MATH1050 Assignment 5

1. We introduce the definition for the notion of *proper subset*:

   Let A, B be sets. A is said to be a **proper subset** *of B if $A \subset B$ and $A \neq B$. We write $A \subsetneq B$.*

   For each of the statements below, determine whether it is true of false. Justify your answer with an appropriate argument (with reference to the appropriate definitions):

   (a) $\{1, 3, 5\}$ *is a proper subset of* $\{1, 3, 5, 7\}$.

   (b) $\{1, 3, 5, 9\}$ *is a proper subset of* $\{1, 3, 5, 7\}$.

   (c) $\{1, 1, 3, 5, 7\}$ *is a proper subset of* $\{1, 3, 5, 7, 7\}$.

2. We introduce the definitions for the notions of *disjointness for sets* and *disjoint union of two sets* below:

   Let A, B be sets.

   - *We say that A, B are* **disjoint** *if $A \cap B = \emptyset$.*
   - *Suppose A, B are indeed disjoint. Then the union of A, B is called the* **disjoint union** *of A, B, and is denoted by $A \sqcup B$.*

   For each of the statements below, determine whether it is true of false. Justify your answer with an appropriate argument (with reference to the appropriate definitions):

   (a)$^\diamondsuit$ $\mathbb{Q}$, $\mathbb{R}\backslash\mathbb{Q}$ *are disjoint.*

   (b) $\{x \mid x$ *is divisible by* $2\ \}$, $\{x \mid x$ *is divisible by* $3\}$ *are disjoint.*

   (c) $\mathbb{Z}$, $\{x \in \mathbb{R} : x = m\sqrt{2}$ *for some* $m \in \mathbb{Z}\}$ *are disjoint.*

   (d)$^\clubsuit$ $\mathbb{Z}\backslash\{0\}$, $\{x \in \mathbb{R}\backslash\{0\} : x = m\sqrt{2}$ *for some* $m \in \mathbb{Z}\}$ *are disjoint.*

3. Fill in the blanks in the blocks below, all labelled by capital-letter Roman numerals, with appropriate words so that they give respectively a proof for the statement (A), a proof for the statement (B), a proof of the statement (C) and a proof of the statement (D). (*The 'underline' for each blank bears no definite relation with the length of the answer for that blank.*)

   (a) We prove the statement (A):

   (A) *Suppose C, D are sets. Then $\mathfrak{P}(C) \cup \mathfrak{P}(D) \subset \mathfrak{P}(C \cup D)$.*

   > Suppose C, D are sets.
   > Pick any object S. ____(I)____ . Then ____(II)____ , by the definition of union.
   >
   > - (Case 1). ____(III)____ Then ____(IV)____ by the definition of power set.
   >   We verify that $S \subset C \cup D$ (according to the definitions of subset relation and union):
   >   - Pick any object x. Suppose $x \in S$.
   >   ____(V)____ Then ____(VI)____ .
   >   Therefore by the definition of union, we have ____(VII)____ .
   >
   >   Since ____(VIII)____ , we have ____(IX)____ by the definition of power set.
   > - (Case 2). Suppose $S \in \mathfrak{P}(D)$. Modifying the argument in Case (1), we also deduce $S \in \mathfrak{P}(C \cup D)$.
   >
   > Hence, in any case, we have ____(X)____
   > It follows that $\mathfrak{P}(C) \cup \mathfrak{P}(D) \subset \mathfrak{P}(C \cup D)$.

   (b) We prove the statement (B):

   (B) *Let A, B be sets. Suppose $A \cup B \subset B$. Then $A\backslash B = \emptyset$.*

Let A, B be sets. Suppose ____(I)____ . Further suppose ____(II)____ .
Take some ____(III)____ . We would have $x_0 \in A$ and $x_0 \notin B$ by definition of complement.
In particular $x_0 \in A$. . Then ____(IV)____ or ____(V)____ .
Therefore ____(VI)____ by the definition of union.
Since $x_0 \in A \cup B$ and $A \cup B \subset B$, we would have ____(VII)____ by the definition of subset relation.
Now we have $x_0 \in B$ and $x_0 \notin B$. ____(VIII)____ .
It follows that $A \backslash B = \emptyset$ in the first place.

(c) We prove the statement (C).

(C) *Suppose A, B are sets. Then $A \cap B \subset A$. Moreover, $A \cap B = A$ iff $A \subset B$.*

Suppose A, B are sets.

[We want to verify '$A \cap B \subset A$'.
According to definition, it is: '*For any object x,* ____*(I)*____ .']

Pick any object x. ____(II)____ .
Then, by the definition of intersection, ____(III)____ .
Therefore ____(IV)____ in particular.
It follows that $A \cap B \subset A$.

- [We want to verify '*if $A \subset B$ then $A \cap B = A$*'.]
  Suppose $A \subset B$.

  [Under this assumption, we try to deduce '$A \subset A \cap B$'.
  According to definition, it is: ' ____*(V)*____ .']

  ____(VI)____

  Since ____(VII)____ , we have $x \in B$ by the definition of subset relation.

  Now we have ____(VIII)____ (simultaneously).

  Then, by the definition of intersection, ____(IX)____ .

  It follows that $A \subset A \cap B$.

  Recall that ____(X)____ also. Then $A \cap B = A$.

- [We want to verify '*if $A \cap B = A$ then $A \subset B$*'.]
  Suppose ____(XI)____ .

  [Under this assumption, we try to deduce '$A \subset B$'.
  According to definition, it is: ' ____(XII)____ ']

  Pick any object x. Suppose ____(XIII)____ .

  Since ____(XIV)____ , we have $x \in A \cap B$.

  Then, ____(XV)____ .

  In particular, ____(XVI)____ .

  It follows that $A \subset B$.

Hence $A \cap B = A$ iff $A \subset B$.

(d) We prove the statement (D):

(D) *Let A, B, C be sets. Suppose $A \subset C$ and $B \subset C$. Then $A \subset B$ iff $C \backslash B \subset C \backslash A$.*

Let A, B, C be sets. Suppose $A \subset C$ and $B \subset C$.

- [We want to deduce '*if $A \subset B$ then $C \backslash B \subset C \backslash A$.*']

  Suppose $A \subset B$.

  [Under this assumption, we verify '$C \backslash B \subset C \backslash A$'.
  It is: '*for any object x, if $x \in C \backslash B$ then $x \in C \backslash A$.*']

  Pick any object x. ______(I)______

  By the definition of __(II)__, we have ______(III)______.

  We verify that $x \notin A$, with the method of proof-by contradiction:

  * ______(IV)______

    Then, ____(V)____ $A \subset B$, we would have __(VI)__ by the definition of subset relation.

    Recall $x \notin B$. Then $x \in B$ __(VII)__ $x \notin B$. Contradiction arises.

  Therefore ______(VIII)______.

  Hence, by the definition of complement, we have ____(IX)____.

  It follows that $C \backslash B \subset C \backslash A$.

- [We want to deduce '*if $C \backslash B \subset C \backslash A$* then $A \subset B$.']

  Suppose $C \backslash B \subset C \backslash A$.

  ______(X)______

  We deduce that $x \in B$ with the method of proof-by-contradiction:

  * ______(XI)______

    Since $x \in A$ ____(XII)____, we have __(XIII)__ by the definition of subset relation.

    Now, since ______(XIV)______, we have ____(XV)____ by the definition of complement.

    Then, since $x \in C \backslash B$ and $C \backslash B \subset C \backslash A$, we have ____(XVI)____ by the definition of subset relation.

    Then, by the definition of __(XVII)__, we have ______(XVIII)______.

    In particular, __(XIX)__.

    Recall that by assumption, $x \in A$. Then ____(XX)____.

    Contradiction arises.

  Hence, in the first place, we have $x \in B$.

  It follows that $A \subset B$.

Hence $A \subset B$ iff $C \backslash B \subset C \backslash A$.

4. (a) Prove the statements below 'from first principles', using the definitions of set equality, subset relation, intersection, union, complement, where appropriate:

   i. *Suppose a, b are two objects (not necessarily distinct). Then $\{a\} = \{b\}$ iff $a = b$.*

   ii. *Suppsoe a, b, c are three objects (not necessarily distinct). Then $\{a, b\} = \{c\}$ iff $a = b = c$.*

   iii.♣ *Suppose a, b, c, d are four objects (not necessarily distinct). Then $\{a, b\} = \{c, d\}$ iff (($a = c$ and $b = d$) or ($a = d$ and $b = c$) or $a = b = c = d$).*

   iv.♢ *Let A, B, C, D be sets. Suppose $\{A, B\} = \{C, D\}$. Then $A \cap B = C \cap D$ and $A \cup B = C \cup D$.*

   Remark. It might help to recall that according to the Method of Specification, $\{p\} = \{x \mid x = p\}$, and $\{p, q\} = \{x \mid x = p \text{ or } x = q\}$ *et cetera*.

(b)$^\diamond$ We introduce the definition for the notion of *Kuratowski ordered pair*:

> Let a, b are objects. We call the set $\{\{a\}, \{a, b\}\}$ the **Kuratowski ordered pair** with first coordinate a and second coordinate b, and denote this object by $(a, b)_K$.

Prove the statement below:

- *Suppose s, t, u, v are four objects (not necessarily distinct). Then $(s, t)_K = (u, v)_K$ iff ($s = u$ and $t = v$).*

5.$^\diamond$ *In this question, you may assume the validity of the statements without proof:*

($\sharp$) *Let a, b be two objects (not necessarily distinct). $\{a\} = \{b\}$ iff $a = b$.*

($\flat$) *Let a, b, c be three objects (not necessarily distinct). $\{a, b\} = \{c\}$ iff $a = b = c$.*

($\natural$) $\emptyset \neq \{\emptyset\}$.

Let $A = \{\emptyset\}$, $B = \{\{\emptyset\}\}$, $C = \{\emptyset, \{\emptyset, \{\emptyset\}\}\}$, $D = \{\emptyset, \{\{\emptyset\}\}\}$.

For each of the statements below, determine whether it is true or false. Justify your answer with an appropriate argument (with reference to the appropriate definitions):

(a) $A \in C$. (b) $A \subset C$. (c) $B \subset D$. (d) $B \in C$. (e) $A \cup B \in C$. (f) $C \cap D = \emptyset$.

6.$^\diamond$ Denote by $P(x)$ the predicate '$x \notin x$' (with variable x).

(a) Denote by R the object $\{x \mid P(x)\}$, obtained from the Method of Specification. (R is called the '**Russell set**'.)

Suppose it were true that R was a set. (Hence it makes sense to discuss whether an arbitrarily given object is an element of R or not.)

i. Can it happen that the object R is an element of the set R? Why?

ii. Can it happen that the object R is not an element of the set R? Why?

Remark. From the answers to the above questions, you would have to conclude that R is not a set in the first place. (Why?) This tells us the above application of the Method of Specification leads to a serious trouble: the construction $\{x \mid P(x)\}$ fails to give a set. This trouble is known as **Russell's Paradox**.

(b) Let A be a set. Denote by B the object $\{x \in A : P(x)\}$, obtained from the Method of Specification. (This time it is guaranteed that B is a set, because we are constructing a subset from the given set A.)

Prove that B is not an element of A. (Apply the proof-by-contradiction method.)

Remark. This shows that given any set, there is always some object which does not belong to it as an element. Hence no set contains every conceivable object as its element. There is no such thing as 'universal set'.