1. (a) Prove each of the statements below:

   i. *There exists some $x \in \mathbb{Z}$ such that $x + 1 < 0$.*

   ii. *There exists some $x \in \mathbb{Z}$ such that $x - 1 > 0$.*

   (b) Dis-prove the statement ($\sharp$):

   ($\sharp$) *There exists some $x \in \mathbb{Z}$ such that ($x + 1 < 0$ and $x - 1 > 0$).*

Remark. It can happen that $[(\exists x)P(x)] \wedge [(\exists y)Q(y)]$ is true while $(\exists x)(P(x) \wedge Q(x))$ is false. In general, $[(\exists x)P(x)] \wedge [(\exists y)Q(y)]$ does not imply $(\exists x)(P(x) \wedge Q(x))$.

2. Fill in the blanks in the block below, all labelled by capital-letter Roman numerals, with appropriate words so that it gives a dis-proof against the statement (D) and a dis-proof for the statement (E). (*The 'underline' for each blank bears no definite relation with the length of the answer for that blank.*)

   (a) We dis-prove the statement (D):

   (D) *There exist some $u \in \mathbb{R}\backslash\{-1, 0, 1\}$, $v \in \mathbb{R}$ such that $u^2 + v^2 \le 2u^4$ and $u^6 + v^6 \le 2v^4$.*

   > [We dis-prove the statement (D) by obtaining a contradiction from it.]
   >
   > ___(I)___ there existed some ___(II)___ , $v \in \mathbb{R}$ such that $u^2 + v^2 \le 2u^4$ and ___(III)___ .
   >
   > For the same u, v, we would have $u^2 + v^2 - 2u^4 \le 0$ and $u^6 + v^6 - 2v^4 \le 0$.
   >
   > Then $u^2(u^2 - 1)^2 + v^2(v^2 - 1)^2 =$ ___(IV)___ .
   >
   > Since u, v are real, $u^2(u^2 - 1)^2 \ge 0$ and $v^2(v^2 - 1)^2 \ge 0$. Then $u^2(u^2 - 1)^2 = 0$ and $v^2(v^2 - 1)^2 = 0$ respectively.
   >
   > In particular, ___(V)___ . Then $u = 0$ or $u = -1$ or $u = 1$. But ___(VI)___ .
   >
   > Contradiction arises.

   (b) We dis-prove the statement (E):

   (E) There exist some $\zeta \in \mathbb{C}\backslash\mathbb{R}$ such that ζ is both an 89-th root of unity and a 55-th root of unity.

   > [We dis-prove the statement (E) by obtaining a contradiction from it.]
   >
   > ___(I)___
   >
   > For the same ζ, we would have $\zeta^{55} =$ ___(II)___ and ___(III)___ by the definition of root of unity.
   >
   > (Note that $\zeta \neq 0$.) Then we would have $\zeta^{34} = \zeta^{89-55} = \zeta^{89}/\zeta^{55} = 1$.
   >
   > Repeating the above argument, we would have:
   >
   > ___(IV)___
   >
   > Recall that by assumption, $\zeta \in$ ___(V)___ . Then $\zeta \neq 1$.
   >
   > Now $\zeta = 1$ ___(VI)___ $\zeta \neq 1$.
   >
   > Contradiction arises.

3. *Here we are going to re-prove of the 'inequality part' of Cauchy-Schwarz Inequality, with the help of mathematical induction and the Triangle Inequality for the reals.*

   (a) Consider the statement (S):

   (S) *Let $n \in \mathbb{N}\backslash\{0, 1\}$. Suppose $a_1, a_2, \cdots, a_n, b_1, b_2, \cdots, b_n$ are non-negative real numbers.*

   Then $\left(\sum_{j=1}^{n} {a_j}^2\right)\left(\sum_{j=1}^{n} {b_j}^2\right) \ge \left(\sum_{j=1}^{n} a_j b_j\right)^2$.

   Fill in the blanks in the block below, all labelled by capital-letter Roman numerals, with appropriate passages so that it gives an argument for the statement (S) by mathematical induction.

Denote by $P(n)$ the proposition below:

(I) ______________________

- Suppose s, t, u, v are non-negative real numbers.

  We have $(s^2+t^2)(u^2+v^2)-(su+tv)^2 =$ ____(II)____ ≥ 0.

  Then ____(III)____ $\geq$ ____(IV)____ .

  Hence $P(2)$ is true.

- Let $m \in \mathbb{N}\backslash\{0,1\}$. Suppose $P(m)$ is true. We verify that $P(m+1)$ is true below:

  Suppose ____(V)____ .

  Define $A = \sqrt{\sum_{j=1}^{m} {a_j}^2}$, $B = \sqrt{\sum_{j=1}^{m} {b_j}^2}$, $C =$ ____(VI)____ .

  Note that A, B, C are non-negative real numbers.

  By $P(2)$, we have $\left(\sum_{j=1}^{m+1} {a_j}^2\right)\left(\sum_{j=1}^{m+1} {b_j}^2\right) =$ ____(VII)____ $\geq (AB + a_{m+1}b_{m+1})^2$.

  By $P(m)$, we have $AB \geq C^2$. Then $AB+$ ____(VIII)____ $\geq$ ____(IX)____ $+a_{m+1}b_{m+1} = \sum_{j=1}^{m+1} a_j b_j \geq 0$.

  Therefore $\left(\sum_{j=1}^{m+1} {a_j}^2\right)\left(\sum_{j=1}^{m+1} {b_j}^2\right) \geq (AB + a_{m+1}b_{m+1})^2 \geq \left(\sum_{j=1}^{m+1} a_j b_j\right)^2$.

  Hence $P(m+1)$ is true.

By the Principle of Mathematical Induction, $P(n)$ is true for any $n \in \mathbb{N}\backslash\{0,1\}$.

(b) By applying the result above together with the Triangle Inequality for the reals, or otherwise, prove the statement (T) below:

(T) *Let $n \in \mathbb{N}\backslash\{0,1\}$. Suppose $a_1, a_2, \cdots, a_n, b_1, b_2, \cdots, b_n$ are real numbers.*

Then $\left(\sum_{j=1}^{n} {a_j}^2\right)\left(\sum_{j=1}^{n} {b_j}^2\right) \geq \left(\sum_{j=1}^{n} a_j b_j\right)^2$.

Remark. Generalize the argument for the statement (T) to give a proof for the statement (T'):

(T) *Let $n \in \mathbb{N}\backslash\{0,1\}$. Suppose $z_1, z_2, \cdots, z_n, w_1, w_2, \cdots, w_n$ are complex numbers.*

Then $\left(\sum_{j=1}^{n} |z_j|^2\right)\left(\sum_{j=1}^{n} |w_j|^2\right) \geq \left|\sum_{j=1}^{n} z_j \overline{w_j}\right|^2$.

4. *The various parts in this question are concerned with applications of the Cauchy-Schwarz Inequality. They are independent of each other.*

   (a) Suppose x, y, z are real numbers. Prove that $yz + zx + xy \leq (y+z-x)^2 + (z+x-y)^2 + (x+y-z)^2$.

   (b) Let $a > 0$. Prove that $\dfrac{a^n}{1+a+a^2+\cdots+a^{2n}} \leq \dfrac{1}{2n+1}$.

   (c) Let n be a positive integer. Prove that $\sum_{k=0}^{n} \sqrt{\binom{n}{k}} \leq \sqrt{2^n(n+1)}$.

5. (a) By applying the Cauchy-Schwarz Inequality, or otherwise, prove the statement $(\sharp)$:

   $(\sharp)$ *Suppose $a_1, a_2, \cdots, a_n$ are positive real numbers. Then* $\dfrac{1}{n}\sum_{k=1}^{n} a_k \leq \sqrt{\dfrac{1}{n}\sum_{k=1}^{n} {a_k}^2}$.

   (b)$^\diamondsuit$ Hence, or otherwise, prove the statements below:

i. *Let* $b_1, b_2, \cdots, b_n$ *be positive real numbers. Suppose* $\sum_{k=1}^{n} b_k = S$. *Then* $\sum_{k=1}^{n} \sqrt{b_k} \leq \sqrt{nS}$.

ii. *Let* $c_1, c_2, \cdots, c_n$ *be positive real numbers. Suppose* $\sum_{k=1}^{n} c_k = 1 + \frac{1}{2n}$. *Then* $\sum_{k=1}^{n} \sqrt{2c_k + 1} \leq n + 1$.

6. (a) Fill in the blanks in the passage below so as to give the statement for the **Arithmetico-geometrical Inequality**:

Let ____(I)____ *be* ____(II)____ *greater than* 1. *Suppose* ____(III)____ *are* ____(IV)____ *real numbers. Then* ____(V)____ $\geq \sqrt[n]{a_1 a_2 \cdot \ldots \cdot a_{n-1} a_n}$. *Moreover, equality holds* ____(VI)____ .

(b) Consider the statement (T):

(T) *Let* $x_1, x_2, \cdots, x_k$ *be positive real numbers, and* $r_1, r_2, \cdots, r_k$ *be positive rational numbers. Suppose* $r_1 + r_2 + \cdots + r_k = 1$. *Then* $r_1 x_1 + r_2 x_2 + \cdots + r_k x_k \geq {x_1}^{r_1} {x_2}^{r_2} \cdot \ldots \cdot {x_k}^{r_k}$.

Fill in the blanks in the block below, all labelled by capital-letter Roman numerals, with appropriate words so that it gives a proof for the statement (T), with the help of the Arithmetic-Geometric Inequality. (*The 'underline' for each blank bears no definite relation with the length of the answer for that blank.*)

Let $x_1, x_2, \cdots, x_k$ be positive real numbers, and $r_1, r_2, \cdots, r_k$ be positive rational numbers.
Suppose $r_1 + r_2 + \cdots + r_k = 1$.
Since ____(I)____ are rational numbers , ____(II)____ some integers ____(III)____ such that $r_1 = \frac{M_1}{N_1}$, ____(IV)____ .
Without loss of generality, we may assume $N_1 = N_2 = \cdots = N_k$. Write $N = N_1$.
Since $r_1, r_2, \cdots, r_k$ are ____(V)____ , we may assume, without loss of generality, that $M_1, M_2, \cdots, M_k, N$ are positive.
Since $r_1 + r_2 + \cdots + r_k = 1$, we have N____(VI)____ .
By the Arithmetic-Geometric Inequality, we have

$$r_1 x_1 + r_2 x_2 + \cdots + r_k x_k = \underline{\qquad (VII) \qquad} \geq \underline{\qquad (VIII) \qquad}$$

(c) By applying the statement (T), or otherwise, prove the statement (T'):

(T') *Suppose* $x_1, x_2, \cdots, x_k$ *are positive real numbers, and* $s_1, s_2, \cdots, s_k$ *are positive rational numbers. Then* $\frac{s_1 x_1 + s_2 x_2 + \cdots + s_k x_k}{s_1 + s_2 + \cdots + s_k} \geq ({x_1}^{s_1} {x_2}^{s_2} \cdot \ldots \cdot {x_k}^{s_k})^{1/(s_1 + s_2 + \cdots + s_k)}$.

7. *The various parts in this question are concerned with applications of the Arithmetico-geometrical Inequality. They are independent of each other.*

(a) Suppose w, x, y, z are real numbers. Prove that $\frac{w^4 + x^2y^2 + y^2z^2 + z^2x^2}{4} \geq wxyz$.

(b) Let a, b, c be positive real numbers. Suppose $a + b + c = 1$. Prove that $\left(\frac{1}{a} - 1\right)\left(\frac{1}{b} - 1\right)\left(\frac{1}{c} - 1\right) \geq 8$.

(c) Let n be a positive integer.

i. Prove that $n^n \geq 1 \cdot 3 \cdot 5 \cdot \ldots \cdot (2n - 3) \cdot (2n - 1)$.

ii. Hence deduce that $(n^2 + n)^n \geq (2n)!$.

8. Here we take for granted the result (†) known as the **Area Comparison Theorem** in the *calculus of one real variable.*

(†) *Let* a, b *be real numbers, with* $a \leq b$, *and let* f, g *be real-valued functions of one real variable whose domains contain the interval* $[a, b]$. *Suppose* f, g *are continuous on* $[a, b]$. *Further suppose that* $f(x) \leq g(x)$ *for any* $x \in [a, b]$. *Then* $\int_a^b f(t)dt \leq \int_a^b g(t)dt$.

Let $h : [0, +\infty) \longrightarrow \mathbb{R}$ be a continuous function. Suppose that for any $x \in [0, +\infty)$,

$$h(x) \geq 0 \quad \text{and} \quad h(x) \geq 1 + \int_0^x 2uh(u)du.$$

(a) Consider the statement (J):

(*J*) *For any* $n \in \mathbb{N}$, $h(x) \geq \sum_{j=0}^{n} \frac{x^{2j}}{j!}$ *for any* $x \in [0, +\infty)$.

Fill in the blanks in the block below, all labelled by capital-letter Roman numerals, with appropriate words so that it gives a proof for the statement (J), with the help of the Area Comparison Theorem. (*The 'underline' for each blank bears no definite relation with the length of the answer for that blank.*)

> Denote by $P(n)$ the proposition that ____(I)____ .
>
> - For any $x \in [0, +\infty)$, $h(x) \geq 0$.
>
>   Then, for any $x \in [0, +\infty)$, we have $h(x) \geq 1 + \int_0^x 2uh(u)du \geq$ ____(II)____ .
>
>   Hence $P(0)$ is true.
> - Let $k \in \mathbb{N}$. ____(III)____
>
>   We have $h(x) \geq \sum_{j=0}^{k} \frac{u^{2j}}{j!}$ for any $x \in [0, +\infty)$.
>
>   For any $x \in (0, +\infty)$, we have $h(x) \geq$ ____(IV)____ $\geq$ ____(V)____
>
>   Hence ____(VI)____ .
>
> By the Principle of Mathematical Induction, $P(n)$ is true for any $n \in \mathbb{N}$.

(b) Prove that $h(\sqrt{e}) \geq e^e$.

9. In this question, we are going to prove the existence of the limit $\lim_{n\to\infty} \sum_{k=1}^{n} \frac{(-1)^{k+1}}{k}$ and find its value.

Here we take for granted the result known as the **Area Comparison Theorem** in the *calculus of one real variable*.

We further take for granted the result (‡) knonw as the **Sandwich Rule** in the *calculus of one real variable*:

(‡) *Let* $\{u_n\}_{n=0}^{\infty}, \{v_n\}_{n=0}^{\infty}, \{w_n\}_{n=0}^{\infty}$ *be infinite sequences of real numbers. Suppose* $u_n \leq v_n \leq w_n$ *for any* $n \in \mathbb{N}$. *Further suppose that both* $\{u_n\}_{n=0}^{\infty}, \{w_n\}_{n=0}^{\infty}$ *converge in* $\mathbb{R}$, *and converge to the same limit, say,* ℓ. *Then* $\{v_n\}_{n=0}^{\infty}$ *converges to* ℓ *also.*

(a)$^\diamondsuit$ Apply mathematical induction to prove that $\sum_{k=1}^{2n} \frac{(-1)^{k+1}}{k} = \sum_{k=n+1}^{2n} \frac{1}{k}$ for any positive integer n.

(b)$^\clubsuit$ Prove the statement (♯), by considering appropriate definite integrals:

(♯) *Let* x *be a real number. Suppose* $x > 1$. *Then* $\ln\left(\frac{x+1}{x}\right) \leq \frac{1}{x} \leq \ln\left(\frac{x}{x-1}\right)$.

(c) Applying the statement (♯), or otherwise, deduce that $\ln\left(\frac{2n+1}{n+1}\right) \leq \sum_{k=n+1}^{2n} \frac{1}{k} \leq \ln(2)$ for any positive integer n.

(d) Hence, or otherwise, prove that the limit $\lim_{n\to\infty} \sum_{k=1}^{2n} \frac{(-1)^{k+1}}{k}$ exists and find its value.

(e)$^\clubsuit$ Let n be a positive integer. Prove that $\left| \sum_{k=n+1}^{2n} \frac{(-1)^{k+1}}{k} \right| \leq \frac{1}{n}$.

Hint.
- When n is even, group up pairs of consecutive terms in the sum, and simplify to give a sum of $n/2$ positive number. Now spot which of these $n/2$ numbers is the largest.
- When n is odd, first leave out the first term, and then proceed in the same way described above on the rest of the terms in the sum.

(f)$^\diamondsuit$ Hence, or otherwise, prove that the limit $\lim_{n\to\infty} \sum_{k=1}^{n} \frac{(-1)^{k+1}}{k}$ exists and find its value.

Hint. What can be said about the difference $\sum_{k=1}^{n} \frac{(-1)^{k+1}}{k} - \sum_{k=1}^{2n} \frac{(-1)^{k+1}}{k}$ for each positive integer n?