MATH1050 Assignment 3 (Answers and selected solutions)

1. **Solution.**

   (a) Suppose ζ, η are complex numbers.
   We have $\zeta\eta = (\mathsf{Re}(\zeta) + i\mathsf{Im}(\zeta))(\mathsf{Re}(\eta) + i\mathsf{Im}(\eta)) = (\mathsf{Re}(\zeta)\mathsf{Re}(\eta) - \mathsf{Im}(\zeta)\mathsf{Im}(\eta)) + i(\mathsf{Re}(\zeta)\mathsf{Im}(\eta) + \mathsf{Im}(\zeta)\mathsf{Re}(\eta))$.
   Then $\overline{\zeta\eta} = (\mathsf{Re}(\zeta)\mathsf{Re}(\eta) - \mathsf{Im}(\zeta)\mathsf{Im}(\eta)) - i(\mathsf{Re}(\zeta)\mathsf{Im}(\eta) + \mathsf{Im}(\zeta)\mathsf{Re}(\eta))$.
   We have $\bar{\zeta} \cdot \bar{\eta} = (\mathsf{Re}(\zeta) - i\mathsf{Im}(\zeta))(\mathsf{Re}(\eta) - i\mathsf{Im}(\eta)) = (\mathsf{Re}(\zeta)\mathsf{Re}(\eta) - \mathsf{Im}(\zeta)\mathsf{Im}(\eta)) - i(\mathsf{Re}(\zeta)\mathsf{Im}(\eta) + \mathsf{Im}(\zeta)\mathsf{Re}(\eta))$.
   Therefore $\overline{\zeta\eta} = \bar{\zeta} \cdot \bar{\eta}$.

   (b) Suppose ζ is a complex numbers.
   Then $\zeta\bar{\zeta} = (\mathsf{Re}(\zeta) + i\mathsf{Im}(\zeta))(\mathsf{Re}(\zeta) - i\mathsf{Im}(\zeta)) = (\mathsf{Re}(\zeta))^2 + (\mathsf{Im}(\zeta))^2 = |\zeta|^2$.

   (c) Let ζ, η be complex numbers. Suppose $\eta \neq 0$. Then $\dfrac{\zeta}{\eta} = \dfrac{\zeta\bar{\eta}}{\eta\bar{\eta}} = \dfrac{\zeta\bar{\eta}}{|\eta|^2}$.

   (d) Suppose ζ, η are complex numbers.
   Then $|\zeta\eta|^2 = (\zeta\eta) \cdot (\overline{\zeta\eta}) = \zeta\eta \cdot \bar{\zeta} \cdot \bar{\eta} = (\zeta\bar{\zeta})(\eta\bar{\eta}) = |\zeta|^2|\eta|^2$.
   Since $|\zeta| \geq 0$, $|\eta| \geq 0$ and $|\zeta\eta| \geq 0$, we have $|\zeta\eta| = |\zeta| \cdot |\eta|$.

2. **Solution.**

   Let $\omega = \dfrac{\sqrt{3}+i}{2}$.

   (a) $\omega^2 = \dfrac{1+\sqrt{3}i}{2}$, $\omega^3 = i$, $\omega^{11} = \dfrac{\sqrt{3}-i}{2}$, $\omega^{12} = 1$.

   (b) $\displaystyle\sum_{k=0}^{2230} \omega^{k+1} = \omega \sum_{k=0}^{2230} \omega^k = \omega \cdot \frac{1-\omega^{2231}}{1-\omega} = \omega \cdot \frac{1-\omega^{-1}}{1-\omega} = -1$.

3. **Answer.**

   (a) $|z|^2 = \dfrac{1+c}{1-c}$.

   (b) $c = \dfrac{z\bar{z}-1}{z\bar{z}+1}$.
   $a = \dfrac{z+\bar{z}}{z\bar{z}+1}$.
   $b = \dfrac{z-\bar{z}}{i(z\bar{z}+1)}$.

4. (a) **Answer.**

   (I) $(z+w)(\bar{z}+\bar{w}) = z\bar{z} + w\bar{w} + z\bar{w} + \bar{z}w = |z|^2 + |w|^2 + z\bar{w} + \bar{z}w$

   (II) $|z|^2 + |-w|^2 + z\overline{(-w)} + \bar{z}(-w) = |z|^2 + |w|^2 - z\bar{w} - \bar{z}w$

   (III) $2|z|^2 + 2|w|^2 + z\bar{w} + \bar{z}w - z\bar{w} - \bar{z}w = 2|z|^2 + 2|w|^2$

   (b) **Answer.**

   (I) r, s, t are complex numbers

   (II) $2|r-s|^2 + 2|r-t|^2 - |(r-s)-(r-t)|^2$

   (III) $2|s-t|^2 + 2|r-s|^2 - |t-r|^2$

   (IV) $|2t-r-s|^2 = 2|t-r|^2 + 2|s-t|^2 - |r-s|^2$

   (V) $3(|s-t|^2 + |t-r|^2 + |r-s|^2)$

(c) **Solution.**

Let ζ, α, β be complex numbers. Suppose $\zeta^2 = \alpha^2 + \beta^2$.

By the Parallelogramic Identity, we have $|\zeta+\alpha|^2 + |\zeta-\alpha|^2 = 2|\zeta|^2 + 2|\alpha|^2$.

Then

$$\begin{aligned}(|\zeta+\alpha|+|\zeta-\alpha|)^2 &= |\zeta+\alpha|^2+|\zeta-\alpha|^2+2|\zeta+\alpha||\zeta-\alpha| \\ &= 2|\zeta|^2+2|\alpha|^2+2|(\zeta+\alpha)(\zeta-\alpha)| \\ &= 2|\zeta|^2+2|\alpha|^2+2|\beta^2| \\ &= 2|\zeta|^2+2|\alpha|^2+2|\beta|^2\end{aligned}$$

Modifying the above argument (by interchanging the roles played by α and β), we have $(|\zeta+\beta|+|\zeta-\beta|)^2 = 2|\zeta|^2+2|\beta|^2+2|\alpha|^2$.

Therefore $(|\zeta+\alpha|+|\zeta-\alpha|)^2 = (|\zeta+\beta|+|\zeta-\beta|)^2$.

Note that $|\zeta+\alpha|+|\zeta-\alpha| \geq 0$ and $|\zeta+\beta|+|\zeta-\beta| \geq 0$. Then $|\zeta+\alpha|+|\zeta-\alpha| = |\zeta+\beta|+|\zeta-\beta|$.

5. **Answer.**

   (a) $|\alpha| = 2$.

   (b) $\alpha = \sqrt{2}+\sqrt{2}i$ or $\alpha = \sqrt{2}-\sqrt{2}i$ or $\alpha = -\sqrt{2}+\sqrt{2}i$ or $\alpha = -\sqrt{2}-\sqrt{2}i$.

6. **Answer.**

   $z = 2$ or $z = 4+2i$.

7. **Answer.**

   (a) $2+2i$, $4i$

   (b) $\sqrt{2}$

   (c) $-1+i$

8. **Answer.**

   (a) $z = 2\cdot\left(\cos\left(-\frac{\pi}{3}\right)+i\sin\left(-\frac{\pi}{3}\right)\right)$.

   (b) i. $-2^{2019}+2^{2019}\cdot\sqrt{3}i$

   ii. $\frac{\sqrt{6}}{2}-\frac{\sqrt{2}}{2}i$, $-\frac{\sqrt{6}}{2}+\frac{\sqrt{2}}{2}i$.

9. **Solution.**

   Let $\zeta = \sin(\frac{2\pi}{3}) + i\cos(\frac{2\pi}{3})$.

   (a) $\zeta = \cos(\frac{\pi}{2}-\frac{2\pi}{3}) + i\sin(\frac{\pi}{2}-\frac{2\pi}{3}) = \cos(-\frac{\pi}{6}) + i\sin(-\frac{\pi}{6})$.

   (b) The three cubic roots of ζ are given by β_1, β_2, β_3, given by

$$\begin{aligned}\beta_1 &= (\cos(-\frac{\pi}{18})+i\sin(-\frac{\pi}{18})), \\ \beta_2 &= (\cos(-\frac{\pi}{18}+\frac{2\pi}{3})+i\sin(-\frac{\pi}{18}+\frac{2\pi}{3})) = (\cos(\frac{11\pi}{18})+i\sin(\frac{11\pi}{18})), \\ \beta_3 &= (\cos(-\frac{\pi}{18}-\frac{2\pi}{3})+i\sin(-\frac{\pi}{18}-\frac{2\pi}{3})) = (\cos(-\frac{13\pi}{18})+i\sin(-\frac{13\pi}{18})).\end{aligned}$$

10. **Answer.**

   (a) z is a solution of the equation concerned iff $z = 2(\cos(\frac{\pi}{2}+N\cdot\frac{2\pi}{5})+i\sin(\frac{\pi}{2}+N\cdot\frac{2\pi}{5}))$ for some N amongst $0, 1, 2, 3, 4$.

(b) $z = 2(\cos(\frac{\pi}{10}) + i\sin(\frac{\pi}{10})$ or $z = 2(\cos(-\frac{3\pi}{10}) + i\sin(-\frac{3\pi}{10})$ or $z = 2(\cos(-\frac{7\pi}{10}) + i\sin(-\frac{7\pi}{10}))$.

11. **Answer.**

(a) There are exactly two elements in $C \cap D$. They are $\{0,1\}$, $\{1,2,3\}$.

(b) There are exactly five elements in $C \cup D$. They are $\{0,1\}$, $\{1\}$, $\{1,2,3\}$, $\{3,4\}$, $\{\{3\},\{4\}\}$.

(c) There are exactly two elements in $C \backslash D$. They are $\{1\}$, $\{3,4\}$.

(d) There is exactly one element in $D \backslash C$. It is $\{\{3\},\{4\}\}$.

(e) There are exactly three elements in $C \triangle D$. They are $\{1\}$,$\{3,4\}$, $\{\{3\},\{4\}\}$.

(f) There are exactly four elements in $\mathfrak{P}(C \backslash D)$. They are $\emptyset$, $\{\{1\}\}$, $\{\{3,4\}\}$, $\{\{1\},\{3,4\}\}$.

12. **Answer.**

(a) Five.

(b) Nine.

(c) Six.

(d) One.

(e) Two.

(f) One.

(g) c, r.

(h) $\emptyset, \{c\}, \{r\}, \{c,r\}$.

13. **Answer.**

(a) $A = \emptyset$.

(b) $B \neq \emptyset$; 125 is an element of B.

14. (b) **Solution.**

We proceed to determine all the solutions of the equation (†) by the sequence of manipulations below.

Reminder.

Line 1. $\sin(2x) + \sin(8x) = \sin(5x)$ —— (†)

Line 2. $\sin(5x - 3x) + \sin(5x + 3x) = \sin(5x)$

Line 3. $2\sin(5x)\cos(3x) = \sin(5x)$

Line 4. $(2\cos(3x) - 1)\sin(5x) = 0$

Line 5. $\cos(3x) = \frac{1}{2}$ or $\sin(5x) = 0$

(Case 1).

Line 6a. $\cos(3x) = \frac{1}{2}$

Line 6b. $3x = \pm\frac{\pi}{3} + K \cdot 2\pi.$ (K may be an arbitrary integer.)

Line 6c. $x = \pm\frac{\pi}{9} + K \cdot \frac{2\pi}{3}$

(Case 2).

Line 7a. $\sin(5x) = 0$

Line 7b. $5x = M\pi.$ (M may be an arbitrary integer.)

Line 7c. $x = M \cdot \frac{\pi}{5}$

Checking:

- Let $x \in \mathbb{R}$. Suppose there exists some $K \in \mathbb{Z}$ such that $x = \frac{\pi}{9} + K \cdot \frac{2\pi}{3}$. Then $\sin(2x) + \sin(8x) - \sin(5x) = \cdots = 0$. Therefore $\sin(2x) + \sin(8x) = \sin(5x)$.
- Let $x \in \mathbb{R}$. Suppose there exists some $L \in \mathbb{Z}$ such that $x = -\frac{\pi}{9} + L \cdot \frac{2\pi}{3}$. Then $\sin(2x) + \sin(8x) - \sin(5x) = \cdots = 0$. Therefore $\sin(2x) + \sin(8x) = \sin(5x)$.
- Let $x \in \mathbb{R}$. Suppose there exists some $M \in \mathbb{Z}$ such that $x = M \cdot \frac{\pi}{5}$. Then $\sin(2x) + \sin(8x) - \sin(5x) = \cdots = 0$. Therefore $\sin(2x) + \sin(8x) = \sin(5x)$.

Define

$$\begin{aligned} A &= \left\{ x \in \mathbb{R} : x = \frac{\pi}{9} + K \cdot \frac{2\pi}{3} \text{ for some } K \in \mathbb{Z} \right\}, \\ B &= \left\{ x \in \mathbb{R} : x = -\frac{\pi}{9} + L \cdot \frac{2\pi}{3} \text{ for some } L \in \mathbb{Z} \right\}, \\ C &= \left\{ x \in \mathbb{R} : x = M \cdot \frac{\pi}{5} \text{ for some } M \in \mathbb{Z} \right\} \end{aligned}$$

The solution set of (†) is given by $A \cup B \cup C$.

(d) **Solution.**

Note that $6^2 + 8^2 = 100 = 10^2$. Take $\alpha = \arcsin(\frac{4}{5})$. Then $10\sin(\alpha) = 8$, $10\cos(\alpha) = 10\sqrt{1 - \sin^2(\alpha)} = 6$.

We proceed to determine all the solutions of the equation (†) by the sequence of manipulations below.

				Reminder.
Line 1.	$6\sin(x) + 8\cos(x)$	$=$	5 —— (†)	
Line 2.	$10\sin(x)\cos(\alpha) + 10\cos(x)\sin(\alpha)$	$=$	5	
Line 3.	$10\sin(x + \alpha)$	$=$	5	
Line 4.	$\sin(x + \alpha)$	$=$	$\frac{1}{2}$	
Line 5.	$x + \alpha$	$=$	$(-1)^N \cdot \frac{\pi}{6} + N\pi.$	(N may be an arbitrary integer.)
Line 6.	x	$=$	$\alpha + (-1)^N \cdot \frac{\pi}{6} + N\pi$	

Checking:

- Let $x \in \mathbb{R}$. Suppose there exists some $N \in \mathbb{Z}$ such that $x = \alpha + (-1)^N \cdot \frac{\pi}{6} + N\pi$. Then $6\sin(x) + 8\cos(x) = \cdots = 5$.

The solution set of (†) is given by $\left\{ x \in \mathbb{R} : x = \alpha + (-1)^N \cdot \frac{\pi}{6} + N\pi \text{ for some } N \in \mathbb{Z} \right\}$.

(f) **Solution.**

We proceed to determine all the solutions of the equation (†) by the sequence of manipulations below.

				Reminder.
Line 1.	$\cos(\frac{1}{2x})$	$=$	1 —— (†)	
Line 2.	$\frac{1}{2x}$	$=$	$2N \cdot \pi.$	(N may be an arbitrary integer.)
Line 3.	x	$=$	$\frac{1}{4N \cdot \pi}.$	(N is required to be non-zero.)

Checking:

- Let $x \in \mathbb{R}$. Suppose there exists some $N \in \mathbb{Z} \backslash \{0\}$ such that $x = \frac{1}{4N \cdot \pi}$. Then $\cos(\frac{1}{2x}) = \cdots = 1$.

The solution set of (†) is given by $\left\{x \in \mathbb{R} : x = \dfrac{1}{4N \cdot \pi} \text{ for some } N \in \mathbb{Z}\backslash\{0\}\right\}$.

Answer.

(a) Define

$$\begin{aligned} A &= \left\{x \in \mathbb{R} : x = -\frac{\pi}{2} + K \cdot 2\pi \text{ for some } K \in \mathbb{Z}\right\}, \\ B &= \left\{x \in \mathbb{R} : x = (-1)^M \cdot \frac{\pi}{6} + M\pi \text{ for some } M \in \mathbb{Z}\right\}. \end{aligned}$$

The solution set of the equation $\cos(2x) = \sin(x)$ is given by $A \cup B$.

(b) Define

$$\begin{aligned} A &= \left\{x \in \mathbb{R} : x = \frac{\pi}{9} + K \cdot \frac{2\pi}{3} \text{ for some } K \in \mathbb{Z}\right\}, \\ B &= \left\{x \in \mathbb{R} : x = -\frac{\pi}{9} + L \cdot \frac{2\pi}{3} \text{ for some } L \in \mathbb{Z}\right\}, \\ C &= \left\{x \in \mathbb{R} : x = M \cdot \frac{\pi}{5} \text{ for some } M \in \mathbb{Z}\right\} \end{aligned}$$

The solution set of the equation $\sin(2x) + \sin(8x) = \sin(5x)$ is given by $A \cup B \cup C$.

(c) Define

$$\begin{aligned} A &= \left\{x \in \mathbb{R} : x = \frac{\pi}{3} + K \cdot 2\pi \text{ for some } K \in \mathbb{Z}\right\}, \\ B &= \left\{x \in \mathbb{R} : x = -\frac{\pi}{3} + K \cdot 2\pi \text{ for some } K \in \mathbb{Z}\right\}, \\ C &= \left\{x \in \mathbb{R} : x = \frac{\pi}{2} + M\pi \text{ for some } M \in \mathbb{Z}\right\} \end{aligned}$$

The solution set of the equation $2\sin(\frac{x}{2})\sin(\frac{3x}{2}) = \cdots = 1$ is given by $A \cup B \cup C$.

(d) Take $\alpha = \arcsin(\frac{4}{5})$. The solution set of the equation $6\sin(x) + 8\cos(x) = 5$ is given by $\left\{x \in \mathbb{R} : x = \alpha + (-1)^N \cdot \frac{\pi}{6} + N\pi \text{ for some } N \in \mathbb{Z}\right\}$.

(e) The solution set of the equation $\tan(3\sqrt{x}) = 1$ is given by $\left\{x \in \mathbb{R} : x = \left(\frac{1}{4} + N\right)^2 \cdot \frac{\pi^2}{9} \text{ for some } N \in \mathbb{N}\right\}$.

(f) The solution set of the equation $\cos(\frac{1}{2x}) = 1$ is given by $\left\{x \in \mathbb{R} : x = \dfrac{1}{4N \cdot \pi} \text{ for some } N \in \mathbb{Z}\backslash\{0\}\right\}$.

15. **Answer.**

(a) Let A, B be sets. We say that A is a subset of B if the statement (†) holds:

(†) For any object x, if $x \in A$ then $x \in B$.

(b) i. (I) $x \in A$

(II) There exists some

(III) $\mathbb{Z}$

(IV) $x = 16m^6$

(V) n

(VI) $2, m \in \mathbb{Z}$

(VII) $2(8m^6) = 2(2m^2)^3 = 2n^3$

(VIII) $x \in B$

ii. (I) $x_0 = 2 \cdot 1^3$

(II) 1

(III) $x_0 \in B$

(IV) Suppose it were true that $x_0 \in A$.

(V) there would exist some $m \in \mathbb{Z}$

(VI) $2 \cdot (4m^6) = 8m^6 = 1$

(VII) $4m^6 \in \mathbb{Z}$

(VIII) divisible

(c) i. (I) Suppose $\zeta \in A$

(II) $|\zeta| \leq 2$

(III) $\leq$

(IV) $(\mathsf{Im}(\zeta))^2$

(V) $|\mathsf{Re}(\zeta)| \leq |\zeta|$

(VI) $|\mathsf{Re}(\zeta)| \leq 2$

(VII) $(\mathsf{Im}(\zeta))^2 \leq (\mathsf{Re}(\zeta))^2 + (\mathsf{Im}(\zeta))^2 = |\zeta|^2$

(VIII) Since $|\mathsf{Im}(\zeta)|$ and $|\zeta|$ are non-negative, we have $|\mathsf{Im}(\zeta)| \leq |\zeta|$. Then by $(\star)$, we have $|\mathsf{Im}(\zeta)| \leq 2$.

(IX) and

(X) $x \in B$

ii. (I) Let $\zeta_0 = 2 + 2i$.

(II) $\mathsf{Re}(\zeta_0)$

(III) $|\mathsf{Im}(\zeta_0)| \leq 2$

(IV) $\zeta_0 \in B$

(V) $|\zeta_0|^2$

(VI) 8

(VII) 4

(VIII) 2

(IX) $\zeta_0 \notin A$

(X) $B \not\subset A$