# MATH1050 Assignment 2

1. *This is a review question on the* summation notation.

   Let $n$ be a positive integer, and $x_1, x_2, \cdots, x_n$ be real numbers. Define $\bar{x} = \dfrac{1}{n}\displaystyle\sum_{k=1}^{n} x_k$.

   (a) i. Prove that $\displaystyle\sum_{j=1}^{n}(x_j - \bar{x}) = 0$.

   ii. Prove that $\displaystyle\sum_{j=1}^{n}(x_j - \bar{x})^2 = \sum_{j=1}^{n} {x_j}^2 - n\bar{x}^2$.

   (b) Let $a, b$ be real numbers, with $a \neq 0$. For each $j = 1, 2, \cdots, n$, define $y_j = ax_j + b$. Define $\bar{y} = \dfrac{1}{n}\displaystyle\sum_{k=1}^{n} y_k$.

   i. Prove that $\bar{y} = a\bar{x} + b$.

   ii. Prove that $\displaystyle\sum_{j=1}^{n}(y_j - \bar{y})^2 = a^2\left(\sum_{j=1}^{n} {x_j}^2 - n\bar{x}^2\right)$.

2. *This is a review question on* binomial coefficients *and the* Binomial Theorem.

   We introduce this definition below:

   - *Let* $n \in \mathbb{N}$. *Define* $\dbinom{n}{k} = \begin{cases} \dfrac{n!}{k! \cdot (n-k)!} & \textit{if} \quad k \in [\![0, n]\!], \\ 0 & \textit{if} \quad k \in \mathbb{Z} \textit{ and } (k < 0 \textit{ or } k > n). \end{cases}$

     *The number* $\dbinom{n}{k}$ *is usually referred to as the* **binomial coefficients** *of* $n$ *over* $k$.

   (a) Let $n \in \mathbb{N}$, and $k \in \mathbb{Z}$.

   i. Verify that $\dbinom{n}{k} = \dbinom{n}{n-k}$.

   ii. Verify that $\dbinom{n}{k} + \dbinom{n}{k+1} = \dbinom{n+1}{k+1}$.

   (b)$^\diamond$ Apply the results above and the Telescopic Method (or mathematical induction) to verify the statements below:

   i. $\displaystyle\sum_{j=0}^{m}\binom{n+j}{j} = \binom{n+m+1}{m}$ *for any* $n, m \in \mathbb{N}$.

   ii. $\displaystyle\sum_{j=0}^{m}\binom{n+j}{n} = \binom{n+m+1}{n+1}$ *for any* $n, m \in \mathbb{N}$.

3. *This is a review question on* binomial coefficients *and the* Binomial Theorem.

   Apply mathematical induction to prove the **Binomial Theorem** (formulated in terms of polynomials):

   - *Suppose* $n \in \mathbb{N}$. *Then* $(1+x)^n = 1 + \dbinom{n}{1}x + \dbinom{n}{2}x^2 + \cdots + \dbinom{n}{k}x^k + \cdots + \dbinom{n}{n-1}x^{n-1} + x^n$ *as polynomials.*

4. *This is a review question on* binomial coefficients *and the* Binomial Theorem.

   (a) Let $n$ be a positive integer. By considering the polynomial $(1+x)^n$, or otherwise, find the respective values of the numbers below. Leave your answer in terms of $n$ where appropriate.

   i. $\displaystyle\sum_{k=0}^{n}\binom{n}{k}$.

   ii. $\displaystyle\sum_{k=0}^{n}(-1)^k\binom{n}{k}$.

   iii. $\displaystyle\sum_{k=0}^{n}\frac{1}{2^k}\binom{n}{k}$.

   iv. $\displaystyle\sum_{k=0}^{n}\frac{(-1)^k \cdot 3^{k-1}}{5^{k+1}}\binom{n}{k}$.

   (b)$^\diamond$ Let $m$ be a positive integer. By consider the polynomial $(1+x)^{2m}$, or otherwise, find the respective values of the numbers below. Leave your answer in terms of $m$ where appropriate.

   i. $\displaystyle\sum_{k=0}^{2m}\binom{2m}{k}$.

   ii. $\displaystyle\sum_{k=0}^{2m}(-1)^k\binom{2m}{k}$.

   iii. $\displaystyle\sum_{k=0}^{m}\binom{2m}{2k}$.

   iv. $\displaystyle\sum_{k=0}^{m-1}\binom{2m}{2k+1}$.

   (c)$^\clubsuit$ Let $p$ be a positive integer. By consider the polynomial $(1+x)^{4p}$, or otherwise, find the respective values of the numbers below. Leave your answer in terms of $p$ where appropriate. (*Hint.* Make good use of complex numbers.)

   i. $\displaystyle\sum_{j=0}^{2p}(-1)^j\binom{4p}{2j}$

   ii. $\displaystyle\sum_{j=0}^{2p-1}(-1)^j\binom{4p}{2j+1}$

   iii. $\displaystyle\sum_{k=0}^{p}\binom{4p}{4k}$.

iv. $\displaystyle\sum_{k=0}^{p-1}\binom{4p}{4k+2}$.  v. $\displaystyle\sum_{k=0}^{p-1}\binom{4p}{4k+1}$.  vi. $\displaystyle\sum_{k=0}^{p-1}\binom{4p}{4k+3}$.

5. *This is a review question on* binomial coefficients *and the* Binomial Theorem.

(a) Let $n \in \mathbb{N}\backslash\{0\}$, and $k \in \mathbb{Z}$. Prove that $k \cdot \binom{n}{k} = n \cdot \binom{n-1}{k-1}$.

(b) Let $n$ be a positive integer. Find the respective values of the numbers below. Leave your answer in terms of $n$ where appropriate.

i. $\displaystyle\sum_{k=0}^{n} k\binom{n}{k}$.  ii.$^{\diamondsuit}$ $\displaystyle\sum_{k=0}^{n}(-1)^{k+1}k\binom{n}{k}$.  iii.$^{\clubsuit}$ $\displaystyle\sum_{k=0}^{n} k(k-1)\binom{n}{k}$.  iv. $\displaystyle\sum_{k=0}^{n} k^2\binom{n}{k}$.

**Remark.** There is an alternative method for computing the sums described here: make use of differentiation.

(c)$^{\diamondsuit}$ Let $m$ be a positive integer. Find the respective values of the numbers below. Leave your answer in terms of $m$ where appropriate.

i. $\displaystyle\sum_{k=0}^{m}\frac{1}{k+1}\binom{m}{k}$.  ii.$^{\diamondsuit}$ $\displaystyle\sum_{k=0}^{m}\frac{(-1)^k}{k+1}\binom{m}{k}$.  iii.$^{\clubsuit}$ $\displaystyle\sum_{k=0}^{m}\frac{1}{(k+2)(k+1)}\binom{m}{k}$.

**Remark.** There is an alternative method for computing the sums described here: make use of integration.

6. (a) Fill in the blanks in the passage below so as to give the definition for the notion of *rational numbers*:

*Suppose $x \in \mathbb{R}$. Then we say that $x$ is* **rational** *if* _____(I)_____ *such that* _____(II)_____ .

(b) Fill in the blanks in the block below, all labelled by capital-letter Roman numerals, with appropriate words so that it gives a proof for the statement $(A)$ and a proof for the statement $(B)$. (*The 'underline' for each blank bears no definite relation with the length of the answer for that blank.*)

i. Here we prove the statement $(A)$:

$(A)$ *Let $x, y \in \mathbb{R}$. Suppose $x, y$ are rational. Then $x + y$ is rational.*

> Let $x, y \in \mathbb{R}$. _____(I)_____ .
>
> [We want to deduce that $x+y$ is rational. This amounts to verifying the statement '*there exist some $s, t \in \mathbb{Z}$ such that $t \neq 0$ and $s = t(x+y)$*'.]
>
> By definition, _____(II)_____ $m, n \in \mathbb{Z}$ such that _____(III)_____ $m = nx$.
> Also, _____(IV)_____ $q \neq 0$ and $p = qy$.
> Note that $mq + pn = nxq + qyn = nq(x+y)$ .
> Since _____(V)_____ and _____(VI)_____ , we have $nq \neq 0$.
> Also, since $m, n, p, q \in \mathbb{Z}$, we have _____(VII)_____ .
> Hence, by definition, _____(VIII)_____ .

ii. Here we prove the statement $(B)$:

$(B)$ *Let $x, y$ be real numbers. Suppose $x, y$ are rational. Then $xy$ is rational.*

> Let $x, y$ be real numbers. Suppose $x, y$ are rational.
>
> [We want to deduce that $xy$ is rational. This amounts to verifying the statement '*there exist some $s, t \in \mathbb{Z}$ such that $t \neq 0$ and $s = t(xy)$*'.]
>
> By definition, _____(I)_____ $n \neq 0$ and $m = nx$.
> Also, _____(II)_____ .
> Note that _____(III)_____ .
> Since $n \neq 0$ _____(IV)_____ , we have _____(V)_____ .
> Also, _____(VI)_____ , we have $mp \in \mathbb{Z}$ and $nq \in \mathbb{Z}$.
> Hence, by definition, $xy$ is rational.

7. (a) Explain the phrase *divisibility for integers* by stating the appropriate definition.

(b) Fill in the blanks in the block below, all labelled by capital-letter Roman numerals, with appropriate words so that it gives a proof for the statement ($C$) and a proof for the statement ($D$). (*The 'underline' for each blank bears no definite relation with the length of the answer for that blank.*)

i. Here we prove the statement ($C$):

($C$) *Let $x, y, n \in \mathbb{Z}$. Suppose $x$ is divisible by $n$ and $y$ is divisible by $n$. Then $x + y$ is divisible by $n$.*

> ______(I)______
> Since $x$ is divisible by $n$, ______(II)______ .
> Since $y$ is divisible by $n$, ______(III)______ .
> We have ______(IV)______ . Then $x + y = kn + \ell n = (k + \ell)n$.
> Since $k \in \mathbb{Z}$ and $\ell \in \mathbb{Z}$, we have ______(V)______ .
> Therefore, by definition, ______(VI)______ .

ii. Here we prove the statement ($D$):

($D$) *Let $x, y, n \in \mathbb{Z}$. Suppose $x$ is divisible by $n$ or $y$ is divisible by $n$. Then $xy$ is divisible by $n$.*

> ______(I)______
>
> - (Case 1). Suppose ______(II)______ . Then ______(III)______ $x = kn$.
>   Note that ______(IV)______ . Also, ______(V)______ .
>   Then ______(VI)______ .
> - (Case 2). ______(VII)______ $y$ is divisible by $n$. Modifying the argument for (Case 1), we also deduce that ______(VIII)______ .
>
> Hence, ______(IX)______ .

8. Fill in the blanks in the block below, all labelled by capital-letter Roman numerals, with appropriate words so that it gives a proof for the statement ($E$), a proof for the statement ($F$), and a proof for the statement ($G$). (*The 'underline' for each blank bears no definite relation with the length of the answer for that blank.*)

(a) Here we prove the statement ($E$):

($E$) *Let $a, b$ be real numbers. Suppose $a > b > 0$. Then $\sqrt{a^2 - b^2} + \sqrt{2ab - b^2} > a$.*

> Let $a, b$ be real numbers. Suppose $a > b > 0$.
> Further suppose that ______(I)______ .
>
> [*Reminder.* Under what we have supposed and what we have further supposed, we try to obtain a contradiction.]
>
> Note that $\sqrt{a^2 - b^2} \geq 0$ ______(II)______ . Then $a \geq \sqrt{a^2 - b^2} + \sqrt{2ab - b^2}$ ___(III)___ 0.
> Since $a > b > 0$, we have $a^2 - b^2 = (a - b)(a + b) \geq 0$. Then $(\sqrt{a^2 - b^2})^2 =$ ______(IV)______ .
> Similarly, ______(V)______ . Then $(\sqrt{2ab - b^2})^2 = 2ab - b^2$.
> Therefore we would have
>
> $$\begin{aligned} \underline{\quad(\text{VI})\quad} &\geq \underline{\quad(\text{VII})\quad} \\ &= (a^2 - b^2) + (2ab - b^2) + 2\sqrt{(a^2 - b^2)(2ab - b^2)} \\ &= a^2 - 2b^2 + 2ab + 2\sqrt{(a - b)(a + b)(2a - b)b}. \end{aligned}$$
>
> Hence
>
> $$0 \leq \underline{\quad(\text{VIII})\quad} \leq \underline{\quad(\text{IX})\quad} = b(b - a).$$
>
> Recall that by assumption, $a > b > 0$. Then ______(X)______ .
> Therefore $0 \leq b(b - a) < 0$. Contradiction arises.
> It follows that, in the first place, $\sqrt{a^2 - b^2} + \sqrt{2ab - b^2} > a$.

(b) Here we prove the statement ($F$):

($F$) *Let $m, n \in \mathbb{Z}$. Suppose $0 < |m| < |n|$. Then $m$ is not divisible by $n$.*

> Let ___(I)___ . Suppose ___(II)___ .
> Further suppose ___(III)___ .
>
> [*Reminder.* Under what we have supposed and what we have further supposed, we try to obtain a contradiction.]
>
> Since $m$ was divisible by $n$, by definition ___(IV)___ .
> By assumption, ___(V)___ . Then $m \neq 0$.
> Since $m \neq 0$ and $m = kn$, we would have ___(VI)___ . Then $|k| \neq 0$.
> ___(VII)___ , $|k|$ would also be an integer. Then $|k| \geq 1$.
> By assumption $|n| > 0$. Then $|m| = |kn| =$ ___(VIII)___ $= |n|$.
> Also, by ___(IX)___ , $|n| > |m|$.
> Then $|m| \geq |n| > |m|$. Therefore $|m| > |m|$. Contradiction arises.
> It follows that, in the first place, ___(X)___ .

(c) Here we prove the statement ($G$):

($G$) *Let $x$ be a positive real number. Suppose $x$ is irrational. Then $\sqrt{x}$ is irrational.*

> Let $x$ be a positive real number.
> Suppose ___(I)___ .
> Further suppose ___(II)___ .
>
> [*Reminder.* Under what we have supposed and what we have further supposed, we try to obtain a contradiction.]
>
> Since ___(III)___ , we have $(\sqrt{x})^2 =$ ___(IV)___ .
> Since ___(V)___ , $(\sqrt{x})^2$ would be rational as well.
> Therefore $x$ would be ___(VI)___ .
> By assumption, $x$ is ___(VII)___ . Then $x$ would be simultaneously ___(VIII)___ .
> ___(IX)___ .
> It follows that, in the first place, ___(X)___ .

9. (a) Explain the phrase *common divisor for integers* by stating the appropriate definition.
   (b) Explain the phrase *prime number* by stating the appropriate definition.
   (c) State, without proof, Euclid's Lemma.
   (d) Here we prove the statement ($H$), with the help of Euclid's Lemma:

   ($H$) $\sqrt[3]{3}$ *is irrational.*

> ___(I)___ .
> Then $\sqrt[3]{3}$ would be a rational number. Therefore ___(II)___ such that ___(III)___ .
> Without loss of generality, we may assume that $m, n$ have no common divisors other than $1, -1$.
> Since $m = n \cdot \sqrt[3]{3}$, we would have $m^3 = 3n^3$.
> Note that $n^3$ was an integer. Then ___(IV)___ .
> Now also note that 3 is a prime number. Then, by ___(V)___ , $m$ would be divisible by 3.
> Therefore ___(VI)___ .
> Then we would have $27k^3 = (3k)^3 = m^3 = 3n^3$. Therefore $n^3 = 9k^3 = 3(3k^3)$.
> ___(VII)___
> Note that ___(VIII)___ . Then, by Euclid's Lemma, ___(IX)___ .
> Therefore both $m, n$ would be divisible by 3. Hence 3 would be a common divisor of $m, n$.
> Recall that we have assumed that ___(X)___ . Contradiction arises.
>
> Therefore the assumption that $\sqrt[3]{3}$ was not irrational is false. It follows that $\sqrt[3]{3}$ is irrational in the first place.