MATH1050 Assignment 3

1. Prove the statements below:

   (a) *Suppose ζ, η are complex numbers. Then $\overline{\zeta\eta} = \bar{\zeta} \cdot \bar{\eta}$.*

   (b) *Suppose ζ is a complex number. Then $|\zeta|^2 = \zeta\bar{\zeta}$.*

   (c) *Let ζ, η be complex numbers. Suppose $\eta \neq 0$. Then $\dfrac{\zeta}{\eta} = \dfrac{\zeta\bar{\eta}}{|\eta|^2}$.*

   (d) *Suppose ζ, η are complex numbers. Then $|\zeta\eta| = |\zeta| \cdot |\eta|$.*

2. Let $\omega = \dfrac{\sqrt{3}+i}{2}$.

   (a) Write down the respective values of ω^2, ω^3, ω^{11}, ω^{12}.

   (b) Hence, or otherwise, find the value of $\displaystyle\sum_{k=0}^{2230} \omega^{k+1}$.

3. Let a, b, c be real numbers. Suppose $a^2 + b^2 + c^2 = 1$ and $c \neq 1$. Define $z = \dfrac{a+bi}{1-c}$.

   (a) Express $|z|^2$ in terms of c alone.

   (b) Express each of a, b, c in terms of $z, \bar{z}$ alone.

4. (a) Fill in the blanks in the block below, all labelled by capital-letter Roman numerals, with appropriate words so that it gives a proof for the statement (A), which is known as the **Parallelogramic Identity**. (*The 'underline' for each blank bears no definite relation with the length of the answer for that blank.*)

   (A) *Suppose z, w are complex numbers. Then $|z+w|^2 + |z-w|^2 = 2|z|^2 + 2|w|^2$.*

   > Suppose z, w are complex numbers.
   > Then $|z+w|^2 = (z+w)\overline{(z+w)} =$ ______(I)______.
   > Also, $|z-w|^2 = |z+(-w)|^2 =$ ______(II)______.
   > Then $|z+w|^2 + |z-w|^2 =$ ______(III)______.

   (b) Fill in the blanks in the block below, all labelled by capital-letter Roman numerals, with appropriate words so that it gives a proof for the statement (B). (*The 'underline' for each blank bears no definite relation with the length of the answer for that blank.*)

   (B) *Suppose r, s, t are complex numbers. Then $|2r-s-t|^2 + |2s-t-r|^2 + |2t-r-s|^2 = 3(|s-t|^2 + |t-r|^2 + |r-s|^2)$.*

   > Suppose ______(I)______.
   > Then, by the Parallelogramic Identity,
   > $$|2r-s-t|^2 = |(r-s)+(r-t)|^2 = \underline{\qquad(\mathrm{II})\qquad} = 2|r-s|^2 + 2|t-r|^2 - |s-t|^2.$$
   > Similarly, $|2s-t-r|^2 =$ ______(III)______.
   > Also, ______(IV)______.
   > Therefore $|2r-s-t|^2 + |2s-t-r|^2 + |2t-r-s|^2 =$ ______(V)______.

   (c)$^\diamond$ Applying the Parallelogramic Identity, or otherwise, prove the statement below.

   - *Let ζ, α, β be complex numbers. Suppose $\zeta^2 = \alpha^2 + \beta^2$. Then $|\zeta+\alpha| + |\zeta-\alpha| = |\zeta+\beta| + |\zeta-\beta|$.*

5. Let α be a non-zero complex number. Suppose $\alpha^5/\overline{\alpha}^3 = 4$. Also suppose that α is neither real nor purely imaginary.

   (a) What is the modulus of α?

   (b) Find all possible values of α. Express your answers in standard form.

6. *There is no need to give any justifications for your answers in this question.*

   Find all the solutions of the system of equations $\left\{ \begin{array}{rcl} |z-2-2i| & = & 2 \\ |z-4+2i| & = & |z-2i| \end{array} \right.$ with unknown z in $\mathbb{C}$.

   Remark. What are the curves described by the respective equations in the Argand plane?

7. *There is no need to give any justifications for your answers in this question.*

   Consider the system of equations $(S_{\alpha,r}) : \left\{ \begin{array}{rcl} |z-2i| & = & 2 \\ |z-4-4i| & = & |z| \\ |z-\alpha| & = & r \end{array} \right.$ with unknown z in $\mathbb{C}$. Here α is some complex number and r is a non-negative real number.

   Suppose that $(S_{\alpha,r})$ has two distinct solutions.

   (a) Write down all solutions of $(S_{\alpha,r})$.

   (b) What is the smallest possible value of r?

   (c) What is the value of α if $|\mathsf{Re}(\alpha)| = |\mathsf{Im}(\alpha)|$?

   Remark. What are the curves described by the respective equations in the Argand plane?

8. Let $z = \dfrac{2(\sqrt{3}-i)}{\sqrt{3}+i}$.

   (a) Express z in polar form.

   (b) Hence, or otherwise, find the numbers below. Express the respective answers in standard form.

      i. z^{2020}.

      ii. The square roots of z.

9. Let $\zeta = \sin\left(\dfrac{2\pi}{3}\right) + i\cos\left(\dfrac{2\pi}{3}\right)$.

   (a) Express ζ in polar form.

   (b) Hence, or otherwise, find the three cubic roots of ζ, expressing your answer in polar form.

10. (a) Solve for all solutions of the equation $z^5 - 32i = 0$ with complex unknown z.
    Express your answer in polar form.

    (b) *In this part, there is no need to justify your answer.*
    Write down all the complex solutions of the system of inequalities

$$\left\{ \begin{array}{rcl} z^5 - 32i & = & 0 \\ \mathsf{Re}(z) & \geq & \mathsf{Im}(z) \end{array} \right.$$

11. *You are not required to justify your answers in this question.*

    Let $C = \{\{0,1\},\{1\},\{1\},\{1,2,3\},\{3,4\}\}$, $D = \{\{0,1,1,1\},\{1,2,3\},\{\{3\},\{4\}\}\}$. Consider each of the sets below:

    (a) $C \cap D$. (b) $C \cup D$. (c) $C \backslash D$. (d) $D \backslash C$. (e) $C \triangle D$. (f) $\mathfrak{P}(C \backslash D)$.

    Decide whether the set concerned is the empty set or not, and list its elements (if any), each exactly once.

    - If the set concerned is the empty set, write your answer as: '*The set blah-blah-blah is the empty set.*'
      (Example: *The set $C \backslash C$ is the empty set.*)
    - If the set concerned is not the empty set, write: '*There are exactly bleh-bleh-bleh elements in the set blah-blah-blah. They are blih-blih-blih.*'
      (Example: *There are exactly four elements in the set C. They are $\{0,1\},\{1\},\{1,2,3\},\{3,4\}$.*)

12. *You are not required to justify your answers in this question.*

    Let $M = \{m,a,r,c,u,s\}$, $T = \{t,u,l,l,i,u,s\}$, $C = \{c,i,c,e,r,o\}$.

    (a) How many elements are there in the set C?

    (b) How many elements are there in the set $M \cup T$?

    (c) How many elements are there in the set $(M \cup T) \backslash C$?

(d) How many elements are there in the set $\{(M \cup T)\backslash C\}$?

(e) How many elements are there in the set $(\{M\} \cup \{T\})\backslash\{C\}$?

(f) How many elements are there in the set $\{M \cup T\}\backslash\{C\}$?

(g) What are the individual elements of $M \cap C$? Give all of them, writing each exactly one.

(h) What are the individual elements of $\mathfrak{P}(M \cap C)$? Give all of them, writing each exactly one.

13. *You are not required to justify your answers in this question.*

Let

$$\begin{aligned} A &= \{x \in \mathbb{N}\backslash\{0,1,2,3,4,5\} : \sqrt{x} = k \cdot \sqrt{5} \text{ for any } k \in \mathbb{N}\}, \\ B &= \{x \in \mathbb{N}\backslash\{0,1,2,3,4,5\} : \sqrt{x} = k \cdot \sqrt{5} \text{ for some } k \in \mathbb{N}\}. \end{aligned}$$

(a) Is A the empty set?

- If *yes*, write: '$A = \emptyset$'.
- If *no*, write: '$A \neq \emptyset$'. Furthermore, name one element of A.

(b) Is B the empty set?

- If *yes*, just write '$B = \emptyset$'.
- If *no*, write '$B \neq \emptyset$'. Furthermore, name one element of B.

14. *This is a review question on* solving equations/inequalities involving trigonometric functions *which can be handled with purely algebraic manipulations.*

For each equation with unknown in the reals below, determine its solution set, presenting them as sets constructed with the help of the method of specification where appropriate.

You are not required to give the 'checking step' explicitly, but be careful not to wrongly include false candidates amongst the solution, nor wrongly ignore a genuine solution.

(a) $\cos(2x) = \sin(x)$.

(b) $\sin(2x) + \sin(8x) = \sin(5x)$.

(c) $2\sin(\frac{x}{2})\sin(\frac{3x}{2}) = 1$.

(d) $6\sin(x) + 8\cos(x) = 5$.

(e)$^\diamondsuit$ $\tan(3\sqrt{x}) = 1$.

(f)$^\diamondsuit$ $\cos(\frac{1}{2x}) = 1$.

15. (a) Explain the phrase *subset of a set* by stating the appropriate definition.

(b) Let $A = \{x \mid x = 16m^6 \text{ for some } m \in \mathbb{Z}\}$, $B = \{x \mid x = 2m^3 \text{ for some } m \in \mathbb{Z}\}$.

Fill in the blanks in the blocks below, all labelled by capital-letter Roman numerals, with appropriate words so that they give respectively a proof for the statement (I) and a proof for the statement (J). (*The 'underline' for each blank bears no definite relation with the length of the answer for that blank.*)

i. Here we prove the statement (I):

(I) $A \subset B$.

> [We intend to deduce: '*For any* x, *if* $x \in A$ *then* $x \in B$.']
>
> Pick any object x. Suppose ____(I)____ .
>
> By the definition of A, ____(II)____ $m \in$ ____(III)____ such that ____(IV)____ .
>
> [We now ask whether it is true that $x \in B$.]
>
> Define ____(V)____ $= 2m^2$.
>
> Since ____(VI)____ , we have $n \in \mathbb{Z}$.
>
> We have $x =$ ____(VII)____ .
>
> Hence, by the definition of B, ____(VIII)____ .
>
> It follows that $A \subset B$.

ii. Here we prove the statement (J):

(J) $B \not\subset A$.

[We intend to deduce: '*There exists some x_0 such that $x_0 \in B$ and $x_0 \notin A$.*']

Let $x_0 = 2$.
We verify that $x_0 \in B$:

- Note that ___(I)___ and ___(II)___ $\in \mathbb{Z}$. Then, by the definition of B, ___(III)___ .

We verify that $x_0 \notin A$, by applying the method of proof-by-contradiction:

- ___(IV)___

  Then ___(V)___ such that $x_0 = 16m^6$.

  Now $2 = x_0 = 16m^6$.

  Then ___(VI)___ .

  Since $4, m \in \mathbb{Z}$, we have ___(VII)___ .

  Then 1 would be ___(VIII)___ by 2. Contradiction arises.

  It follows that, in the first place, $x_0 \notin A$.

Hence $B \not\subset A$.

(c) Let $A = \{\zeta \in \mathbb{C} : |\zeta| \leq 2\}$, $B = \{\zeta \in \mathbb{C} : |\mathsf{Re}(\zeta)| \leq 2 \text{ and } |\mathsf{Im}(\zeta)| \leq 2\}$.
Fill in the blanks in the blocks below, all labelled by capital-letter Roman numerals, with appropriate words so that they give respectively a proof for the statement (K) and a proof for the statement (L). (*The 'underline' for each blank bears no definite relation with the length of the answer for that blank.*)

i. Here we prove the statement (K):
(K) $A \subset B$.

[We intend to deduce: '*For any $\zeta \in \mathbb{C}$, if $\zeta \in A$ then $\zeta \in B$.*']

Pick any $\zeta \in \mathbb{C}$. ___(I)___ .
By the definition of A, ___(II)___ . —— $(\star)$

[We now ask whether it is true that $\zeta \in B$.]

We have $|\mathsf{Re}(\zeta)|^2 = (\mathsf{Re}(\zeta))^2$ ___(III)___ $(\mathsf{Re}(\zeta))^2 +$ ___(IV)___ $= |\zeta|^2$.
Since $|\mathsf{Re}(\zeta)|$ and $|\zeta|$ are non-negative, we have ___(V)___ . Then by $(\star)$, we have ___(VI)___ .
We also have $|\mathsf{Im}(\zeta)|^2 =$ ___(VII)___ .
___(VIII)___
Therefore $|\mathsf{Re}(\zeta)| \leq 2$ ___(IX)___ $|\mathsf{Im}(\zeta)| \leq 2$. Hence, by the definition of B, ___(X)___ .
It follows that $A \subset B$.

ii. Here we prove the statement (L):
(L) $B \not\subset A$.

[We intend to deduce: '*There exists some $\zeta_0 \in \mathbb{C}$ such that $\zeta_0 \in B$ and $\zeta_0 \notin A$.*']

___(I)___
We verify that $\zeta_0 \in B$:

- We have ___(II)___ $= 2$. Then $|\mathsf{Re}(\zeta_0)| \leq 2$.

  We have $\mathsf{Im}(\zeta_0) = 2$. Then ___(III)___ .

  Therefore $|\mathsf{Re}(\zeta_0)| \leq 2$ and $|\mathsf{Im}(\zeta_0)| \leq 2$. Hence, by the definition of B, we have ___(IV)___ .

We verify that $\zeta_0 \notin A$:

- We note that ___(V)___ $= (\mathsf{Re}(\zeta_0))^2 + (\mathsf{Im}(\zeta_0))^2 =$ ___(VI)___ $>$ ___(VII)___.

  Then $|\zeta_0| >$ ___(VIII)___ . Therefore, by the definition of A, ___(IX)___ .

Hence ___(X)___