MATH1050 Assignment 6 (Answers and selected solutions)

1. **Solution.**

   (a) Take $n = 3$. Note that $3 \in \mathbb{N}$. Note that $3 + 2 = 5$, $3 + 4 = 7$. The integers $3, 5, 7$ are prime numbers.

   (b) Take $x = \sqrt{2}$. Note that $x \in \mathbb{R}$. We have $x^2 - 2 = (\sqrt{2})^2 - 2 = 2 - 2 = 0$.

   (c) Take $z_0 = \dfrac{1+i}{\sqrt{2}}$. Note that $z_0 \in \mathbb{C}$.

   Also note that $z_0{}^4 = \left(\dfrac{1+i}{\sqrt{2}}\right)^4 = \dfrac{1 + 4i + 6i^2 + 4i^3 + i^4}{4} = \dfrac{1 + 4i - 6 - 4i + 1}{4} = 1$.

2. **Answer.**

   (a) *There are many correct answers for (II), (III), ..., (IX) collectively, dependent on the choices made in (II).*

   (I) There exist some $x, y, z \in \mathbb{Z}$ such that each of xy, xz is divisible by 4 and xyz is not divisible by 8.

   (II) $y = z = 1$

   (III) 4

   (IV) 4

   (V) $4 = 1 \cdot 4$ and $1 \in \mathbb{Z}$

   (VI) 4

   (VII) 4 were divisible by 8

   (VIII) $4 = 8k$

   (IX) $\dfrac{1}{2}$

   (b) (I) There exist some sets A, B, C such that $A \cap B \neq \emptyset$ and $A \cap B \subset C$ and $A \not\subset C$ and $B \not\subset C$.

   (II) $C = \{3\}$

   (III) $\emptyset$

   (IV) $A \cap B \subset C$

   (V) and $1 \notin C$

   (VI) $A \not\subset C$

   (VII) $2 \in B$ and $2 \notin C$

   (VIII) $B \not\subset C$

   (c) (I) There exist some $x, y \in \mathbb{R}$ such that $x > 0$ and $y > 0$ and $|x^2 - 2x| < |y^2 - 2y|$ and $x^2 > y^2$.

   (II) $y = 1$

   (III) $x > 0$ and $y > 0$

   (IV) 0

   (V) $|y^2 - 2y| = 1$

   (VI) $|x^2 - x|$

   (VII) $|y^2 - y|$

   (VIII) $x^2 = 4$

   (IX) $x^2 > y^2$

   (d) (I) There exist some $m, n \in \mathbb{N}\backslash\{0, 1, 2\}$, $\zeta, \omega \in \mathbb{C}$ such that $m \neq n$ and $\zeta \neq \omega$ and ζ is an m-th root of unity and ω is an n-th root of unity and $\zeta\omega$ is not an $(m+n)$-th root of unity.

   (II) Take

   (III) $\omega = \cos\left(\dfrac{\pi}{4}\right) + i\sin\left(\dfrac{\pi}{4}\right)$

   (IV) $m \neq n$ and $\zeta \neq \omega$

   (V) ζ is an m-th root of unity

   (VI) $\omega^n = \left(\cos\left(\dfrac{\pi}{4}\right) + i\sin\left(\dfrac{\pi}{4}\right)\right)^8 = \cos\left(8 \cdot \dfrac{\pi}{4}\right) + i\sin\left(8 \cdot \dfrac{\pi}{4}\right) = \cos(2\pi) + i\sin(2\pi) = 1$

(VII) 12

(VIII) $(\zeta\omega)^{m+n} = \left(\cos\left(\frac{3\pi}{4}\right) + i\sin\left(\frac{3\pi}{4}\right)\right)^{12} = \cos\left(12\cdot\frac{3\pi}{4}\right) + i\sin\left(12\cdot\frac{3\pi}{4}\right) = \cos(9\pi) + i\sin(9\pi) = -1$

(IX) $\neq$

(X) $\zeta\omega$ is not an $(m+n)$-th root of unity

3. **Solution.**

(a) Let $z \in \mathbb{C}\backslash\{0\}$. Suppose it were true that $\mathsf{Re}(z) = 0$ and $\mathsf{Im}(z) = 0$. Then $z = \mathsf{Re}(z) + i\mathsf{Im}(z) = 0 + i \cdot 0 = 0$. Contradiction arises. Hence $\mathsf{Re}(z) \neq 0$ or $\mathsf{Im}(z) \neq 0$ in the first place.

(b) The statement 'for any $z \in \mathbb{C}\backslash\{0\}$, $\mathsf{Re}(z) \neq 0$' is false: we have $i \in \mathbb{C}\backslash\{0\}$ and $\mathsf{Re}(i) = 0$. The statement 'for any $w \in \mathbb{C}\backslash\{0\}$, $\mathsf{Im}(w) \neq 0$' is also false: we have $1 \in \mathbb{C}$ and $\mathsf{Im}(1) = 0$.

Hence the statement '(for any $z \in \mathbb{C}\backslash\{0\}$, $\mathsf{Re}(z) \neq 0$) or (for any $w \in \mathbb{C}\backslash\{0\}$, $\mathsf{Im}(w) \neq 0$)' is false.

4. (a) **Answer.**

(I) Suppose

(II) Suppose s is not divisible by 2.

(III) there exist some $k, r \in \mathbb{Z}$ such that $s = 2k + r$ and $0 \leq r < 2$

(IV) s is not divisible by 2

(V) $0 < r < 2$

(VI) $r \in \mathbb{Z}$

(VII) $s = 2k + 1$

(VIII) if there exists some $k \in \mathbb{Z}$ such that $s = 2k + 1$ then s is not divisible by 2

(IX) Suppose it were true that s was divisible by 2.

(X) there would exist some $\ell \in \mathbb{Z}$ such that $s = 2\ell$

(XI) $s = 2k + 1$ and $s = 2\ell + 0$

(XII) By the Division Algorithm for Integers

(XIII) $0 = 1$

(b) ——

(c) ——

5. ——

6. **Solution.**

Let n be a positive integer.

Since n is a positive integer, we have $n^7 + n^6 + n^5 + n^4 + n^3 + n^2 + n + 1 > n^4 + n^3 + n^2 + n + 1 > 0$.

Repeatedly applying Division Algorithm, we obtain:

$$\begin{cases} n^7 + n^6 + n^5 + n^4 + n^3 + n^2 + n + 1 & = & n^3(n^4 + n^3 + n^2 + n + 1) & + & (n^2 + n + 1) \\ n^4 + n^3 + n^2 + n + 1 & = & n^2(n^2 + n + 1) & + & (n + 1) \\ n^2 + n + 1 & = & n(n + 1) & + & 1 \end{cases}$$

Since n is a positive integer, we indeed have the inequalities $n^4 + n^3 + n^2 + n + 1 > n^2 + n + 1 > n + 1 > 1 > 0$.

Hence the greatest common divisor of $n^7 + n^6 + n^5 + n^4 + n^3 + n^2 + n + 1$ and $n^4 + n^3 + n^2 + n + 1$ is 1.

7. ——

8. (a) **Answer.**

(I) Suppose a, c are relatively prime and ab is divisible by c

(II) ab is divisible by c

(III) $k \in \mathbb{Z}$

(IV) $\gcd(a, c) = 1$

(V) there exist some $s, t \in \mathbb{Z}$

(VI) $sa + tc$

(VII) $\gcd(a, c)$

(VIII) $(sa + tc)b = sab + tbc = skc + tbc = (sk + tb)c$

(IX) $sk + tb$

(X) b is divisible by c

(b) i. ——

ii. *Hint.*
Apply the result in part (b.i).

iii. ——

iv. *Hint.*
Apply the result in part (b.iii).

v. ——

9. (a) i. ——

ii. ——

iii. ——

iv. ——

v. *Hint.*
You may find the equality $xy - uv = x(y - v) + (x - u)v$ useful. (Or you may choose $xy - uv = (x - u)y + u(y - v)$ as an alternative.)

(b) ——

(c) **Answer.**

i. The solutions of the equation $3x \equiv 1(\mathsf{mod}\ 5)$ are given by $x \equiv 2(\mathsf{mod}\ 5)$.

ii. The solutions of the equation $6x \equiv 4(\mathsf{mod}\ 7)$ are given by $x \equiv 3(\mathsf{mod}\ 7)$.

iii. The solutions of the equation $4x \equiv 2(\mathsf{mod}\ 9)$ are given by $x \equiv 5(\mathsf{mod}\ 9)$.

(d) **Answer.**

i. The only solutions of the equation $4x \equiv 2(\mathsf{mod}\ 6)$ are given by $x \equiv 2(\mathsf{mod}\ 3)$.

ii. The equation $4x \equiv 1(\mathsf{mod}\ 6)$ has no solution.