MATH1050 Assignment 8 (Answers and selected solutions)

1. **Answer.**

   (a) Least element: -1.
   Greatest element: *None*.
   The set concerned is bounded above by 1 in $\mathbb{R}$. (Every real number no less than 1 is an upper bound.)

   (b) Least element: *None*.
   The set concerned is bounded below by -1 in $\mathbb{R}$. (Every real number no greater than -1 is a lower bound.)
   Greatest element: *None*.
   The set concerned is bounded above by 1 in $\mathbb{R}$. (Every real number no less than 1 ia an upper bound.)

   (c) Least element: *None*.
   The set concerned is bounded below by 0 in $\mathbb{R}$. (Every real number no greater than 0 is a lower bound.)
   Greatest element: 1.

   (d) Least element: *None*.
   The set concerned is bounded below by -1 in $\mathbb{R}$. (Every real number no greater than -1 is a lower bound.)
   Greatest element: 2.

   (e) Least element: *None*.
   The set concerned is bounded below by 1 in $\mathbb{R}$. (Every real number no greater than 1 is a lower bound.)
   Greatest element: *None*.
   The set concerned is not bounded above in $\mathbb{R}$.

   (f) Least element: *None*.
   The set concerned is bounded below by 1 in $\mathbb{R}$. (Every real number no greater than 1 is a lower bound.)
   Greatest element: *None*.
   The set concerned is not bounded above in $\mathbb{R}$.

   (g) Least element: *None*.
   The set concerned is bounded below by $-\frac{3}{2}$ in $\mathbb{R}$. (Every real number no greater than $-\frac{3}{2}$ is a lower bound.)
   Greatest element: *None*
   The set concerned is not bounded above in $\mathbb{R}$.

   (h) Least element: -1.
   Greatest element: 2.

   (i) Least element: *None*.
   The set concerned is not bounded below in $\mathbb{R}$.
   Greatest element: *None*.
   The set concerned is bounded above by 1 in $\mathbb{R}$. (Every real number no less than 1 is a upper bound.)

   (j) Least element: *None*.
   The set concerned is bounded below by -3 in $\mathbb{R}$. (Every real number no greater than -3 is a lower bound.)
   Greatest element: *None*.
   The set concerned is bounded above by 1 in $\mathbb{R}$. (Every real number no less than 1 is a upper bound.)

   (k) Least element: *None*.
   The set concerned is bounded below by 0 in $\mathbb{R}$. (Every real number no greater than 0 is a lower bound.)
   Greatest element: 2.

   (l) Least element: *None*.
   The set concerned is bounded below by -1 in $\mathbb{R}$. (Every real number no greater than -1 is a lower bound.)
   Greatest element: *None*.
   The set concerned is bounded below by 1 in $\mathbb{R}$. (Every real number no less than 1 is a upper bound.)

2. **Answer.**

   (a) (I) $\frac{1}{\sqrt{2}}$

(II) $\lambda = 0 \cdot 1 + \frac{1}{2} \cdot \sqrt{2}$

(III) $\mathbb{Q}$

(IV) $\frac{1}{\sqrt{2}} \leq \lambda < \sqrt{2}$

(V) $\lambda \in B$

(VI) and

(VII) Pick any $x \in C$

(VIII) and $x \in B$

(IX) $\frac{1}{\sqrt{2}} \leq x < \sqrt{2}$

(X) $x \geq \lambda$

(b) (I) Suppose

(II) a greatest element in $\mathbb{R}$

(III) $\mu \in A$ and $\mu \in B$

(IV) there would exist some $a, b \in \mathbb{Q}$ such that

(V) $\mu \in B$

(VI) $\frac{1}{\sqrt{2}} \leq \mu < x_0 < \sqrt{2}$

(VII) $\frac{a}{2} + \frac{b+1}{2}\sqrt{2}$

(VIII) $a \in \mathbb{Q}$

(IX) $\frac{b+1}{2} \in \mathbb{Q}$

(X) $x_0 \in A$

(XI) $x_0 \in C$

(XII) μ was a greatest element of C

3. **Answer.**

   (a) ——

   (b) $\frac{1}{25}$ is the least element of T.

   (c) *Hint.* $\frac{1}{125}$ is an element of S and is not an element of T.

   (d) *Hint.* Given that $u, v \in S$ and $u < v$, is it true that $\frac{4u+v}{5} \in S$ and $u < \frac{4u+v}{5} < v$?

4. **Answer.**

   (a) This infinite sequence is strictly decreasing.
   0 is a lower bound for this infinite sequence.

   (b) This infinite sequence is strictly decreasing.
   0 is a lower bound for this infinite sequence.

   (c) This infinite sequence is strictly decreasing.
   0 is a lower bound for this infinite sequence.

   (d) This infinite sequence is strictly decreasing.
   0 is a lower bound for this infinite sequence.

   (e) This infinite sequence is strictly decreasing.
   0 is a lower bound for this infinite sequence

   (f) This infinite sequence is strictly decreasing.
   0 is a lower bound for this infinite sequence.

   (g) This infinite sequence is strictly decreasing.
   0 is a lower bound for this infinite sequence.

(h) This infinite sequence is strictly increasing.
$3/2$ is an upper bound of this infinite sequence.

(i) This infinite sequence is strictly increasing.
$3/2$ is an upper bound of this infinite sequence.

(j) This infinite sequence is strictly decreasing.
0 is a lower bound for this infinite sequence.

(k) This infinite sequence is strictly increasing.
1 is an upper bound of this infinite sequence.

(l) This infinite sequence is strictly decreasing.
0 is a lower bound of this infinite sequence.

(m) This infinite sequence is strictly increasing.
1 is an upper bound of this infinite sequence.

(n) This infinite sequence is strictly increasing.
1 is an upper bound of this infinite sequence.

5. ——

6. (a) ——

(b) **Answer.**
$\lim_{n\to\infty} a_n = \alpha$.

7. (a) **Answer.**

(I) $c_n = \dfrac{a_n + b_n}{2}$ and $a_n \leq c_n \leq b_n$ and $b_n - a_n = \dfrac{b-a}{2^n}$ and $g(a_n) < 0$ and $g(b_n) > 0$

(II) By definition, $a_0 = a < b = b_0$ and $c_0 = \dfrac{a+b}{2} = \dfrac{a_0+b_0}{2}$. Then $a_0 \leq c_0 \leq b_0$.

Also, $b_0 - a_0 = b - a = \dfrac{b-a}{2^0}$.
By definition, $g(a_0) = g(a) < 0$ and $g(b_0) = g(b) > 0$.

(III) Then, by definition, $a_{k+1} = c_k$, $b_{k+1} = b_k$ and $c_{k+1} = \dfrac{c_k + b_k}{2} = \dfrac{a_{k+1} + b_{k+1}}{2}$. Since $c_k \leq b_k$, we have $a_{k+1} \leq c_{k+1} \leq b_{k+1}$.

Moreover, $b_{k+1} - a_{k+1} = b_k - c_k = b_k - \dfrac{a_k + b_k}{2} = \dfrac{b_k - a_k}{2} = \dfrac{b-a}{2^{k+1}}$.
Also, by definition, $g(a_{k+1}) = g(c_k) < 0$ and $g(b_{k+1}) = g(b_k) > 0$.

(IV) Suppose $g(c_k) > 0$. Then, by definition, $a_{k+1} = a_k$, $b_{k+1} = c_k$ and $c_{k+1} = \dfrac{a_k + c_k}{2} = \dfrac{a_{k+1} + b_{k+1}}{2}$. Since $a_k \leq c_k$, we have $a_{k+1} \leq c_{k+1} \leq b_{k+1}$.

Moreover, $b_{k+1} - a_{k+1} = c_k - a_k = \dfrac{a_k + b_k}{2} - a_k = \dfrac{b_k - a_k}{2} = \dfrac{b-a}{2^{k+1}}$.
Also, by definition, $g(a_{k+1}) = g(a_k) < 0$ and $g(b_{k+1}) = g(c_k) > 0$.

(V) $c_{k+1} = \dfrac{a_{k+1} + b_{k+1}}{2}$ and $a_{k+1} \leq c_{k+1} \leq b_{k+1}$ and $b_{k+1} - a_{k+1} = \dfrac{b-a}{2^{k+1}}$ and $g(a_{k+1}) < 0$ and $g(b_{k+1}) > 0$

(VI)

- Let $n \in \mathbb{N}$. By definition, $a_{n+1} = c_n$ or $a_{n+1} = a_n$.
  * (Case 1). Suppose $a_{n+1} = c_n$. Note that $a_n \leq c_n \leq b_n$. Then $a_{n+1} = c_n = \dfrac{a_n + b_n}{2} \leq \dfrac{a_n + a_n}{2} = a_n$.
  * (Case 2). Suppose $a_{n+1} = a_n$. Then $a_{n+1} \leq a_n$.

  Therefore $a_{n+1} \leq a_n$ in any case.
  Hence $\{a_n\}_{n=0}^{\infty}$ is increasing.
- Let $n \in \mathbb{N}$. We have $a_n \leq b_n \leq b_{n-1} \leq \cdots \leq b_1 \leq b_0 = b$.
  Hence $\{a_n\}_{n=0}^{\infty}$ is bounded above by b.
- By the Bounded-Monotone Theorem, $\{a_n\}_{n=0}^{\infty}$ converges in $\mathbb{R}$.

(VII)

- Let $n \in \mathbb{N}$. By definition, $b_{n+1} = c_n$ or $b_{n+1} = b_n$.
  - (Case 1). Suppose $b_{n+1} = c_n$. Note that $a_n \leq c_n \leq b_n$. Then $b_{n+1} = c_n = \dfrac{a_n + b_n}{2} \geq \dfrac{b_n + b_n}{2} = b_n$.
  - (Case 2). Suppose $b_{n+1} = b_n$. Then $b_{n+1} \leq b_n$.

  Therefore $b_{n+1} \leq b_n$ in any case.
  Hence $\{b_n\}_{n=0}^{\infty}$ is decreasing.
- Let $n \in \mathbb{N}$. We have $b_n \geq a_n \geq a_{n-1} \geq \cdots \leq a_1 \leq a_0 = a$.
  Hence $\{b_n\}_{n=0}^{\infty}$ is bounded below by a.
- By the Bounded-Monotone Theorem, $\{b_n\}_{n=0}^{\infty}$ converges in $\mathbb{R}$.

(VIII) We have $\ell_b - \ell_a = \lim\limits_{n\to\infty} b_n - \lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} (b_n - a_n) = \lim\limits_{n\to\infty} \dfrac{b-a}{2^n} = 0$.
Then $\lim\limits_{n\to\infty} b_n = \lim\limits_{n\to\infty} a_n$.

(IX) We have $a_n \leq c_n \leq b_n$ for each $n \in \mathbb{N}$.
We have just verified that $\lim\limits_{n\to\infty} a_n$, $\lim\limits_{n\to\infty} b_n$ exist in $\mathbb{R}$ and are equal to ℓ.
Then by the Sandwich Rule, we conclude that $\lim\limits_{n\to\infty} c_n$ exists in $\mathbb{R}$, and is equal to ℓ.

(X) We have $a = a_0 = \lim\limits_{n\to\infty} a_0 \leq \lim\limits_{n\to\infty} a_n = \ell = \lim\limits_{n\to\infty} b_n \leq \lim\limits_{n\to\infty} b_n = b_0 = b$.

(XI) $\lim\limits_{n\to\infty} g(a_n) = g(\ell) = \lim\limits_{n\to\infty} g(a_n)$

(XII) $g(\ell) = 0$ and $g(\ell) \neq 0$

(b) ——

(c) ——

8. (a) ——

   (b) i. **Answer.**
   $f(0) = 1$.

   ii. ——

   (c) ——

9. (a) ——

   (b) i. **Answer.**
   $R = 1$, $S = 1$, $T = 1$.

   ii. ——

   iii. ——

   iv. ——

   (c) ——

   (d) ——