MATH1050 Assignment 1 (Answers and selected solutions)

1. (a) **Solution.**

We proceed to solve the equation $(\star)$:

$$\begin{aligned}
x+\sqrt{x+1} &= 11 \quad \text{——} \quad (\star) \\
\sqrt{x+1} &= 11-x \\
(\sqrt{x+1})^2 &= (11-x)^2 \\
x+1 &= x^2-22x+121 \\
x^2-23x+120 &= 0 \\
(x-8)(x-15) &= 0 \\
x=8 \quad &\text{or} \quad x=15
\end{aligned}$$

Checking:

- $8+\sqrt{8+1}=11$.
- $15+\sqrt{15+1}=19\neq 11$.

The only solution of $(\star)$ is $x=8$.

(c) **Solution.**

We proceed to solve the equation $(\star)$:

$$\begin{aligned}
\log_{5-x}(215-x^3) &= 3 \quad \text{——} \quad (\star) \\
\frac{\ln(215-x^3)}{\ln(5-x)} &= 3 \\
\ln(215-x^3) &= 3\ln(5-x) \\
215-x^3 &= (5-x)^3 \\
x^2-5x-6 &= 0 \\
(x+1)(x-6) &= 0 \\
x=-1 \quad &\text{or} \quad x=6
\end{aligned}$$

Checking:

- $5-(-1)=6>0$ and $215-(-1)^3=216>0$. We have $\log_{5-(-1)}(215-(-1)^3)=\log_6(216)=3$.
- $5-6<0$. Then $\log_{5-6}(u)$ is not well-defined for whatever real value of u.

The only solution of $(\star)$ is $x=-1$.

(f) **Solution.**

We proceed to solve the equation $(\star)$:

$$\begin{aligned}
(x-4)^2-5|x-4|+6 &= 0 \quad \text{——} \quad (\star) \\
|x-4|^2-5|x-4|+6 &= 0 \\
(|x-4|-2)(|x-4|-3) &= 0 \\
|x-4|=2 \quad &\text{or} \quad |x-4|=3 \\
x-4=2 \quad \text{or} \quad x-4=-2 \quad &\text{or} \quad x-4=3 \quad \text{or} \quad x-4=-3 \\
x=6 \quad \text{or} \quad x=2 \quad &\text{or} \quad x=7 \quad \text{or} \quad x=1
\end{aligned}$$

(Every line is logically equivalent to the next. No checking of solution is needed.)

The solutions of $(\star)$ are given by $x=1$ or $x=2$ or $x=6$ or $x=7$.

(m) **Solution.**

We proceed to solve the inequality $(\star)$:

$$\begin{aligned}
\frac{2}{3-x} &\leq 1 \quad \text{——} \quad (\star) \\
2(3-x) &\leq (3-x)^2 \quad \text{and } x \neq 3 \\
(x-3)^2 + 2(x-3) &\geq 0 \quad \text{and } x \neq 3 \\
(x-1)(x-3) &\geq 0 \quad \text{and } x \neq 3 \\
(x \leq 1 &\quad \text{or} \quad x \geq 3) \quad \text{and } x \neq 3 \\
x \leq 1 &\quad \text{or} \quad x > 3
\end{aligned}$$

(Every line is logically equivalent to the next. No checking of solution is needed.)

The solutions of the inequality $(\star)$ are given by $x \leq 1$ or $x > 3$.

(p) **Solution.**

We proceed to solve the inequality $(\star)$:

$$\begin{aligned}
|x^2 - 5x| &< 6 \quad \text{——} \quad (\star) \\
x^2 - 5x > -6 &\quad \text{and} \quad x^2 - 5x < 6 \\
x^2 - 5x + 6 > 0 &\quad \text{and} \quad x^2 - 5x - 6 < 0 \\
(x-2)(x-3) > 0 &\quad \text{and} \quad (x+1)(x-6) < 0 \\
(x < 2 \text{ or } x > 3) &\quad \text{and} \quad -1 < x < 6 \\
(x < 2 \text{ and } -1 < x < 6) &\quad \text{or} \quad (x > 3 \text{ and } -1 < x < 6) \\
-1 < x < 2 &\quad \text{or} \quad 3 < x < 6
\end{aligned}$$

(Every line is logically equivalent to the next. No checking of solution is needed.)

The solutions of the inequality $(\star)$ are given by $-1 < x < 2$ or $3 < x < 6$.

(s) **Solution.**

We proceed to solve the inequality $(\star)$:

$$\begin{aligned}
|x^2 - 3| &\leq 2|x| \quad \text{——} \quad (\star) \\
(x^2 - 3)^2 &\leq 4x^2 \\
x^4 - 6x^2 + 9 &\leq 4x^2 \\
x^4 - 10x^2 + 9 &\leq 0 \\
(x^2 - 1)(x^2 - 9) &\leq 0 \\
(x+3)(x+1)(x-1)(x-3) &\leq 0 \\
-3 \leq x \leq -1 &\quad \text{or} \quad 1 \leq x \leq 3
\end{aligned}$$

(Every line is logically equivalent to the next. No checking of solution is needed.)

The solutions of the inequality $(\star)$ are given by $-3 \leq x \leq -1$ or $1 \leq x \leq 3$.

Answer.

(a) The only solution of the equation $x + \sqrt{x+1} = 11$ is $x = 8$.

(b) The only solution of the equation $2(4^x + 4^{-x}) - 7(2^x + 2^{-x}) + 10 = 0$ is $x = 0$.

(c) The only solution of the equation $\log_{5-x}(215 - x^3) = 3$ is $x = -1$.

(d) The solutions of the equation $|x^2 - 5x + 6| = x$ are given by $x = 3 - \sqrt{3}$ or $x = 3 + \sqrt{3}$.

(e) The only solution of the equation $x|x| + 5x + 6 = 0$ is $x = -1$.

(f) The solutions of the equation $(x-4)^2 - 5|x-4| + 6 = 0$ are given by $x = 1$ or $x = 2$ or $x = 6$ or $x = 7$.

(g) The solutions of the system $\begin{cases} xy + x = 6 \\ xy - y = 2 \end{cases}$ are given by $(x, y) = (2, 2)$ or $(x, y) = (3, 1)$.

(h) The solutions of the system $\begin{cases} xy & = & 35 \\ x^{\log_5(y)} & = & 7 \end{cases}$ are given by $(x, y) = (5, 7)$ or $(x, y) = (7, 5)$.

(i) The solutions of the inequality $x^2 - 3x < 10$ are given by $-2 < x < 5$.

(j) The solutions of the system $\begin{cases} (x+1)(x-6) & \geq & 8 \\ 3x - 1 & \geq & 5 \end{cases}$ are given by $x \geq 7$.

(k) The solutions of the system $(x + 1)^2 > 16$ or $2x + 5 > 7$ are given by $x < -5$ or $x > 1$.

(l) The solutions of the inequality $(x - 1)(x - 2)(x - 3) \geq 0$ are given by $1 \leq x \leq 2$ or $x \geq 3$.

(m) The solutions of the inequality $\dfrac{2}{3-x} \leq 1$ are given by $x \leq 1$ or $x > 3$.

(n) The solutions of the inequality $2x - \dfrac{3}{x} \geq 1$. are given by $-1 \leq x < 0$ or $x \geq 1.5$.

(o) The solutions of the inequality $\dfrac{x^2-1}{x^2-4} \leq -2$ are given by $-2 < x \leq -\sqrt{3}$ or $\sqrt{3} \leq x < 2$.

(p) The solutions of the inequality $|x^2 - 5x| < 6$ are given by $-1 < x < 2$ or $3 < x < 6$.

(q) The solutions of the inequality $\left|\dfrac{3x+11}{x+2}\right| < 2$ are given by $-7 < x < -3$.

(r) The solutions of the inequality $\Big|\ |x| - 4\ \Big| > 3$ are given by $-1 < x < 1$ or $x < -7$ or $x > 7$.

(s) The solutions of the inequality $|x^2 - 3| \leq 2|x|$ are given by $-3 \leq x \leq -1$ or $1 \leq x \leq 3$.

(t) The solutions of the inequality $|2x + 1| < 3x - 2$ are given by $x > 3$.

2. **Answer.**

(a) (I) Suppose $x + y > 1$ and $x > y$
(II) $(x - y)(x + y - 1)$
(III) Since
(IV) $x > y$
(V) $x + y - 1 > 0$ and $x - y > 0$
(VI) $(x - y)(x + y) - (x - y) > 0$
(VII) $x^2 - y^2 > x - y$

(b) (I) Let $x, y \in \mathbb{R}$. Suppose $x > 0$ and $y > 0$.
(II) $(x + y)(x^2 - xy + y^2) - xy(x + y) = (x + y)(x^2 - 2xy + y^2) = (x + y)(x - y)^2$
(III) $x - y$ is (also) a real number
(IV) ≥ 0
(V) $(x^3 + y^3) - xy(x + y) \geq 0$

(c) (I) Suppose $y > x > 0$ and $z > -y$
(II) $z > -y$
(III) > 0
(IV) Suppose
(V) $zy > zx$
(VI) $\dfrac{(x+z)y - x(y+z)}{y(y+z)} > 0$
(VII) Suppose $\dfrac{x+z}{y+z} > \dfrac{x}{y}$
(VIII) $\dfrac{x+z}{y+z} \cdot y(y+z) > \dfrac{x}{y} \cdot y(y+z)$
(IX) $zy - zx = (xy + zy) - (xy + zx) > 0$
(X) and
(XI) $z < 0$ and $y - x < 0$

(XII) $\frac{x+z}{y+z} > \frac{x}{y}$ iff $z > 0$

3. (a) (I) Suppose
(II) $t^4 - s^4 = (t^2 - s^2)(t^2 + s^2) = (t-s)(t+s)(t^2+s^2)$
(III) $s < t$
(IV) $s \geq 0$ and
(V) $s + t > 0$
(VI) $t^2 > 0$
(VII) $s \geq 0$
(VIII) $t^2 + s^2 > 0$
(IX) $f(t) - f(s) > 0$
(X) f is strictly increasing on $[0, +\infty)$

(b) (I) Suppose f is strictly decreasing on $\mathbb{R}$.
(II) Pick any $s, t \in \mathbb{R}$. Suppose $s < t$.
(III) $g(s) - g(t) = (f(s) - 2s^3) - (f(t) - 2t^3) = (f(s) - f(t)) + 2(t-s)(t^2 + st + s^2)$
(IV) Since f is strictly decreasing and $s < t$
(V) $t^2 + st + s^2$
(VI) 0
(VII) $f(s) - f(t) + 2(t-s)(t^2 + st + s^2) > 0$
(VIII) $g(s) > g(t)$

4. (a) **Answer.**

(I) There exists some non-zero complex number r

(II) for any $n \in \mathbb{N}$, $\frac{b_{n+1}}{b_n} = r$

(b) **Answer.**

(I) there exists some non-zero complex number r such that for any $n \in \mathbb{N}$, $\frac{b_{n+1}}{b_n} = r$

(II) $\frac{b_1}{b_0} = r$

(III) $\frac{b_m}{b_{m-1}} = r$

(IV) $\frac{b_1}{b_0} \cdot \frac{b_2}{b_1} \cdot \frac{b_3}{b_2} \cdot \ldots \cdot \frac{b_{m-1}}{b_{m-2}} \cdot \frac{b_m}{b_{m-1}} = r^m$

(V) $b_m = b_0 r^m$

(c) **Solution.**

Let $\{a_n\}_{n=0}^{\infty}$ be a geometric progression. Suppose $k, \ell, m \in \mathbb{N}$, and $a_k = A$, $a_\ell = B$ and $a_m = C$.
By the result in the previous part, there exists some non-zero complex number r such that for any $n \in \mathbb{N}$, $a_n = a_0 r^n$.
In particular, $a_k = a_0 r^k$, $a_\ell = a_0 r^\ell$ and $a_m = a_0 r^m$.
Then

$$\begin{aligned} A^{\ell-m} B^{m-k} C^{k-\ell} &= (a_0 r^k)^{\ell-m} (a_0 r^\ell)^{m-k} (a_0 r^m)^{k-\ell} \\ &= {a_0}^{(\ell-m)+(m-k)+(k-\ell)} r^{k(\ell-m)+\ell(m-k)+m(k-\ell)} \\ &= {a_0}^0 r^0 = 1 \end{aligned}$$

5. (a) **Answer.**

(I) Suppose a, b, c are in arithmetic progression.
(II) d
(III) $c - b = d$
(IV) $b^2 - a^2 - ca + bc = (b-a)(b+a) + (b-a)c = d(a+b+c)$

(V) $[(c^2-ab)-(b^2-ca)] = c^2-b^2-ab+ca = (c-b)(c+b)+(c-b)a = d(a+b+c)$

(VI) $(b^2-ca)-(a^2-bc) = (c^2-ab)-(b^2-ca)$

(b) **Solution.**

Let a, b, c be numbers. Suppose a^2-bc, b^2-ca, c^2-ab are in arithmetic progression. Further suppose $a+b+c \neq 0$.

By assumption, $b^2 - ca = \dfrac{(a^2-bc)+(c^2-ab)}{2}$.

Therefore $2b^2 - 2ca = a^2+c^2-ab-bc$.

Hence $0 = (a^2-b^2)+(c^2-b^2)+(ac-ab)+(ac-bc) = \cdots = (a+c-2b)(a+b+c)$.

Since $a+b+c \neq 0$, we have $a+c-2b=0$. Then $b = \dfrac{a+b}{2}$.

Therefore a, b, c are in arithmetic progression.

6. **Answer.**

$P = Q = 1$.

Hint. The key is to make use of the relations

$$\begin{cases} \alpha+\beta & = & -b/a \\ \alpha\beta & = & c/a \end{cases}$$

with which the assumption $\alpha = r\beta$ is combined.