$, $f(x) = \mathsf{id}_A(x)$'.]
>
> - (Case 1.) Suppose f is injective.
>
>   Pick any $\underline{\quad(\text{II})\quad}$. By the definition of the function f, we have $f(x) \in A$.
>
>   By $(\star)$, we have $(f \circ f)(x) = \underline{\;(\text{III})\;}$.
>
>   By the definition of composition, we have $\underline{\;(\text{IV})\;} = f(f(x))$.
>
>   Then $f(f(x)) = f(x)$. Now, by $\underline{\quad\quad(\text{V})\quad\quad}$, we have $\underline{\quad(\text{VI})\quad}$.
>
>   It follows that $f = \mathsf{id}_A$.
>
> - (Case 2.) $\underline{\quad\quad(\text{VII})\quad\quad}$
>
> $\underline{\quad(\text{VIII})\quad}$ $x \in A$. By the definition of surjectivity, $\underline{\quad\quad\quad(\text{IX})\quad\quad\quad}$.
>
>   Then we have $f(x) = f(f(u)) = \underline{\;(\text{X})\;}$ by the definition of composition.
>
>   By $(\star)$, we have $\underline{\quad(\text{XI})\quad} = x$. Then $f(x) = x = \mathsf{id}_A(x)$.
>
>   It follows that $\underline{\;(\text{XII})\;}$.
>
> Hence, in any case, $f = \mathsf{id}_A$.

(c) Hence, or otherwise, prove the statement $(\sharp)$:

$(\sharp)$ *Let B be a set, K be a subset of B, and $\varphi : \mathfrak{P}(B) \longrightarrow \mathfrak{P}(B)$ be the function defined by $\varphi(S) = S \cap K$ for any $S \in \mathfrak{P}(B)$. Suppose φ is injective or φ is surjective. Then $K = B$.*

6. We introduce the definition for the notion of *zero of a function* below:

Let D be a subset of $\mathbb{C}$, and $f : D \longrightarrow \mathbb{C}$ be a function. Let $\zeta \in D$. We say ζ is a **zero of** f **in** D *if $f(\zeta) = 0$.*

In this question, you may take for granted that every polynomial function is continuous on $\mathbb{R}$. You may also take for granted the validity of **Bolzano's Intermediate Value Theorem (BIVT)***:*

(BIVT) *Let $a, b \in \mathbb{R}$, with $a < b$, and $f : [a, b] \longrightarrow \mathbb{R}$ be a function. Suppose f is continuous on $[a, b]$. Further suppose $f(a)f(b) < 0$. Then f has a zero in (a, b).*

(a)$^\clubsuit$ Let $p, q, r \in \mathbb{R}$, and $f : \mathbb{R} \longrightarrow \mathbb{R}$ be the function defined by $f(x) = x^3 + px^2 + qx + r$ for any $x \in \mathbb{R}$. Define $b = 1 + 2(|p| + |q| + |r|)$, and $a = -b$.

i. Prove that $f(b) \geq \dfrac{b^3}{2}$ and $f(a) \leq -\dfrac{b^3}{2}$.

Remark. You may need to apply the Triangle Inequality at some point.

ii. Hence apply Bolzano's Intermediate Value Theorem to deduce that f has a zero in (a, b).

(b)$^\diamondsuit$ By applying the result in the previous part, or otherwise, prove the statement $(\sharp)$:

$(\sharp)$ *Suppose $g : \mathbb{R} \longrightarrow \mathbb{R}$ is a cubic polynomial function with real coefficients. Then g is surjective.*

(c)♣ Determine whether the statement (♮) is true or false. Justify your answer.

(♮) *Let A, B, C, K, L be real numbers, with $L \neq 0$. Suppose $h : \mathbb{R} \longrightarrow \mathbb{R}$ is the rational function defined by $h(x) = \dfrac{x^3 + Ax^2 + Bx + C}{(x-K)^2 + L^2}$ for any $x \in \mathbb{R}$. Then h is surjective.*

7. *Take for granted the validity of the* **Mean-Value Theorem (MVT)***:*

(MVT) *Let D be a subset of $\mathbb{R}$ and $f : D \longrightarrow \mathbb{R}$ be a function. Let $a, b \in D$, with $a < b$, and with $[a, b] \subset D$. Suppose f is continuous on $[a, b]$, and f is differentiable on (a, b). Then there exists some $x_0 \in (a, b)$ such that $f(b) - f(a) = (b - a)f'(x_0)$.*

Let p, q be real numbers, with p, q and $g : (p, q) \longrightarrow \mathbb{R}$ be a function. Suppose g is differentiable on (p, q), and $g'(x) > 0$ for any $x \in (p, q)$. Prove the statements below:

(a)◇ g is strictly increasing on (p, q).

(b) g is injective.

8.♣ We introduce the notation for the *set of all functions from a given set to a given set*:

Let D, R be sets. The **set of all functions with domain** *D* **and range** *R is denoted by $\mathsf{Map}(D, R)$.*

Let A, B be non-empty sets. For any $x \in A$, define the function $E_x : \mathsf{Map}(A, B) \longrightarrow B$ by $E_x(f) = f(x)$ for any $x \in A$. Fill in the blanks in the blocks below, all labelled by capital-letter Roman numerals, with appropriate words so that they give respectively a proof for the statement (P) and a proof for the statement (Q). (*The 'underline' for each blank bears no definite relation with the length of the answer for that blank.*)

(a) Here we prove the statement (P):

(P) *For any $x \in A$, the function E_x is surjective.*

> ___(I)___ . We verify that E_x is surjective:
>
> - ___(II)___ . Define the function $f : A \longrightarrow B$ by ___(III)___ .
>
>   By definition, $f \in$ ___(IV)___ . By definition of E_f, we have ___(V)___ .
>
>   It follows that E_x is surjective.

(b) Here we prove the statement (Q):

(Q) *Suppose B has more than one element. Also suppose there exists some $u \in A$ such that E_u is injective. Then A is a singleton.*

> Suppose B has more than one element. Pick ___(I)___ $y, z \in B$.
>
> Also suppose ___(II)___ .
>
> Note that $\{u\} \subset A$. We now verify $A \subset \{u\}$:
>
> - Pick any $x \in A$. Suppose it were true that $x \notin \{u\}$. Then by definition of complement, ___(III)___ .
>
>   ___(IV)___ $f(t) = y$ ___(V)___ $t \in A$.
>
>   Define $g : A \longrightarrow B$ by $g(t) = \begin{cases} y & \text{if } \_\_\_(VI)\_\_\_ \\ z & \text{if } t \in A \backslash \{u\} \end{cases}$.
>
>   By definition, $f, g \in \mathsf{Map}(A, B)$.
>
>   We have $E_u(f) =$ ___(VII)___ . Then, since E_u is injective, we have ___(VIII)___ .
>
>   Recall that $x \in A\backslash\{u\}$. Since $f(x) = y$ and $g(x) = z$ and ___(IX)___ , we have ___(X)___ .
>
>   Now $f = g$ and $f \neq g$. Contradiction arises.
>
>   It follows that in the first place, we have $x \in \{u\}$.
>
> Hence $A = \{u\}$.