MATH1050 Assignment 2 (Answers and selected solutions)

1. ——

2. ——

3. ——

4. **Solution.**

(a) Let n be a positive integer, and $f(x)$ be the polynomial $f(x) = (1+x)^n$.

Note that $f(x) = \sum_{k=0}^{n} \binom{n}{k} x^k$ as polynomials.

i. $\sum_{k=0}^{n} \binom{n}{k} = \sum_{k=0}^{n} \binom{n}{k} \cdot 1^k = f(1) = (1+1)^n = 2^n$.

ii. $\sum_{k=0}^{n} (-1)^k \binom{n}{k} = f(-1) = (1-1)^n = 0$.

iii. $\sum_{k=0}^{n} \frac{1}{2^k} \binom{n}{k} = f(\frac{1}{2}) = \left(1+\frac{1}{2}\right)^n = \frac{3^n}{2^n}$.

iv. $\sum_{k=0}^{n} \frac{(-1)^k \cdot 3^{k-1}}{5^{k+1}} \binom{n}{k} = \frac{1}{15} \sum_{k=0}^{n} \frac{(-1)^k \cdot 3^k}{5^k} \binom{n}{k} = \frac{1}{15} f(-\frac{3}{5}) = \frac{1}{15}\left(1-\frac{3}{5}\right)^n = \frac{2^n}{15 \cdot 5^n}$.

(b) Let m be a positive integer. Then $2m$ is a positive integer.
Let $g(x)$ be the polynomial $g(x) = (1+x)^{2m}$.

Note that $g(x) = \sum_{k=0}^{2m} \binom{2m}{k} x^k$ as polynomials.

i. $\sum_{k=0}^{2m} \binom{2m}{k} = g(1) = 2^{2m}$.

ii. $\sum_{k=0}^{2m} (-1)^k \binom{2m}{k} = g(-1) = 0$.

iii.

$$\begin{aligned}
\sum_{k=0}^{m} \binom{2m}{2k} &= \sum_{j=0}^{2m} \frac{1}{2}\left(\binom{2m}{j} + (-1)^j \binom{2m}{j}\right) \\
&= \frac{1}{2}\left(\sum_{j=0}^{2m} \binom{2m}{j} + \sum_{j=0}^{2m} (-1)^j \binom{2m}{j}\right) = \frac{1}{2}(2^{2m}+0) = 2^{2m-1}
\end{aligned}$$

iv.

$$\begin{aligned}
\sum_{k=0}^{m-1} \binom{2m}{2k+1} &= \sum_{j=0}^{2m} \frac{1}{2}\left(\binom{2m}{j} - (-1)^j \binom{2m}{j}\right) \\
&= \frac{1}{2}\left(\sum_{j=0}^{2m} \binom{2m}{j} - \sum_{j=0}^{2m} (-1)^j \binom{2m}{j}\right) = \frac{1}{2}(2^{2m}-0) = 2^{2m-1}
\end{aligned}$$

Answer.

(a) i. 2^n.

ii. 0.

iii. $\frac{3^n}{2^n}$.

iv. $\dfrac{2^n}{15 \cdot 5^n}$.

(b) i. 2^{2m}.

ii. 0.

iii. 2^{2m-1}.

iv. 2^{2m-1}.

(c) i. $(-1)^p \cdot 2^{2p-1}$.

ii. 0.

iii. $2^{4p-2} + (-1)^p \cdot 2^{2p-1}$.

iv. $2^{4p-2} - (-1)^p \cdot 2^{2p-1}$.

v. 2^{4p-2}.

vi. 2^{4p-2}.

5. **Solution.**

(a) Let $n \in \mathbb{N}\backslash\{0\}$, and $k \in \mathbb{Z}$.

- (Case 1.) Suppose $0 < k \leq n$. Then

$$k \cdot \binom{n}{k} = k \cdot \frac{n!}{k! \cdot (n-k)!} = \frac{n!}{(k-1)! \cdot (n-k)!} = n \cdot \frac{(n-1)!}{(k-1)! \cdot [(n-1)-(k-1)]!} = n \cdot \binom{n-1}{k-1}.$$

- (Case 2.) Suppose $k \leq 0$ or $k > n$. Then $k \cdot \binom{n}{k} = 0 = n \cdot \binom{n-1}{k-1}$.

Hence in any case, $k \cdot \binom{n}{k} = n \cdot \binom{n-1}{k-1}$.

(b) i. Let n be a positive integer.

$$\sum_{k=0}^{n} k\binom{n}{k} = \sum_{k=1}^{n} k\binom{n}{k} = \sum_{k=1}^{n} n\binom{n-1}{k-1} = n\sum_{k=1}^{n}\binom{n-1}{k-1} = n\sum_{j=0}^{n-1}\binom{n-1}{j} = n \cdot 2^{n-1}.$$

(c) i. Let m be a positive integer.

$$\begin{aligned}
\sum_{k=0}^{m} \frac{1}{k+1}\binom{m}{k} &= \sum_{k=0}^{m} \frac{1}{m+1}\binom{m+1}{k+1} \\
&= \frac{1}{m+1}\sum_{k=0}^{m}\binom{m+1}{k+1} \\
&= \frac{1}{m+1}\sum_{j=1}^{m+1}\binom{m+1}{j} \\
&= \frac{1}{m+1}\left(\sum_{j=0}^{m+1}\binom{m+1}{j} - 1\right) = \frac{2^{m+1}-1}{m+1}
\end{aligned}$$

Answer.

(a) ——

(b) i. $n \cdot 2^{n-1}$

ii. When $n = 1$, $\sum_{k=0}^{n} k(k-1)\binom{n}{k} = 1$. Whenever $n \geq 2$, $\sum_{k=0}^{n} k(k-1)\binom{n}{k} = 0$.

iii. $n(n-1) \cdot 2^{n-2}$

iv. $n(n+1) \cdot 2^{n-2}$

(c) i. $\dfrac{2^{m+1}-1}{m+1}$

ii. $\dfrac{1}{m+1}$

iii. $\dfrac{2^{m+2}-m-3}{(m+2)(m+1)}$

6. **Answer.**

   (a) (I) there exist some $m, n \in \mathbb{Z}$

   (II) $n \neq 0$ and $m = nx$

   (b) i. (I) Suppose x, y are rational

   (II) there exist some

   (III) $n \neq 0$ and

   (IV) there exist some $p, q \in \mathbb{Z}$ such that

   (V) $n \neq 0$

   (VI) $q \neq 0$

   (VII) $mq + pn \in \mathbb{Z}$ and $nq \in \mathbb{Z}$

   (VIII) $x + y$ is rational

   ii. (I) there exist some $m, n \in \mathbb{Z}$ such that

   (II) there exists some $p, q \in \mathbb{Z}$ such that $q \neq 0$ and $p = qy$

   (III) $mp = nxqy = nq(xy)$

   (IV) and $q \neq 0$

   (IV) $nq \neq 0$

   (VI) since $m, n, p, q \in \mathbb{Z}$

7. **Answer.**

   (a) Suppose x, y are integers. Then we say that x is divisible by y if there exists some $n \in \mathbb{Z}$ such that $x = ny$.

   (b) i. (I) Let $x, y, n \in \mathbb{Z}$. Suppose x is divisible by n and y is divisible by n.

   (II) there exists some $k \in \mathbb{Z}$ such that $x = kn$

   (III) there exists some $\ell \in \mathbb{Z}$ such that $y = \ell n$

   (IV) $x = kn$ and $y = \ell n$

   (V) $k + \ell \in \mathbb{Z}$

   (VI) $x + y$ is divisible by n

   ii. (I) Let $x, y, n \in \mathbb{Z}$. Suppose x is divisible by n or y is divisible by n.

   (II) x is divisible by n

   (III) there exists some $k \in \mathbb{Z}$ such that

   (IV) $xy = (kn)y = (ky)n$

   (V) since $k \in \mathbb{Z}$ and $y \in \mathbb{Z}$, we have $ky \in \mathbb{Z}$

   (VI) xy is divisible by n

   (VII) Suppose

   (VIII) xy is divisible by n

   (IX) in any case, xy is divisible by n

   Remark. The entries for (IV), (V) may be interchanged.

8. **Answer.**

   (a) (I) it were true that $\sqrt{a^2-b^2} + \sqrt{2ab-b^2} \leq a$

   (II) and $\sqrt{2ab-b^2} \geq 0$

   (III) $\geq$

   (IV) $a^2 - b^2$

   (V) since $a > b > 0$, we have $2ab - b^2 = (2a-b)b \geq 0$

   (VI) a^2

(VII) $(\sqrt{a^2-b^2}+\sqrt{2ab-b^2})^2$

(VIII) $\sqrt{(a-b)(a+b)(2a-b)b}$

(IX) b^2-ab

(X) $b(b-a)<0$

(b) (I) $m, n \in \mathbb{Z}$

(II) $0<|m|<|n|$

(III) it were true that m was divisible by n

(IV) there would exist some $k \in \mathbb{Z}$ such that $m=kn$

(V) $|m|>0$

(VI) $k \neq 0$

(VII) Since k was an integer

(VIII) $|k||n| \geq 1 \cdot |n|$

(IX) assumption

(X) m is not divisible by n

(c) (I) x is irrational

(II) it were true that $\sqrt{x}$ was rational

(III) x is positive

(IV) x

(V) $\sqrt{x}$ was rational

(VI) rational

(VII) irrational

(VIII) rational and irrational

(IX) Contradiction arises

(X) $\sqrt{x}$ is irrational

9. (a) Let $m, n, c \in \mathbb{Z}$. We say that c is a common divisor of m, n if m is divisible by c and n is divisible by c.

(b) Let $p \in \mathbb{Z}$. Suppose p is not amongst $0, 1, -1$. Then we say that p is a prime number if the statement $(\star)$ holds:

For any $n \in \mathbb{Z}$, if p is divisible by n then $n=1$ or $n=-1$ or $n=p$ or $n=-p$.

Acceptable answer.

Let $p \in \mathbb{Z}$. Suppose p is not amongst $0, 1, -1$. Then we say that p is a prime number if p is divisible by no integer other than $1, -1, p, -p$.

(c) Let $h, k, p \in \mathbb{Z}$. Suppose p is a prime number. Further suppose hk is divisible by p. Then at least one of h, k is divisible by p.

(d) (I) Suppose it were true that $\sqrt[3]{3}$ was not irrational

(II) there would exist some $m, n \in \mathbb{Z}$

(III) $n \neq 0$ and $m = n \cdot \sqrt[3]{3}$

(IV) m^3 would be divisible by 3

(V) Euclid's Lemma

(VI) there would exist some $k \in \mathbb{Z}$ such that $m=3k$

(VII) Note that $3k^3$ was an integer. Then n^3 would be divisible by 3.

(VIII) 3 is a prime number

(IX) n would be divisible by 3

(X) m, n have no common divisors other than $-1, 1$