MATH1050 Assignment 4

1. Apply mathematical induction to justify each of the statements below:

   (a) $1 \cdot 2 + 2 \cdot 5 + 3 \cdot 8 + \cdots + n(3n-1) = n^2(n+1)$ *for any positive integer* n.

   (b) $1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \cdots + \frac{1}{\sqrt{n}} \geq \sqrt{n}$ *whenever* n *is a positive integer.*

   (c) $n^2 < 2^{n-1}$ *whenever* n *is an integer greater than* 6.

   (d) $n(n^2+2)$ *is divisible by* 3 *for any* $n \in \mathbb{N}$.

   (e) $7^n(3n+1) - 1$ *is divisible by* 9 *for any* $n \in \mathbb{N}$.

2. (a) i. Consider the statement (A):

   (A) *Suppose* $\{a_n\}_{n=0}^{\infty}$ *is an infinite sequence of complex numbers. Then* $\sum_{k=0}^{n}(a_{k+1} - a_k) = a_{n+1} - a_0$.

   Fill in the blanks in the blocks below, all labelled by capital-letter Roman numerals, with appropriate words so that they give a proof for the statement (A). (*The 'underline' for each blank bears no definite relation with the length of the answer for that blank.*)

   > Suppose $\{a_n\}_{n=0}^{\infty}$ is an infinite sequence of complex numbers.
   > Denote by $P(n)$ the proposition ______(I)______ .
   >
   > - We verify that $P(0)$ is true:
   >
   >   ______(II)______
   >
   >   Hence $P(0)$ is true.
   >
   > - Let $m \in \mathbb{N}$. ______(III)______ . Then $\sum_{k=0}^{m}(a_{k+1} - a_k) =$ ______(IV)______ .
   >
   >   We verify $P(m+1)$:
   >
   >   ______(V)______
   >
   >   Hence $P(m+1)$ is true.
   >
   > ______(VI)______

   ii. Prove the statement (B):

   (B) *Suppose* $\{a_n\}_{n=0}^{\infty}$ *is an infinite sequence of complex numbers. Further suppose* $a_j \neq 0$ *for each* $j \in \mathbb{N}$. *Then* $\prod_{k=0}^{n} \frac{a_{k+1}}{a_k} = \frac{a_{n+1}}{a_0}$.

   Remarks. The statements $(A), (B)$ give the mechanism for a useful method for computing sums/products of consecutive terms of sequences. This method is known as the **Telescopic Method**.

   (b)$^\diamond$ Apply the result described in the statement (A) to prove the statement $(\sharp)$:

   $(\sharp)$ *Let* $\{c_n\}_{n=0}^{\infty}$ *be an infinite sequence of numbers. Let* α, β *be numbers, with* $\alpha \neq 1$. *Suppose* $c_{n+1} = \alpha c_n + \beta$ *for each* $n \in \mathbb{N}$. *Then* $c_n = \alpha^n c_0 + \frac{\beta(1-\alpha^n)}{1-\alpha}$ *for each* $n \geq 1$.

   (c) Prove the statements below. (The Telescopic Method may be useful.)

   i. *Let* $\theta \in \mathbb{R}$. *Suppose* $\sin(\theta) \neq 0$. *Then* $\cos(\theta)\cos(2\theta)\cos(2^2\theta) \cdot \ldots \cdot \cos(2^n\theta) = \frac{\sin(2^{n+1}\theta)}{2^{n+1}\sin(\theta)}$ *for any* $n \in \mathbb{N}$.

   ii. *Let* $\theta \in \mathbb{R}$. *Suppose* $\sin\left(\frac{\theta}{2}\right) \neq 0$. *Then* $1 + 2\sum_{k=1}^{n}\cos(k\theta) = \frac{\sin((n+1/2)\theta)}{\sin(\theta/2)}$ *for any* $n \in \mathbb{N}$.

   iii. *Let* $\theta \in \mathbb{R}$. *Suppose* $\sin(2^p\theta) \neq 0$ *for any* $p \in \mathbb{N}$. *Then* $\sum_{k=0}^{n} 2^k \tan(2^k\theta) = \cot(\theta) - 2^{n+1}\cot(2^{n+1}\theta)$ *for any* $n \in \mathbb{N}$.

   iv. *Let* $\theta \in \mathbb{R}$. *Suppose* $\sin(2^p\theta) \neq 0$ *for any* $p \in \mathbb{N}$. *Then* $\sum_{k=1}^{n} \csc(2^k\theta) = \cot(\theta) - \cot(2^n\theta)$ *for any* $n \in \mathbb{N}\backslash\{0\}$.

3. Fill in the blanks in the blocks below, all labelled by capital-letter Roman numerals, with appropriate words so that they give respectively a proof for the statement (E) and a proof for the statement (F). (*The 'underline' for each blank bears no definite relation with the length of the answer for that blank.*)

(a) We prove the statement (E):

(E) *Let* $\{a_n\}_{n=0}^{\infty}$ *be an infinite sequence of positive real numbers. Suppose* $\displaystyle\sum_{j=0}^{n} a_j = \left(\frac{1+a_n}{2}\right)^2$ *for each* $n \in \mathbb{N}$.

Then $a_n = 2n+1$ *for each* $n \in \mathbb{N}$.

> Let $\{a_n\}_{n=0}^{\infty}$ be an infinite sequence of positive real numbers. ________ (I) ________
>
> Denote by $P(n)$ the proposition below:
>
> ________ (II) ________
>
> - We verify that $P(0)$ is true:
>
>   ________ (III) ________
>
>   Hence $P(0)$ is true.
>
> - ________ (IV) ________
>
>   We verify that $P(k+1)$ is true:
>
>   ________ (V) ________
>
>   Therefore $P(k+1)$ is true.
>
> ________ (VI) ________

(b) We prove the statement (F):

(F) *Let* α, β *are the two distinct roots of the polynomial* $f(x) = x^2 - x - 1$. *Suppose* $\{a_n\}_{n=1}^{\infty}$ *is the infinite sequence of real numbers defined by*

$$\begin{cases} a_1 = 1, & a_2 = 3, \\ & a_{n+2} = a_{n+1} + a_n \quad \text{if} \quad n \geq 1 \end{cases}.$$

Then $a_n = \alpha^n + \beta^n$ *for each positive integer* n.

> ________ (I) ________
>
> Denote by $P(n)$ the proposition below:
>
> $a_n = \alpha^n + \beta^n$ and $a_{n+1} = \alpha^{n+1} + \beta^{n+1}$.
>
> - We verify that $P(1)$ is true:
>
>   We have $a_1 =$ ________ (II) ________ .
>
>   We also have $a_2 =$ ________ (III) ________ .
>
>   Hence $P(1)$ is true.
>
> - ________ (IV) ________
>
>   Then $a_k = \alpha^k + \beta^k$, and $a_{k+1} = \alpha^{k+1} + \beta^{k+1}$.
>
>   We verify that $P(k+1)$ is true:
>
>   We have $a_{k+1} =$ ____ (V) ____ by ____ (VI) ____ immediately.
>
>   Now we verify that $a_{(k+1)+1} = \alpha^{(k+1)+1} + \beta^{(k+1)+1}$:
>
>   ________ (VII) ________
>
>   Therefore $P(k+1)$ is true.
>
> ________ (VIII) ________

4. (a) Fill in the blanks in the blocks below, all labelled by capital-letter Roman numerals, with appropriate words so that they give respectively a proof for the statement (K), and a proof for the statement (L). Both statements are

referred to as (the 'inequality part' of) the **Triangle Inequality for complex numbers**. (*The 'underline' for each blank bears no definite relation with the length of the answer for that blank.*)

i. We prove the statement (K):

(K) *Suppose* $\mu, \nu \in \mathbb{C}$. *Then* $|\mu + \nu| \leq |\mu| + |\nu|$.

> Suppose $\mu, \nu \in \mathbb{C}$. We have
>
> $$\begin{aligned}(|\mu| + |\nu|)^2 - |\mu + \nu|^2 &= \underline{\qquad\qquad (\text{I}) \qquad\qquad} \\ &= 2|\mu\overline{\nu}| - 2\mathsf{Re}(\mu\overline{\nu}) \\ &= 2(|\mu\overline{\nu}| - \mathsf{Re}(\mu\overline{\nu})) \quad \text{——}(\star)\end{aligned}$$
>
> Note that $\underline{\quad (\text{II}) \quad} \leq \underline{\quad (\text{III}) \quad} = |\mu\overline{\nu}|^2$.
> Then $\mathsf{Re}(\mu\overline{\nu}) \leq |\mathsf{Re}(\mu\overline{\nu})| \leq |\mu\overline{\nu}|$. Therefore $|\mu\overline{\nu}| - \mathsf{Re}(\mu\overline{\nu}) \geq 0$.
> Then by $(\star)$, $\underline{\qquad (\text{IV}) \qquad}$.
> Since $|\mu + \nu| \geq 0$ and $\underline{\quad (\text{V}) \quad}$, we have $|\mu + \nu| \leq |\mu| + |\nu|$.

ii. We prove the statement (L):

(L) *Let* $n \in \mathbb{N}\backslash\{0, 1\}$. *Suppose* $\mu_1, \mu_2, \cdots, \mu_n \in \mathbb{C}$. *Then* $\left|\sum_{j=1}^{n} \mu_j\right| \leq \sum_{j=1}^{n} |\mu_j|$.

> Denote by $P(n)$ the proposition below:
>
> $\underline{\qquad (\text{I}) \qquad}$. Then $\underline{\qquad (\text{II}) \qquad}$.
>
> - By the statement (L), $\underline{\qquad (\text{III}) \qquad}$.
> - $\underline{\qquad\qquad (\text{IV}) \qquad\qquad}$
>
>   We verify that $P(k+1)$ is true:
>
>   Suppose $\underline{\qquad (\text{V}) \qquad}$. Then $\underline{\qquad\qquad (\text{VI}) \qquad\qquad}$
>
>   Therefore $P(k+1)$ is true.
>
> By $\underline{\qquad (\text{VII}) \qquad}$, $P(n)$ is true for any $n \in \mathbb{N}\backslash\{0, 1\}$.

(b) By applying the results above, or otherwise, prove the statement $(\sharp)$ below:

$(\sharp)$ *Let* $\zeta \in \mathbb{C}$. *Suppose* $0 < |\zeta| < 1$. *Then* $\left|\sum_{k=1050}^{4060} \zeta^k\right| < \dfrac{|\zeta|^{1050}}{1 - |\zeta|}$.

5. Fill in the blanks in the block below, all labelled by capital-letter Roman numerals, with appropriate words so that it gives a proof for the statement (U), a proof for the statement (V), and a proof for the statement (W). (*The 'underline' for each blank bears no definite relation with the length of the answer for that blank.*)

(a) We prove the statement (U).

(U) *Suppose* r *is a real number. Then there are at most one rational number* a *and at most one rational number* b *satisfying* $r = a + b\sqrt{2}$.

Suppose r is a real number.

[The desired conclusion is: '*there are at most one rational number a and at most one rational number b satisfying $r = a + b\sqrt{2}$.*' It can be re-formulated as:
'*For any $a, a', b, b' \in \mathbb{Q}$, if* ____(I)____ *then* ____(II)____ .']

Pick any $a, a', b, b' \in \mathbb{Q}$. Suppose $r = a + b\sqrt{2}$ ____(III)____ .

Then we have $a + b\sqrt{2} = a' + b'\sqrt{2}$. Therefore ____(IV)____ $= a' - a$.

We verify that $b = b'$ with the help of proof-by-contradiction:

* ____(V)____ .

  Then $b - b' \neq 0$. We would have ____(VI)____ .

  Since a, a', b, b' are rational numbers, ____(VII)____ . However, $\sqrt{2}$ is an irrational number. Contradiction arises.

Now we have verified that ____(VIII)____ . Then $a' - a = (b - b')\sqrt{2} = 0$. Therefore $a = a'$.

(b) We prove the statement (V):

(V) *Suppose $\zeta \in \mathbb{C} \backslash \mathbb{R}$, and $\eta \in \mathbb{C}$. Then there are at most one $a \in \mathbb{R}$ and at most one $b \in \mathbb{R}$ satisfying $\eta = a\zeta + b\zeta^2$.*

Suppose ____(I)____ .

[The desired conclusion is: '*there are at most one $a \in \mathbb{R}$ and at most one $b \in \mathbb{R}$ satisfying $\eta = a\zeta + b\zeta^2$.*' It can be re-formulated as:
' ____(II)____ ']

____(III)____ .

____(IV)____ $\eta = a\zeta + b\zeta^2$ and ____(V)____ . Then $a\zeta + b\zeta^2 = a'\zeta + b'\zeta^2$.

Since $\zeta \notin \mathbb{R}$, we have ____(VI)____ in particular. Then $a + b\zeta = a' + b'\zeta$. Therefore $(b - b')\zeta =$ ____(VII)____ .

We verify $b = b'$ with the help of proof-by-contradiction:

* ____(VIII)____ .

  Then $b - b' \neq 0$. Therefore $\zeta =$ ____(IX)____ .

  Since a, a', b, b' are real, ζ would be ____(X)____ . However, by assumption, ____(XI)____ . Contradiction arises.

Now we have verified that $b = b'$. Then $a' - a = (b - b')\zeta = 0$. Therefore ____(XII)____ also.

(c) We prove the statement (W):

(W) *Suppose $r \in \mathbb{R}$. Then there is at most one integer n satisfying $n \leq r < n + 1$.*

____(I)____

[The desired conclusion to be deduced is: '*there is at most one integer n satisfying $n \leq r < n + 1$.*' It can be re-formulated as:
' ____(II)____ ']

____(III)____

Then $0 \leq r - n < 1$ and $0 \leq r - n' < 1$.
Therefore $n - n' =$ ____(IV)____ $= 1$.

Also, $n - n' =$ ____(V)____ $= -1$.

Hence $-1 < n - n' < 1$.

By assumption, ____(VI)____ . Then $n - n'$ is also ____(VII)____ .

The ____(VIII)____ integer strictly between $-1, 1$ is ____(IX)____ . Then ____(X)____ . Therefore $n = n'$.