MATH1050 Assignment 7 (Answers and selected solutions)

1. **Solution.**

   (a) i. $-2 \in \mathbb{Z}$. $-2+1=-1<0$.

   ii. $2 \in \mathbb{Z}$. $2-1=1>0$.

   (b) Suppose it were true that there existed some $x \in \mathbb{Z}$ such that ($x+1<0$ and $x-1>0$). For this x, we would have $x<-1$ and $x>1$. Then $x<-1<1<x$. Therefore $x \neq x$. Contradiction arises.

   Hence it is false that there exists some $x \in \mathbb{Z}$ such that ($x+1<0$ and $x-1>0$).

   Alterntative argument: The negation of the statement 'there exists some $x \in \mathbb{Z}$ such that ($x+1<0$ and $x-1>0$)' is given by:

   - For any $x \in \mathbb{Z}$, ($x+1 \geq 0$ or $x-1 \leq 0$).

   We give a proof of the latter:

   - Let $x \in \mathbb{Z}$. We have $x \geq 0$ or $x \leq 0$. Where $x \geq 0$, we have $x+1 \geq 1 \geq 0$. Where $x \leq 0$, we have $x-1 \leq -1 \leq 0$.

2. **Answer.**

   (a) (I) Suppose

   (II) $u \in \mathbb{R}\backslash\{-1,0,1\}$

   (III) $u^6+v^6 \leq 2v^4$

   (IV) $u^6-2u^4+u^2+v^6-2v^4+v^2 \leq 0$

   (V) $u^2(u^2-1)^2=0$

   (VI) $u \in \mathbb{R}\backslash\{-1,0,1\}$

   (b) (I) Suppose there existed some $\zeta \in \mathbb{C}\backslash\mathbb{R}$ such that ζ was both an 89-th root of unity and a 55-th root of unity.

   (II) 1

   (III) $\zeta^{89}=1$

   (IV) $\zeta^{21}=\zeta^{55}/\zeta^{34}=1$, $\zeta^{13}=\zeta^{34}/\zeta^{21}=1$, $\zeta^{8}=\zeta^{21}/\zeta^{13}=1$, $\zeta^{5}=\zeta^{13}/\zeta^{8}=1$, $\zeta^{3}=\zeta^{8}/\zeta^{5}=1$, $\zeta^{2}=\zeta^{5}/\zeta^{3}=1$, $\zeta=\zeta^{3}/\zeta^{2}=1$.

   (V) $\mathbb{C}\backslash\mathbb{R}$

   (VI) and

3. (a) **Answer.**

   (I) Suppose $a_1, a_2, \cdots, a_n, b_1, b_2, \cdots, b_n$ are non-negative real numbers.

   Then $\left(\sum_{j=1}^{n} {a_j}^2\right)\left(\sum_{j=1}^{n} {b_j}^2\right) \geq \left(\sum_{j=1}^{n} a_j b_j\right)^2$.

   (II) $(sv-tu)^2$

   (III) $(s^2+t^2)(u^2+v^2)$

   (IV) $(su+tv)^2$

   (V) $a_1, a_2, \cdots, a_m, a_{m+1}, b_1, b_2, \cdots, b_m, b_{m+1}$ are non-negative real numbers

   (VI) $\sqrt{\sum_{j=1}^{m} a_j b_j}$

   (VII) $\left(A^2+{a_{m+1}}^2\right)\left(B^2+{b_{m+1}}^2\right)$

   (VIII) $a_{m+1}b_{m+1}$

   (IX) C^2

   (b) **Solution.**

   Let $n \in \mathbb{N}\backslash\{0,1\}$. Suppose $a_1, a_2, \cdots, a_n, b_1, b_2, \cdots, b_n$ are real numbers.

   Note that $|a_1|, |a_2|, \cdots, |a_n|, |b_1|, |b_2|, \cdots, |b_n|$ are non-negative real numbers.

   By the result in part (a), we have $\left(\sum_{j=1}^{n} |a_j|^2\right)\left(\sum_{j=1}^{n} |b_j|^2\right) \geq \left(\sum_{j=1}^{n} |a_j b_j|\right)^2$.

Note that $\sum_{j=1}^{n} |a_j|^2 = \sum_{j=1}^{n} a_j{}^2$, $\sum_{j=1}^{n} |b_j|^2 = \sum_{j=1}^{n} b_j{}^2$.

By the Triangle Inequality for real numbers, we have $\sum_{j=1}^{n} |a_j b_j| \geq \left|\sum_{j=1}^{n} a_j b_j\right|$.

Then $\left(\sum_{j=1}^{n} |a_j b_j|\right)^2 \geq \left|\sum_{j=1}^{n} a_j b_j\right|^2 = \left(\sum_{j=1}^{n} a_j b_j\right)^2$.

Therefore $\left(\sum_{j=1}^{n} a_j{}^2\right)\left(\sum_{j=1}^{n} b_j{}^2\right) = \left(\sum_{j=1}^{n} |a_j|^2\right)\left(\sum_{j=1}^{n} |b_j|^2\right) \geq \left(\sum_{j=1}^{n} |a_j b_j|\right)^2 \geq \left(\sum_{j=1}^{n} a_j b_j\right)^2$.

4. (a) **Solution.**

Suppose x, y, z are real numbers.

By the Cauchy-Schwarz Inequality, $|yz + zx + xy| \leq (y^2 + z^2 + x^2)^{\frac{1}{2}}(z^2 + x^2 + y^2)^{\frac{1}{2}} = x^2 + y^2 + z^2$.

Then

$$\begin{aligned}
& (y+z-x)^2 + (z+x-y)^2 + (x+y-z)^2 - (yz+zx+xy) \\
= \; & (y^2+z^2+x^2+2yz-2xy-2zx) + (z^2+x^2+y^2+2zx-2xy-2yz) \\
& +(x^2+y^2+z^2+2xy-2yz-2zx) - (yz+zx+xy) \\
= \; & 3[(x^2+y^2+z^2) - (yz+zx+xy)] \\
\geq \; & 3[(x^2+y^2+z^2) - |yz+zx+xy|] \\
\geq \; & 0
\end{aligned}$$

Hence $yz + zx + xy \leq (y+z-x)^2 + (z+x-y)^2 + (x+y-z)^2$.

(b) ——

(c) ——

5. ——

6. (a) **Answer.**

(I) n

(II) an integer

(III) $a_1, a_2, \cdots, a_{n-1}, a_n$

(IV) positive

(V) $\dfrac{a_1 + a_2 + \cdots + a_{n-1} + a_n}{n}$

(VI) iff $a_1 = a_2 = \cdots = a_{n-1} = a_n$

(b) **Answer.**

(I) $r_1, r_2, \cdots, r_k$

(II) there exist

(III) $M_1, M_2, \cdots, M_k, N_1, N_2, \cdots, N_k$

(IV) $r_2 = \dfrac{M_2}{N_2}, \cdots, r_k = \dfrac{M_k}{N_k}$

(V) positive

(VI) $= M_1 + M_2 + \cdots + M_k$

(VII) $\dfrac{M_1}{N}x_1 + \dfrac{M_2}{N}x_2 + \cdots + \dfrac{M_k}{N}x_k = \dfrac{M_1x_1 + M_2x_2 + \cdots + M_kx_k}{N}$

(VIII) $\left(x_1{}^{M_1} x_2{}^{M_2} \cdot \ldots \cdot x_k{}^{M_k}\right)^{1/N} = x_1{}^{M_1/N} x_2{}^{M_2/N} \cdot \ldots \cdot x_k{}^{M_k/N}$

(c) ——

7. (a) **Solution.**

Suppose w, x, y, z are real numbers. Then $w^4, x^2y^2, y^2z^2, z^2x^2$ are all non-negative.

- (Case 1). Suppose some of w, x, y, z is zero. Then $\dfrac{w^4 + x^2y^2 + y^2z^2 + z^2x^2}{4} \geq 0 = wxyz$.
- (Case 2). Suppose none of w, x, y, z is zero. Then $w^4, x^2y^2, y^2z^2, z^2x^2$ are all positive real numbers. By the Arithmetico-geometrical Inequality,

$$\begin{aligned}\frac{w^4 + x^2y^2 + y^2z^2 + z^2x^2}{4} &\geq \left(w^4 \cdot x^2y^2 \cdot y^2z^2 \cdot z^2x^2\right)^{\frac{1}{4}} \\ &= \left(w^4x^4y^4z^4\right)^{\frac{1}{4}} \\ &= |wxyz| \\ &\geq wxyz.\end{aligned}$$

Hence, in any case, we have $\dfrac{w^4 + x^2y^2 + y^2z^2 + z^2x^2}{4} \geq wxyz$.

(b) ——

(c) ——

8. (a) **Answer.**

(I) $h(x) \geq \sum_{j=0}^{n} \frac{x^{2j}}{j!}$ for any $x \in [0, +\infty)$

(II) $1 + \int_0^x 0du = 1 + 0 = 1$

(III) Suppose $P(k)$ is true.

(IV) $1 + \int_0^x 2uh(u)du$

(V) $1 + \int_0^x 2u \sum_{j=0}^{k} \frac{u^{2j}}{j!} du = 1 + \sum_{j=0}^{k} \int_0^x \frac{2u^{2j+1}}{j!} du = 1 + \sum_{j=0}^{k} \left. \frac{u^{2j+2}}{(j+1)!} \right|_0^x = 1 + \sum_{j=0}^{k} \frac{x^{2j+2}}{(j+1)!} = \sum_{j=0}^{k+1} \frac{x^{2j}}{j!}$

(VI) $P(k+1)$ is true

(b) ——

9. ——