MATH1050 Assignment 4 (Answers and selected solutions)

1. **Solution.**

(a) Denote by $P(n)$ the proposition

$$1 \cdot 2 + 2 \cdot 5 + 3 \cdot 8 + \cdots + n(3n-1) = n^2(n+1).$$

- Note that $1 \cdot 2 = 2 = 1^2(1+1)$. Then $P(1)$ is true.
- Let k be a positive integer. Suppose $P(k)$ is true. Then

$$1 \cdot 2 + 2 \cdot 5 + 3 \cdot 8 + \cdots + k(3k-1) = k^2(k+1).$$

We verify that $P(k+1)$ is true:

We have

$$\begin{aligned} & 1 \cdot 2 + 2 \cdot 5 + 3 \cdot 8 + \cdots + k(3k-1) + (k+1)[3(k+1)-1] \\ = \; & k^2(k+1) + (k+1)(3k+2) = (k+1)[k^2 + (3k+2)] = \cdots = (k+1)^2[(k+1)+1] \end{aligned}$$

Hence $P(k+1)$ is true.

By the Principle of Mathematical Induction, $P(n)$ is true for any positive integer n.

(b) Denote by $P(n)$ the proposition

$$1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \cdots + \frac{1}{\sqrt{n}} \geq \sqrt{n}.$$

- We have $1 \geq \sqrt{1}$. Hence $P(1)$ is true.
- Let k be a positive integer. Suppose $P(k)$ is true. Then $1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \cdots + \frac{1}{\sqrt{k}} \geq \sqrt{k}$.

We verify that $P(k+1)$ is true:

$$\begin{aligned} 1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \cdots + \frac{1}{\sqrt{k}} + \frac{1}{\sqrt{k+1}} &\geq \sqrt{k} + \frac{1}{\sqrt{k+1}} \\ &= \frac{\sqrt{k} \cdot \sqrt{k+1} + 1}{\sqrt{k+1}} \\ &\geq \frac{\sqrt{k} \cdot \sqrt{k} + 1}{\sqrt{k+1}} \\ &= \sqrt{k+1} \end{aligned}$$

Hence $P(k+1)$ is true.

By the Principle of Mathematical Induction, $P(n)$ is true whenever n is a positive integer.

(c) Denote by $P(n)$ the proposition

$$n^2 < 2^{n-1}.$$

- We have $7^2 = 49 < 64 = 27 - 1$. Then $P(7)$ is true.
- Let k be an integer greater than 6. Suppose $P(k)$ is true. Then $k^2 < 2^{k-1}$. Therefore $2^{k-1} > 2^k$.
We have

$$\begin{aligned} 2^{(k+1)-1} - (k+1)^2 &= 2^k - (k^2 + 2k + 1) \\ &\geq 2^k - k^2 - 2k - k \\ &= 2^k - k^2 - 3k \\ &\geq 2^k k^2 - k \cdot k = 2(2^{k-1} - k^2) >\geq 0. \end{aligned}$$

Then $(k+1)^2 < 2^{(k+1)-1}$. Hence $P(k+1)$ is true.

By the Principle of Mathematical Induction, $P(n)$ is true whenever k is an integer greater than 6.

(d) Denote by $P(n)$ the proposition that $n(n^2+2)$ is divisible by 3.

- We have $0 \cdot (0^2+2) = 0 = 3 \cdot 0$ and $0 \in \mathbb{Z}$. Hence $0 \cdot (0^2+2)$ is divisible by 3.
  Then $P(0)$ is true.
- Let k be a positive integer. Suppose $P(k)$ is true. Then $k(k^2+2)$ is divisible by 3. Therefore there exists some $q \in \mathbb{Z}$ such that $k(k^2+2) = 3q$.
  We verify that $P(k+1)$ is true:
  We have
  $$(k+1)[(k+1)^2+2] = k^3+3k^2+5k+3 = k(k^2+2)+3k^2+3 = 3q+3k^2+3 = 3(q+k^2+1).$$
  Note that $q+k^2+1 \in \mathbb{Z}$. Then $(k+1)[(k+1)^2+2]$ is divisible by 3.
  Hence $P(k+1)$ is true.

By the Principle of Mathematical Induction, $P(n)$ is true for any $n \in \mathbb{N}$.

(e) Denote by $P(n)$ the proposition that $7^n(3n+1)-1$ is divisible by 9.

- We have $7^0(3 \cdot 0+1)-1 = 0$. 0 is divisible by 9. Then $P(0)$ is true.
- Let $k \in \mathbb{N}$. Suppose $P(k)$ is true. Then $7^k(3k+1)-1$ is divisible by 9. Therefore there exists some $q \in \mathbb{Z}$ such that $7^k(3k+1)-1 = 9q$.
  We verify that $P(k+1)$ is true:
  $$\begin{aligned} 7^{k+1}[3(k+1)+1]-1 &= 7 \cdot 7^k[(3k+1)+3]-1 \\ &= 7 \cdot 7^k(3k+1)+3 \cdot 7^{k+1}-1 \\ &= 7 \cdot [7^k(3k+1)-1]+3(7^{k+1}-1)+9 \\ &= 7 \cdot 9q+3(7-1)\sum_{j=0}^{k} 7^j+9 = 9\left(7q+2\sum_{j=0}^{k} 7^j+1\right) \end{aligned}$$
  Since $q \in \mathbb{Z}$ and $k \in \mathbb{N}$, we have $7q+2\sum_{j=0}^{k} 7^j+1 \in \mathbb{Z}$. Therefore $7^{k+1}[3(k+1)+1]-1$ is divisible by 9.
  Hence $P(k+1)$ is true.

By the Principle of Mathematical Induction, $P(n)$ is true for any $n \in \mathbb{N}$.

2. (a) i. **Answer.**

(I) $\sum_{k=0}^{n}(a_{k+1}-a_k) = a_{n+1}-a_0$

(II) $\sum_{k=0}^{0}(a_{k+1}-a_k) = a_1-a_0 = a_{0+1}-a_0$.

(III) Suppose $P(m)$ is true

(IV) $a_{m+1}-a_0$

(V) $\sum_{k=0}^{m+1}(a_{k+1}-a_k) = \sum_{k=0}^{m}(a_{k+1}-a_k)+(a_{m+2}-a_{m+1}) = (a_{m+1}-a_0)+(a_{m+2}-a_{m+1}) = a_{m+2}-a_0 = a_{(m+1)+1}-a_0$.

(VI) By the Principle of Mathematical Induction, $P(n)$ is true for any $n \in \mathbb{N}$.

ii. ——

(b) **Solution.**

Let $\{c_n\}_{n=0}^{\infty}$ be an infinite sequence of numbers. Let α, β be numbers, with $\alpha \neq 1$. Suppose $c_{n+1} = \alpha c_n + \beta$ for each $n \in \mathbb{N}$.

For each $n \in \mathbb{N}$, define $a_n = \dfrac{c_n}{\alpha^n}$.

Then by definition, for each $n \in \mathbb{N}$, we have $a_{n+1} = \dfrac{c_n}{\alpha^{n+1}} = \dfrac{\alpha c_n+\beta}{\alpha^{n+1}} = \dfrac{c_n}{\alpha^n}+\dfrac{\beta}{\alpha^{n+1}} = a_n+\dfrac{\beta}{\alpha^{n+1}}$.

By the result described in the statement (A), for each $n \in \mathbb{N}$, we have

$$a_{n+1} - a_0 = \sum_{k=0}^{n}(a_{k+1} - a_k) = \sum_{k=0}^{n}\frac{\beta}{\alpha^{k+1}} = \alpha^{-n-1}\beta\sum_{k=0}^{n}\alpha^k = \alpha^{-n-1}\beta \cdot \frac{1-\alpha^{n+1}}{1-\alpha}$$

Therefore

$$c_{n+1} - \alpha^{n+1}c_0 = \alpha^{n+1}a_{n+1} - \alpha^{n+1}a_0 = \alpha^{n+1}(a_{n+1} - a_0) = \frac{\beta(1-\alpha^{n+1})}{1-\alpha}.$$

Remark. The key step in the application of the telescopic method is displayed below:

By assumption, for each n, we have

$$\begin{cases} & c_{n+1} & - & \alpha & c_n & = & \beta \\ \alpha & c_n & - & \alpha^2 & c_{n-1} & = & \beta\alpha \\ \alpha^2 & c_{n-1} & - & \alpha^3 & c_{n-2} & = & \beta\alpha^2 \\ & \vdots & & & \vdots & & \vdots \\ \alpha^{n-2} & c_3 & - & \alpha^{n-1} & c_2 & = & \beta\alpha^{n-2} \\ \alpha^{n-1} & c_2 & - & \alpha^n & c_1 & = & \beta\alpha^{n-1} \\ \alpha^n & c_1 & - & \alpha^{n+1} & c_0 & = & \beta\alpha^n \end{cases}$$

Then $c_{n+1} - \alpha^{n+1}c_0 = \sum_{k=0}^{n}\alpha^{n-k}\beta = \sum_{j=0}^{n}\alpha^j\beta = \dfrac{\beta(1-\alpha^{n+1})}{1-\alpha}.$

(c) i. ——

ii. **Solution.**

Let $\theta \in \mathbb{R}$. Suppose $\sin\left(\frac{\theta}{2}\right) \neq 0$. Pick any $n \in \mathbb{N}$.

$$\begin{aligned} \left(1 + 2\sum_{k=1}^{n}\cos(k\theta)\right)\sin\left(\frac{\theta}{2}\right) &= \sin\left(\frac{\theta}{2}\right) + \sum_{k=1}^{n}2\cos(k\theta)\sin\left(\frac{\theta}{2}\right) \\ &= \sin\left(\frac{\theta}{2}\right) + \sum_{k=1}^{n}\left(\sin\left((k+\frac{1}{2})\theta\right) - \sin\left((k-\frac{1}{2})\theta\right)\right) \\ &= \sin\left((n+\frac{1}{2})\theta\right) \end{aligned}$$

By assumption $\sin(\frac{\theta}{2}) \neq 0$. Then $= 1 + 2\sum_{k=1}^{n}\cos(k\theta) = \dfrac{\sin((n+1/2)\theta)}{\sin(\theta/2)}$.

iii. ——

iv. ——

3. **Answer.**

(a) (I) Suppose $\sum_{j=0}^{n} a_j = \left(\dfrac{1+a_n}{2}\right)^2$ for each $n \in \mathbb{N}$.

(II) $a_n = 2n + 1$.

(III) We have $a_0 = \sum_{j=0}^{0} a_j = \left(\dfrac{1+a_0}{2}\right)^2 = \dfrac{1}{4}(1 + 2a_0 + {a_0}^2)$. Then $(a_0 - 1)^2 = {a_0}^2 - 2a_0 + 1 = 0$. Therefore $a_0 = 1 = 2 \cdot 0 + 1$.

(IV) Let $k \in \mathbb{N}$. Suppose $P(k)$ is true.

(V) We have

$$\left(\frac{1+a_{k+1}}{2}\right)^2 = \sum_{j=0}^{k+1} a_j = \sum_{j=0}^{k} a_j + a_{k+1} = \left(\frac{1+a_k}{2}\right)^2 + a_{k+1} = \left[\frac{1+(2k+1)}{2}\right]^2 + a_{k+1} = (k+1)^2 + a_{k+1}.$$

Then $\frac{1}{4}(1+2a_{k+1}+{a_{k+1}}^2)=(k+1)^2+a_{k+1}$.

Therefore $(a_{k+1}-1)^2={a_{k+1}}^2-2a_{k+1}+1=(2k+2)^2$.

Hence $a_{k+1}=2k+3$ or $a_{k+1}=-2k-1$. Since $a_{k+1}>0$, we have $a_{k+1}=2k+3=2(k+1)+1$.

(VI) By the Principle of Mathematical Induction, $P(n)$ is true for any $n\in\mathbb{N}$.

(b) (I) Let α,β are the two distinct roots of the polynomial $f(x)=x^2-x-1$. Suppose $\{a_n\}_{n=1}^{\infty}$ is the infinite sequence of real numbers defined by

$$\begin{cases} a_1 = 1, & a_2 = 3, \\ & a_{n+2} = a_{n+1}+a_n \quad \text{if} \quad n\geq 1 \end{cases}.$$

(II) $1=-(-1)=\alpha+\beta$

(III) $3=[-(-1)]^2-2(-1)=(\alpha+\beta)^2-2\alpha\beta=\alpha^2+\beta^2$

(IV) Let k be a positive integer. Suppose $P(k)$ is true.

(V) $\alpha^{k+1}+\beta^{k+1}$

(VI) $P(k)$

(VII) $a_{k+2}=a_{k+1}+a_k=(\alpha^{k+1}+\beta^{k+1})+(\alpha^k+\beta^k)=\alpha^k(\alpha+1)+\beta^k(\beta+1)=\alpha^k\cdot\alpha^2+\beta^k\cdot\beta^2=\alpha^{k+2}+\beta^{k+2}$.

(VIII) By the Principle of Mathematical Induction, $P(n)$ is true for each positive integer n.

4. (a) **Answer.**

i. (I) $|\mu|^2+|\nu|^2+2|\mu|\cdot|\nu|-(\mu+\nu)\overline{(\mu+\nu)}=|\mu|^2+|\nu|^2+2|\mu|\cdot|\overline{\nu}|-\mu\overline{\mu}-\nu\overline{\nu}-\mu\overline{\nu}-\overline{\mu}\nu$

(II) $(\mathsf{Re}(\mu\overline{\nu}))^2$

(III) $(\mathsf{Re}(\mu\overline{\nu}))^2+(\mathsf{Im}(\mu\overline{\nu}))^2$

(IV) $|\mu+\nu|^2\leq(|\mu|+|\nu|)^2$

(V) $|\mu|+|\nu|\geq 0$

ii. (I) Suppose $\mu_1,\cdots,\mu_n\in\mathbb{C}$.

(II) $\left|\sum_{j=1}^{n}\mu_j\right|\leq\sum_{j=1}^{n}|\mu_j|$.

(III) $P(2)$ is true

(IV) Let $k\in\mathbb{N}\backslash\{0,1\}$. Suppose $P(k)$ is true.

(V) $\nu_1,\cdots,\nu_k,\nu_{k+1}$ be complex numbers

(VI)

$$\left|\sum_{j=1}^{k+1}\nu_j\right|=\left|\sum_{j=1}^{k}\nu_j+\nu_{k+1}\right|\leq\left|\sum_{j=1}^{k}\nu_j\right|+|\nu_{k+1}|\leq\sum_{j=1}^{k}|\nu_j|+|\nu_{k+1}|\leq\sum_{j=1}^{k+1}|\nu_j|$$

(VII) the Principle of Mathematical Induction

(b) **Solution.**

Let $\zeta\in\mathbb{C}$. Suppose $0<|\zeta|<1$. Then we have

$$\left|\sum_{k=1050}^{4060}\zeta^k\right|\leq\sum_{k=1050}^{4060}|\zeta^k|=\sum_{k=1050}^{4060}|\zeta|^k=|\zeta|^{1050}\cdot\sum_{k=0}^{3010}|\zeta|^k=|\zeta|^{1050}\cdot\frac{1-|\zeta|^{3011}}{1-|\zeta|}<\frac{|\zeta|^{1050}}{1-|\zeta|}.$$

The first inequality is a consequence of Statement (T).

The last inequality follows from $|\zeta|^{1050}>0$ and $0<|\zeta|^{3011}<1$.

5. **Answer.**

(a) (I) $r=a+b\sqrt{2}$ and $r=a'+b'\sqrt{2}$

(II) $a=a'$ and $b=b'$

(III) and $r=a'+b'\sqrt{2}$

(IV) $(b - b')\sqrt{2}$

(V) Suppose it were true that $b \neq b'$

(VI) $\sqrt{2} = \dfrac{a' - a}{b - b'}$

(VII) $\sqrt{2}$ would be a rational number

(VIII) $b = b'$

(b) (I) $\zeta \in \mathbb{C}\backslash\mathbb{R}$ and $\eta \in \mathbb{C}$

(II) For any $a, a', b, b' \in \mathbb{R}$, if $\eta = a\zeta + b\zeta^2$ and $\eta = a'\zeta + b'\zeta^2$ then $a = a'$ and $b = b'$.

(III) Pick any $a, a', b, b' \in \mathbb{R}$.

(IV) Suppose

(V) $\eta = a'\zeta + b'\zeta^2$

(VI) $\zeta \neq 0$

(VII) $a' - a$

(VIII) Suppose it were true that $b \neq b'$

(IX) $\dfrac{a' - a}{b - b'}$

(X) real

(XI) ζ is not real

(XII) $a' = a$

(c) (I) Suppose $r \in \mathbb{R}$.

(II) For any $n, n' \in \mathbb{Z}$, if $n \leq r < n + 1$ and $n' \leq r < n' + 1$ then $n = n'$.

(III) Pick any $n, n' \in \mathbb{Z}$. Suppose $n \leq r < n + 1$ and $n' \leq r < n' + 1$.

(IV) $(r - n') - (r - n) < 1 - 0$

(V) $(r - n') - (r - n) > 0 - 1$

(VI) n, n' are integers

(VII) an integer

(VIII) only

(IX) 0

(X) $n - n' = 0$