MATH1050 Assignment 1

1. *This is a review question on* solving equations/inequalities *which can be handled with purely algebraic manipulations.*

   Solve each of the equations/inequalities/systems below for all its real solutions. 'Check solution' when indeed you have to do so. [1]

   (a) $x + \sqrt{x+1} = 11$.

   (b) $2(4^x + 4^{-x}) - 7(2^x + 2^{-x}) + 10 = 0$.

   (c) $\log_{5-x}(215 - x^3) = 3$.

   (d) $|x^2 - 5x + 6| = x$.

   (e) $x|x| + 5x + 6 = 0$.

   (f) $(x-4)^2 - 5|x-4| + 6 = 0$.

   (g) $\begin{cases} xy + x = 6 \\ xy - y = 2 \end{cases}$

   (h) $\begin{cases} xy = 35 \\ x^{\log_5(y)} = 7 \end{cases}$

   (i) $x^2 - 3x < 10$.

   (j) $\begin{cases} (x+1)(x-6) \geq 8 \\ 3x - 1 \geq 5 \end{cases}$

   (k) $(x+1)^2 > 16$ or $2x + 5 > 7$.

   (l) $(x-1)(x-2)(x-3) \geq 0$.

   (m) $\dfrac{2}{3-x} \leq 1$.

   (n) $2x - \dfrac{3}{x} \geq 1$.

   (o) $\dfrac{x^2-1}{x^2-4} \leq -2$.

   (p) $|x^2 - 5x| < 6$.

   (q) $\left|\dfrac{3x+11}{x+2}\right| < 2$.

   (r)$^\diamondsuit$ $\big|\ |x| - 4\ \big| > 3$.

   (s) $|x^2 - 3| \leq 2|x|$.

   (t)$^\diamondsuit$ $|2x+1| < 3x - 2$.

   Remark. Now suppose you are not required to give any step of algebraic manipulation. Can you modify the 'graphical method' for solving equations in *school mathematics* to determine the answer for each part as quickly as possible?

2. Fill in the blanks in the blocks below, all labelled by capital-letter Roman numerals, with appropriate words so that they give respectively a proof for the statement (A), a proof for the statement (B) and a proof for the statement (C). (*The 'underline' for each blank bears no definite relation with the length of the answer for that blank.*)

   (a) Here we prove the statement (A):

   (A) *Let $x, y \in \mathbb{R}$. Suppose $x + y > 1$ and $x > y$. Then $x^2 - y^2 > x - y$.*

   > Let $x, y \in \mathbb{R}$. ____(I)____ .
   > We note that $(x-y)(x+y) - (x-y) =$ ____(II)____ . —— $(\star)$
   > ____(III)____ $x + y > 1$, we have $x + y - 1 > 0$.
   > Since ____(IV)____ , we have $x - y > 0$.
   > Since ____(V)____ , we have $(x-y)(x+y-1) > 0$.
   > Then, by $(\star)$, ____(VI)____ .
   > Therefore ____(VII)____ .

   (b) Here we prove the statement (B):

   (B) *Let $x, y \in \mathbb{R}$. Suppose $x > 0$ and $y > 0$. Then $x^3 + y^3 \geq xy(x+y)$.*

[1] In various situations, you may need apply some special rules about the words '*and*', '*or*', known as the *Distributive Laws for 'and', 'or'*, (with or without your being aware of them). They may be in-formally stated as below:

A. *The pair of statements below are the same in the sense that one holds exactly when the other holds:*

- ∗ *(blah-blah-blah* or *bleh-bleh-bleh)* and *bloh-bloh-bloh.*
- ∗ *(blah-blah-blah* and *bloh-bloh-bloh)* or *(bleh-bleh-bleh* and *bloh-bloh-bloh).*

B. *The pair of statements below are the same in the sense that one holds exactly when the other holds:*

- ∗ *(blah-blah-blah* and *bleh-bleh-bleh)* or *bloh-bloh-bloh.*
- ∗ *(blah-blah-blah* or *bloh-bloh-bloh)* and *(bleh-bleh-bleh* or *bloh-bloh-bloh).*

More will be said of them in the discussion on *logic*.

________(I)________ .
We have $(x^3 + y^3) - xy(x + y) =$ ________(II)________ . —— $(\star)$
Since $x > 0$ and $y > 0$, we have $x + y > 0$.
Since x, y are real numbers, ____(III)____ . Then $(x - y)^2 \geq 0$.
Since $x + y > 0$ and $(x - y)^2 \geq 0$, we have $(x + y)(x - y)^2$ __(IV)__ .
Then, by $(\star)$, ____(V)____ .
Hence $x^3 + y^3 \geq xy(x + y)$.

(c) Here we prove the statement (C):

(C) *Let $x, y, z \in \mathbb{R}$. Suppose $y > x > 0$ and $z > -y$. Then $\dfrac{x+z}{y+z} > \dfrac{x}{y}$ iff $z > 0$.*

Let $x, y, z \in \mathbb{R}$. ____(I)____ .
Since __(II)__ , we have $y + z > 0$. Then, since $y > 0$ also, we have $y(y + z)$ __(III)__ .
• [We want to deduce: '*If $z > 0$ then $\dfrac{x+z}{y+z} > \dfrac{x}{y}$*.']
__(IV)__ $z > 0$.
Then, since $z > 0$ and $y > x$, we have ____(V)____ .
Therefore $(x + z)y = xy + zy > xy + zx = x(y + z)$.
Then $\dfrac{x+z}{y+z} - \dfrac{x}{y} =$ ____(VI)____ .
Therefore $\dfrac{x+z}{y+z} > \dfrac{x}{y}$.
• [We want to deduce: '*If $\dfrac{x+z}{y+z} > \dfrac{x}{y}$ then $z > 0$*.']
________(VII)________ .
Then $xy + zy = (x + z)y =$ ____(VIII)____ $= x(y + z) = xy + zx$.
Therefore $z(y - x) =$ ____(IX)____ .
Then ($z > 0$ __(X)__ $y - x > 0$) or (____(XI)____).
Since $y > x$, we have $y - x > 0$.
Hence $z > 0$ and $y - x > 0$. In particular, $z > 0$.
It follows that ________(XII)________ .

3.$^\diamond$ We introduce/recall the definitions on *strict monotonicity* for real-valued functions of one real variable:

Let I be an interval, and $h : D \longrightarrow \mathbb{R}$ be a real-valued function of one real variable with domain D which contains I as a subset entirely.

- *h is said to be* **strictly increasing** *on I if the statement* (StrIncr) *holds:*
  (StrIncr) *For any $s, t \in I$, if $s < t$ then $h(s) < h(t)$.*
- *h is said to be* **strictly decreasing** *on I if the statement* (StrDecr) *holds:*
  (StrDecr) *For any $s, t \in I$, if $s < t$ then $h(s) > h(t)$.*

Fill in the blanks in the blocks below, all labelled by capital-letter Roman numerals, with appropriate words so that they give respectively a proof for the statement (D) and a proof for the statement (E). (*The 'underline' for each blank bears no definite relation with the length of the answer for that blank.*)

(a) Define the function $f : \mathbb{R} \longrightarrow \mathbb{R}$ by $f(x) = x^4$ for any $x \in \mathbb{R}$.

Here we prove the statement (D):

(D) *f is strictly increasing on the interval $[0, +\infty)$.*

[We are going to verify the statement (†): '*For any* $s, t \in [0, +\infty)$, *if* $s < t$ *then* $f(s) < f(t)$.']

Pick any $s, t \in [0, +\infty)$. ___(I)___ $s < t$.

[We want to deduce $f(t) - f(s) > 0$.]

We have $f(t) - f(s) =$ ___________(II)___________. —— $(\star)$

[We want to check that each of $t - s$, $t + s$, $t^2 + s^2$ is positive.
First we ask whether it is true that $t - s > 0$.]

Since ___(III)___ , we have $t - s > 0$.

[Next we ask whether it is true that $t + s > 0$.]

Since ___(IV)___ $s < t$, we have $t > 0$.
Then, since $s \geq 0$ and $t > 0$, we have ___(V)___ .

[Finally we ask whether it is true that $t^2 + s^2 > 0$.]

Since $t > 0$, we have ___(VI)___ . Since ___(VII)___ , we have $s^2 \geq 0$. Then ___(VIII)___ .
Now, since $t - s > 0$ and $t + s > 0$ and $t^2 + s^2 > 0$, we have $(t - s)(t + s)(t^2 + s^2) > 0$.
Then by $(\star)$, we have ___(IX)___ .
Therefore $f(s) < f(t)$.
It follows from definition that ___________(X)___________ .

(b) Here we prove the statement (E):

(E) *Let* $f : \mathbb{R} \longrightarrow \mathbb{R}$ *be a function. Define the function* $g : \mathbb{R} \longrightarrow \mathbb{R}$ *by* $g(x) = f(x) - 2x^3$ *for any* $x \in \mathbb{R}$. *Suppose* f *is strictly decreasing on* $\mathbb{R}$. *Then* g *is strictly decreasing on* $\mathbb{R}$.

Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be a function. Define the function $g : \mathbb{R} \longrightarrow \mathbb{R}$ by $g(x) = f(x) - 2x^3$ for any $x \in \mathbb{R}$.
___________(I)___________

[We are going to verify (possibly with the help of the assumption 'f *is strictly decreasing on* $\mathbb{R}$') the statement (†): '*For any* $s, t \in \mathbb{R}$, *if* $s < t$ *then* $g(s) > g(t)$.']

_______________(II)_______________

[We want to deduce $g(s) - g(t) > 0$.]

We have ___________(III)___________ . —— $(\star)$
___________(IV)___________ , we have $f(s) - f(t) > 0$.

Since $s < t$, we have $t - s > 0$. Also, ___(V)___ $= \frac{1}{2}[t^2 + (t + s)^2 + s^2] \geq$ ___(VI)___ . Then $2(t - s)(t^2 + st + s^2) \geq 0$.
Since $f(s) - f(t) > 0$ and $2(t - s)(t^2 + st + s^2) \geq 0$, we have ___________(VII)___________ .
Then by $(\star)$, we have $g(s) - g(t) > 0$.
Therefore ___(VIII)___ .
It follows from definition that g is strictly decreasing on $\mathbb{R}$.

4. *This is a review question on* geometric progressions. *If you are not confident with your understanding on the notion of* geometric progressions, *you are advised to read the handout* Arithmetic progressions and geometric progressions.

(a) Fill in the blanks in the passage below so as to give the definition for the notion of *geometric progression*:

Let $\{b_n\}_{n=0}^{\infty}$ *be an infinite sequence of non-zero complex numbers.*
The infinite sequence $\{b_n\}_{n=0}^{\infty}$ *is said to be a* **geometric progression** *if the statement (GP) holds:*

(GP) ___________(I)___________ such that ___________(II)___________ .

The number r *is called the* **common ratio** *of this geometric progression.*

(b) Consider the statement (F):

(F) *Let $\{b_n\}_{n=0}^{\infty}$ be an infinite sequence of non-zero complex numbers. Suppose $\{b_n\}_{n=0}^{\infty}$ is a geometric progression. Then there exists some non-zero complex number r such that for any $m \in \mathbb{N}$, $b_m = b_0 r^m$.*

Fill in the blanks in the blocks below, all labelled by capital-letter Roman numerals, with appropriate words so that they give respectively a proof for the statement (F). (*The 'underline' for each blank bears no definite relation with the length of the answer for that blank.*)

> Let $\{b_n\}_{n=0}^{\infty}$ is an infinite sequence of non-zero complex numbers. Suppose $\{b_n\}_{n=0}^{\infty}$ is a geometric progression.
>
> Then _____(I)_____ .
>
> Pick any $m \in \mathbb{N}$.
>
> We have _____(II)_____ , $\dfrac{b_2}{b_1} = r$, $\dfrac{b_3}{b_2} = r$, ..., $\dfrac{b_{m-1}}{b_{m-2}} = r$, and _____(III)_____ .
>
> Then $\dfrac{b_m}{b_0} =$ _____(IV)_____ .
>
> Therefore _____(V)_____ .

(c) Hence, or otherwise, prove the statement ($\sharp$):

($\sharp$) *Let $\{a_n\}_{n=0}^{\infty}$ be a geometric progression. Suppose $k, \ell, m \in \mathbb{N}$, and $a_k = A$, $a_\ell = B$ and $a_m = C$. Then $A^{\ell-m}B^{m-k}C^{k-\ell} = 1$.*

5. *This is a review question on* arithmetic progressions. *If you are not confident with your understanding on the notion of* arithmetic progressions*, you are advised to read the handout* Arithmetic progressions and geometric progressions*.*

(a) Consider the statement (G):

(G) *Let a, b, c be numbers. Suppose a, b, c are in arithmetic progression. Then a^2-bc, b^2-ca, c^2-ab are in arithmetic progression.*

Fill in the blanks in the blocks below, all labelled by capital-letter Roman numerals, with appropriate words so that they give respectively a proof for the statement (G). (*The 'underline' for each blank bears no definite relation with the length of the answer for that blank.*)

> Let a, b, c be numbers. _____(I)_____
>
> Denote the common difference by __(II)__ .
>
> [We want to verify that $a^2 - bc, b^2 - ca, c^2 - ab$.
> By definition, it sufficies to verify that $(b^2 - ca) - (a^2 - bc) = (c^2 - ab) - (b^2 - ca)$.]
>
> By definition, we have $b - a = d$ and _____(III)_____ .
>
> Note that $(b^2 - ca) - (a^2 - bc) =$ _____(IV)_____ .
>
> Also note that _____(V)_____ .
>
> Then _____(VI)_____ .
>
> Hence $a^2 - bc, b^2 - ca, c^2 - ab$ are in arithmetic progression.

(b) Prove the statement ($\sharp$):

($\sharp$) *Let a, b, c be numbers. Suppose a^2-bc, b^2-ca, c^2-ab are in arithmetic progression. Further suppose $a+b+c \neq 0$. Then a, b, c are in arithmetic progression.*

Remark. You may take for granted the result ($\sharp$):

($\sharp$) *Suppose u, v, w are numbers. Then u, v, w are in arithmetic progression iff $v = \dfrac{u+w}{2}$.*

6. *This is a review question on* quadratic polynomials.

Let a, b, c, r be numbers, with $a \neq 0$ and $c \neq 0$ and $r \neq 0$. Let $f(x)$ be the quadratic polynomial given by $f(x) = ax^2 + bx + c$. Suppose α, β are the roots of $f(x)$. Further suppose $\alpha = r\beta$.

Prove that $rb^2 = (Pr + Q)^2 ac$. Here P, Q are some integers whose values you have to determined explicitly.