MATH1050 Assignment 8

1. Consider each of the subsets of $\mathbb{R}$ below.
   - Determine whether it has any least element. If *yes*, name it as well. If it has no least element, determine whether it has a lower bound in $\mathbb{R}$.
   - Determine whether it has any greatest element. If *yes*, name it as well. If it has no greatest element, determine whether it has an upper bound in $\mathbb{R}$.

   There is no need to justify your answers. (Drawing appropriate pictures, on the real line or on the coordinate plane, may help you find the answers.)

   (a) $[-1,1)\cap\mathbb{Q}$

   (b) $[-1,1)\backslash\mathbb{Q}$

   (c) $\left\{\frac{1}{n+1}\,\middle|\, n\in\mathbb{N}.\right\}$

   (d)$^\diamondsuit$ $\left\{\frac{1}{n+1}+(-1)^n\,\middle|\, n\in\mathbb{N}.\right\}$

   (e) $(1,+\infty)\cap\mathbb{Q}$

   (f) $(1,+\infty)\backslash\mathbb{Q}$

   (g) $\{x\in\mathbb{R} : 2x+3>0.\}$

   (h) $\{x\in\mathbb{R} : x+2\geq x^2.\}$

   (i) $\{x\in\mathbb{R} : x<x^{-1}.\}$

   (j) $\{x\in\mathbb{R} : x^2-2x-3<0.\}$

   (k)$^\diamondsuit$ $\left\{\frac{1}{2^m}+\frac{1}{3^n}\,\middle|\, m,n\in\mathbb{N}.\right\}$

   (l)$^\clubsuit$ $\left\{\frac{1}{2^m}-\frac{1}{3^n}\,\middle|\, m,n\in\mathbb{N}.\right\}$

2. Let $A=\{x\in\mathbb{R} : x=a+b\sqrt{2}\text{ for some } a,b\in\mathbb{Q}\}$, $B=\left[\frac{1}{\sqrt{2}},\sqrt{2}\right)$ and $C=A\cap B$.

   Fill in the blanks in the blocks below, all labelled by capital-letter Roman numerals, with appropriate words so that they gives a proof for the statement (P) and a dis-proof against the statement (Q).

   (a) Here we prove the statement (P):

   (P) *C has a least element.*

   > Take $\lambda=$ ____(I)____.
   >
   > - We have ____(II)____, and $0,\frac{1}{2}\in$ ____(III)____. Then by the definition of A, we have $\lambda\in A$.
   >
   >   Note that ____(IV)____. Then by the definition of B, we have ____(V)____.
   >
   >   Now we have $\lambda\in A$ ____(VI)____ $\lambda\in B$.
   >
   >   Therefore, by the definition of C, we have $\lambda\in C$.
   > - ____(VII)____. Then by the definition of C, we have $x\in A$ ____(VIII)____. Therefore $x\in B$ in particular.
   >
   >   Then by the definition of B, we have ____(IX)____.
   >
   >   Recall that $\lambda=\frac{1}{\sqrt{2}}$. Therefore ____(X)____.
   >
   > It follows that λ is a least element of C.

   (b) Here we dis-prove the statement (Q):

   (Q) *C has a greatest element.*

   > ____(I)____ it were true that C had ____(II)____, which we denote by μ.
   >
   > Then, by definition, $\mu\in C$. Therefore ____(III)____ by the definition of C.
   >
   > Since $\mu\in A$, ____(IV)____ $\mu=a+b\sqrt{2}$.
   >
   > Since ____(V)____, we would have $\frac{1}{\sqrt{2}}\leq\mu<\sqrt{2}$.
   >
   > Define $x_0=\frac{\mu+\sqrt{2}}{2}$.
   >
   > By definition, we would have ____(VI)____. Then $x_0>\mu$ and $x_0\in B$.
   >
   > Also by definition, $x_0=\frac{\mu+\sqrt{2}}{2}=\frac{a+b\sqrt{2}+\sqrt{2}}{2}=$ ____(VII)____.
   >
   > Since ____(VIII)____, we would have $\frac{a}{2}\in\mathbb{Q}$. Since $b\in\mathbb{Q}$, we would have ____(IX)____. Then ____(X)____.
   >
   > Now $x_0\in A$ and $x_0\in B$. Then ____(XI)____ by definition.
   >
   > But $x_0>\mu$, and ____(XII)____. Contradiction arises.

3. Let $S = \left\{ x \in \left(0, \frac{1}{24}\right) : x = \frac{b}{5^a} \text{ for some } a, b \in \mathbb{N} \right\}$, and $T = \left\{ y \in \mathbb{R} : y = \sum_{k=1}^{n} \frac{1}{25^k} \text{ for some } n \in \mathbb{N}\backslash\{0\} \right\}$.

   (a) Verify that $T \subset S$.

   (b) Does T have a least element? Justify your answer.

   (c) Prove that $S \not\subset T$.

   Remark. The result you obtain in part (b) may be useful.

   (d)$^\diamondsuit$ Prove the statement ($\sharp$):

   ($\sharp$) *For any $u, v \in S$, if $u < v$ then there exists some $w \in S$ such that $u < w < v$.*

4. Consider each of the infinite sequences (of non-negative real numbers) below. Determine whether it is strictly increasing or strictly decreasing or neither.

   - Where it is strictly increasing, determine whether it is bounded above in $\mathbb{R}$; if it is bounded above in $\mathbb{R}$, name an upper bound for it.
   - Where it is strictly decreasing, determine whether it is bounded below in $\mathbb{R}$; if it is bounded below in $\mathbb{R}$, name a lower bound for it.

   There is no need to justify your answers.

   (a) $\left\{\frac{1}{n}\right\}_{n=1}^{\infty}$

   (b) $\left\{\frac{1}{n^2}\right\}_{n=1}^{\infty}$

   (c) $\left\{\frac{n+1}{n^2+n+1}\right\}_{n=0}^{\infty}$

   (d) $\left\{\frac{2}{3^n}\right\}_{n=0}^{\infty}$

   (e) $\left\{\frac{5^n}{n!}\right\}_{n=5}^{\infty}$

   (f) $\left\{\frac{(n!)^2}{(2n)!}\right\}_{n=0}^{\infty}$

   (g) $\left\{\sqrt[n]{2}\right\}_{n=2}^{\infty}$

   (h) $\left\{\sum_{k=0}^{n} \frac{1}{3^k}\right\}_{n=0}^{\infty}$

   (i)$^\diamondsuit$ $\left\{\sum_{k=2}^{n} \frac{2}{k^2-1}\right\}_{n=2}^{\infty}$

   (j) $\left\{\frac{n^2}{3^n}\right\}_{n=2}^{\infty}$

   (k) $\left\{\sqrt[n]{\frac{2}{3}}\right\}_{n=2}^{\infty}$

   (l) $\left\{1-\prod_{k=2}^{n}\cos\left(\frac{\pi}{2^k}\right)\right\}_{n=2}^{\infty}$

   (m) $\left\{\prod_{k=1}^{n}\frac{1}{2^k}\right\}_{n=1}^{\infty}$

   (n) $\left\{1-\frac{1}{2^n}+\frac{(-1)^n}{5^n}\right\}_{n=1}^{\infty}$

5. Define $a_n = \sum_{k=1}^{n} \frac{1}{k^2}$, $b_n = \sum_{k=1}^{n} \frac{1}{k^3}$ for each $n \geq 1$.

   (a)$^\clubsuit$ Prove that $a_{2^{m+1}-1} - a_{2^m-1} \leq \frac{1}{2^m}$ for each $m \geq 1$. Hence deduce that $\{a_n\}_{n=1}^{\infty}$ is bounded above by 2.

   (*Hint.* It help to observe that $a_n \leq a_{2^n-1}$ for each n. But is this observation true? Why?)

   (b)$^\diamondsuit$ By applying the result in the previous part, or otherwise, prove that $\{b_n\}_{n=1}^{\infty}$ is bounded above by 2.

6. Let p be a positive real number, and $\alpha = \sqrt[3]{p}$. Suppose $b \in (\alpha, +\infty)$. Define infinite sequence $\{a_n\}_{n=0}^{\infty}$ recursively by

$$\begin{cases} a_0 & = & b \\ a_{n+1} & = & \frac{1}{3}\left(2a_n + \frac{\alpha^3}{a_n{}^2}\right) \quad \text{for any } n \in \mathbb{N} \end{cases}$$

   (a)$^\clubsuit$ Prove the statements below:

   i. *$\{a_n\}_{n=0}^{\infty}$ is bounded below by α in $\mathbb{R}$.*

   ii. *$\{a_n\}_{n=0}^{\infty}$ is strictly decreasing.*

   (b) Prove that $\{a_n\}_{n=0}^{\infty}$ converges in $\mathbb{R}$, and determine the value of $\lim_{n\to\infty} a_n$.

7. (a) Fill in the blanks in the block below, all labelled by capital-letter Roman numerals, with appropriate words so that it gives a proof for the statement (G):

   (G) *Let $a, b \in \mathbb{R}$, with $a < b$. Let $g : [a, b] \longrightarrow \mathbb{R}$ be a function. Suppose g is continuous on $[a, b]$, and $g(a) < 0 < g(b)$. Then there exists some $x_0 \in (a, b)$ such that $g(x_0) = 0$.*

Let $a, b \in \mathbb{R}$, with $a < b$. Let $g : [a, b] \longrightarrow \mathbb{R}$ be a function.
Suppose g is continuous on $[a, b]$, and $g(a) < 0 < g(b)$.
Further suppose for any $x \in (a, b)$, $g(x) \neq 0$. [We are going to obtain a contradiction.]
Let $\{a_n\}_{n=0}^{\infty}$, $\{b_n\}_{n=0}^{\infty}$, $\{c_n\}_{n=0}^{\infty}$ be three recursively defined infinite sequences of real numbers, by

$$\begin{cases} (a_0, b_0, c_0) & = \left(a, b, \dfrac{a+b}{2}\right) \\ (a_{n+1}, b_{n+1}, c_{n+1}) & = \begin{cases} \left(c_n, b_n, \dfrac{c_n + b_n}{2}\right) & \text{if} \quad g(c_n) < 0 \\ \left(a_n, c_n, \dfrac{a_n + c_n}{2}\right) & \text{if} \quad g(c_n) > 0 \end{cases} \end{cases}$$

We apply mathematical induction to prove that for any $n \in \mathbb{N}$,

$$c_n = \frac{a_n + b_n}{2} \text{ and } a_n \leq c_n \leq b_n \text{ and } b_n - a_n = \frac{b-a}{2^n} \text{ and } g(a_n) < 0 \text{ and } g(b_n) > 0.$$

Denote by $P(n)$ the proposition ______ (I) ______ .

- ______ (II) ______ Hence $P(0)$ is true.
- Let $k \in \mathbb{N}$. Suppose $P(k)$ is true. Then $c_k = \dfrac{a_k + b_k}{2}$ and $a_k \leq c_k \leq b_k$ and $b_k - a_k = \dfrac{b-a}{2^k}$ and $g(a_k) < 0$ and $g(b_k) > 0$. We verify that $P(k+1)$ is true:
  - (∗) (Case 1). Suppose $g(c_k) < 0$. ______ (III) ______
  - (∗) (Case 2). ______ (IV) ______

  Therefore, in any case, ______ (V) ______ . Hence $P(k+1)$ is true.

By the Principle of Mathematical Induction, $P(n)$ is true for any $n \in \mathbb{N}$.

We prove that $\{a_n\}_{n=0}^{\infty}$ converges in $\mathbb{R}$, with the help of the Bounded-Monotone Theorem:
______ (VI) ______

We also prove that $\{b_n\}_{n=0}^{\infty}$ converges in $\mathbb{R}$: ______ (VII) ______

We prove that $\{a_n\}_{n=0}^{\infty}$, $\{b_n\}_{n=0}^{\infty}$ converge to the same limit:

- Write $\lim_{n\to\infty} a_n = \ell_a$. Write $\lim_{n\to\infty} b_n = \ell_b$. ______ (VIII) ______

Now write $\ell = \ell_a = \ell_b$. We prove that $\{c_n\}_{n=0}^{\infty}$ converges to ℓ with the help of the Sandwich Rule:
______ (IX) ______

We prove that $a \leq \ell \leq b$: ______ (X) ______

We apply the continuity of g on $[a, b]$ to verify that $g(\ell) = 0$:

- Since g is continuous on $[a, b]$, g is continuous at ℓ.

  Since $\{a_n\}_{n=0}^{\infty}$, $\{b_n\}_{n=0}^{\infty}$ both converge to ℓ, we have ______ (XI) ______ by the continuity of g.
- We have $g(a_n) < 0$ for any $n \in \mathbb{N}$. Then $\lim_{n\to\infty} g(a_n) \leq 0$.

  We have $g(b_n) > 0$ for any $n \in \mathbb{N}$. Then $\lim_{n\to\infty} g(b_n) \geq 0$.

  Then $0 \leq \lim_{n\to\infty} g(b_n) = g(\ell) = \lim_{n\to\infty} g(a_n) \leq 0$. Therefore we would have $g(\ell) = 0$.

Recall that by assumption, $g(a) \neq 0$ and $g(b) \neq 0$ and for any $x \in (a, b)$, $g(x) \neq 0$.
Then, since $a \leq \ell \leq b$, we also have $g(\ell) \neq 0$.

Now we have obtained ______ (XII) ______ . Contradiction arises.
It follows that, in the first place, there exists some $x_0 \in (a, b)$ such that $g(x_0) = 0$.

(b) Hence, or otherwise, deduce **Bolzano's Intermediate Value Theorem**:

Let $a, b \in \mathbb{R}$, with $a < b$. Let $f : [a, b] \longrightarrow \mathbb{R}$ be a function. Suppose $f(a)f(b) < 0$. Suppose f is continuous on $[a, b]$. Then there exists some $x_0 \in (a, b)$ such that $f(x_0) = 0$.

(c) Hence, or otherwise, deduce the **Intermediate Value Theorem**:

Let $a, b \in \mathbb{R}$, with $a < b$. Let $h : [a, b] \longrightarrow \mathbb{R}$ be a function. Suppose $h(a) \neq h(b)$. Suppose h is continuous on $[a, b]$. Then, for any $\gamma \in \mathbb{R}$, if γ is strictly between $h(a)$ and $h(b)$ then there exists some $c \in (a, b)$ such that $h(c) = \gamma$.

8. *In this question, take for granted what you have learnt in the calculus of one real variable.*

Let $f : [0, 1] \longrightarrow \mathbb{R}$ be a function.

Suppose f is continuous on $[0, 1]$, and $x^2 + (f(x))^2 = 1$ for any $x \in [0, 1]$.

(a) Let $p \in [0, 1]$.
Prove that $f(p) = 0$ iff $p = 1$.

(b) Suppose $f(0) \geq 0$.

i. What is the value of $f(0)$?

ii.♣ Prove that $f(x) = \sqrt{1 - x^2}$ for any $x \in [0, 1]$.
Remark. You will need Bolzano's Intermediate Value Theorem at some stage in the argument.

(c)♢ Suppose $f(0) \leq 0$.
Prove that $f(x) = -\sqrt{1 - x^2}$ for any $x \in [0, 1]$.

9. *This question is about the construction of the exponential function* $\exp : \mathbb{R} \longrightarrow \mathbb{R}$, *which is formally defined by* $\exp(x) = \lim_{n\to\infty} \left(1 + \frac{x}{n}\right)^n$ *for any* $x \in \mathbb{R}$. *The first problem in such a construction is the quesetion on the existence of the limit* $\lim_{n\to\infty} \left(1 + \frac{\alpha}{n}\right)^n$ *for each individual* $\alpha \in \mathbb{R}$.

Let α be a positive real number. For each $n \geq 2$, define

$$a_n = \left(1 + \frac{\alpha}{n}\right)^n, \qquad b_n = \sum_{k=0}^{n} \frac{\alpha^k}{k!}, \qquad c_n = \left(1 - \frac{\alpha^2}{2n}\right) \sum_{k=0}^{n} \frac{\alpha^k}{k!}.$$

(a) Prove that $\{b_n\}_{n=2}^{\infty}$ is strictly increasing.

(b) i. By applying the Binomial Theorem, or otherwise, prove that whenever $n \geq 2$,

$$a_n = R + S\alpha + \sum_{k=2}^{n} \frac{T\alpha^k}{k!} \cdot T \cdot \left(T - \frac{1}{n}\right)\left(T - \frac{2}{n}\right) \cdot \ldots \cdot \left(T - \frac{k-1}{n}\right).$$

Here R, S, T are positive integers which are independent of n. You have to determine the respective values of R, S, T explicitly.

ii. Prove that $a_n < b_n$ whenever $n \geq 2$.

iii.♣ *Take for granted the validity of* **Archimedean Principle (AP)** *(in the formulation below):*

(AP) *For any positive real number u, there exists some $M \in \mathbb{N}\backslash\{0\}$ such that $M > u$.*

Prove the statement (♯):

(♯) *There exists some integer $N \geq 3$ such that for any $n \in \mathbb{N}$, if $n \geq 2$ then* $b_n \leq b_{N-1} + \dfrac{\alpha^N}{(1 - \alpha/N) \cdot (N!)}$.

iv.♢ Prove that $c_n < a_n$ whenever $n \geq \dfrac{\alpha^2}{2}$.
Remark. You may need to apply Weierstrass' Product Inequality at some stage of the argument.

(c)♢ Prove that $\{a_n\}_{n=2}^{\infty}$ is strictly increasing.
Remark. You may need to apply Bernoulli's Inequality at some stage of the argument.

(d) *Take for granted the validity of the Bounded-Monotone Theorem and the Sandwich Rule.*
Prove that $\lim_{n\to\infty} a_n$, $\lim_{n\to\infty} b_n$, $\lim_{n\to\infty} c_n$ exist and are equal to each other.
Remark. You may need to apply the Archimedean Principle at some stage of the argument.