Lecture 9

February 7, 2021

1 Diffusion on the half-line

We are trying to solve the diffusion equation on the half-line:

$$\begin{aligned} v_t - kv_{xx} = 0 \quad & in \quad 0 < x < \infty, 0 < t < \infty \\ v(x,0) = \phi(x) \quad & for \quad t = 0 \\ v(0,t) = 0 \quad & for \quad x = 0. \end{aligned} \tag{1}$$

Let ϕ_{odd} be the unique odd extension of ϕ to the whole line. That is,

$$\phi_{odd}(x) = \begin{cases} \phi(x) & for\ x > 0 \\ -\phi(-x) & for\ x < 0 \\ 0 & for\ x = 0. \end{cases}$$

We first solve the diffusion equation on the whole line

$$\begin{aligned} u_t - ku_{xx} = 0 \quad & for \quad -\infty < x < \infty, 0 < t < \infty \\ u(x,0) = \phi_{odd}(x). & \end{aligned}$$

According to Lec 7, it is given by the formula

$$u(x,t) = \int_{-\infty}^{+\infty} \Phi(x-y,t)\phi_{odd}(y)dy,$$

where $\Phi(x,t) = \frac{1}{\sqrt{4\pi kt}}e^{-x^2/4kt}$.
Because $u(x,t)$ is also an odd function of x, we have $u(0,t) = 0$.
Thus its restriction on the half-line will be the solution to (1)

$$v(x,t) = u(x,t) \quad for \quad x > 0.$$

Hence for $0 < x < \infty$, $0 < t < \infty$, we have

$$v(x,t) = \frac{1}{\sqrt{4\pi kt}} \int_0^{\infty} [e^{-(x-y)^2/4kt} - e^{-(x+y)^2/4kt}]\phi(y)dy.$$

Exercise 1. Solve (1) with $\phi(x) = 1$.

Now let's play the same game with the Neumann problem

$$\begin{aligned} w_t - kw_{xx} = 0 \quad for \quad 0 < x < \infty, 0 < t < \infty \\ w(x, 0) = \phi(x) \\ w_x(0, t) = 0. \end{aligned} \tag{2}$$

In this case, we consider an even extension

$$\phi_{even}(x) \quad = \quad \begin{cases} \phi(x) & for\ x \geq 0 \\ \phi(-x) & for\ x < 0. \end{cases}$$

By the same reason, the solution

$$\begin{aligned} w(x,t) &= \frac{1}{\sqrt{4\pi kt}}[\int_0^\infty e^{-(x-y)^2/4kt}\phi(y)dy + \int_{-\infty}^0 e^{-(x-y)^2/4kt}\phi(-y)dy] \\ &= \frac{1}{\sqrt{4\pi kt}}\int_0^\infty [e^{-(x-y)^2/4kt} + e^{-(x+y)^2/4kt}]\phi(y)dy. \end{aligned}$$

$w(x, t)$ is an even function of x, so $w_x(0, t) = 0$.

Exercise 2. Solve (2) with $\phi(x) = 1$.

2 Diffusion with a source

In this section we solve the inhomogeneous diffusion equation on the whole line,

$$\begin{aligned} u_t - ku_{xx} = f(x,t) \quad -\infty < x < \infty \quad 0 < t < \infty \\ u(x, 0) = \phi(x) \end{aligned} \tag{3}$$

with $f(x, t)$ and $\phi(x)$ arbitrary given functions.
Motivation: The simplest analogy is the ODE

$$\begin{cases} \frac{du(t)}{dt} + Au(t) = f(t) \\ u(0) = \phi. \end{cases}$$

where A is a constant. Using the integrating factor to get the solution

$$u(t) \quad = \quad e^{-tA}\phi + \int_0^t e^{(s-t)A}f(s)ds.$$

If we denote the operator $S(t)\phi = e^{-tA}\phi$ (the solution to the homogenous ODE), then the solution u to the inhomogenous ODE is

$$u(t) \quad = \quad S(t)\phi + \int_0^t S(t-s)f(s)ds.$$

Similarly, the solution to the homogenous diffusion equation is

$$\mathscr{S}(t)\phi(x) = \int_{-\infty}^{\infty} \Phi(x-y,t)\phi(y)dy.$$

The solution to the inhomogenous equation (3) may be analogous to the ODE case

$$\begin{aligned} u(x,t) &= \mathscr{S}(t)\phi(x) + \int_0^t \mathscr{S}(t-s)f(x,s)ds. \\ &= \int_{-\infty}^{\infty} \Phi(x-y,t)\phi(y)dy + \int_0^t \int_{-\infty}^{\infty} \Phi(x-y,t-s)f(y,s)dyds. \end{aligned}$$

We begin to verify that the function $u(x,t)$ is the solution.

$$\begin{aligned} \frac{\partial u}{\partial t} &= \int_{-\infty}^{\infty} \Phi_t(x-y,t)\phi(y)dy + \lim_{s\to t}\int_{-\infty}^{\infty} \Phi(x-y,t-s)f(y,s)dy \\ &\quad + \int_0^t \int_{-\infty}^{\infty} \Phi_t(x-y,t-s)f(y,s)dyds \\ &= k\int_{-\infty}^{\infty} \Phi_{xx}(x-y,t)\phi(y)dy + f(y,t) \\ &\quad +k\int_0^t \int_{-\infty}^{\infty} \Phi_{xx}(x-y,t-s)f(y,s)dyds \\ &= ku_{xx} + f(y,t). \end{aligned}$$

and

$$\begin{aligned} u(x,0) &= \lim_{t\to 0}\int_{-\infty}^{\infty} \Phi(x-y,t)\phi(y)dy \\ &= \phi(x). \end{aligned}$$

3 Source on a half-line

Now consider the Dirichlet problem

$$\begin{aligned} v_t - kv_{xx} = f(x,t) \quad for \quad 0 < x < \infty, 0 < t < \infty \\ v(0,t) = h(t) \\ v(x,0) = \phi(x). \end{aligned}$$

Let $V(x,t) := v(x,t) - h(t)$. Then it satisfies

$$\begin{aligned} V_t - kV_{xx} = f(x,t) - h'(t) \quad for \quad 0 < x < \infty, 0 < t < \infty \\ V(0,t) = 0 \\ V(x,0) = \phi(x) - h(0). \end{aligned} \tag{4}$$

Exercise 3. Using the method of reflection to solve the above problem (4).

For the inhomogeneous Neumann problem on the half-line,

$$\begin{aligned} w_t - kw_{xx} &= f(x,t) \quad for \quad 0 < x < \infty, 0 < t < \infty \\ w_x(0,t) &= h(t) \\ w(x,0) &= \phi(x), \end{aligned}$$

we would subtract off the function $xh(t)$. That is $W(x,t) = w(x,t) - xh(t)$ which satisfies the equation

$$\begin{aligned} W_t - kW_{xx} &= f(x,t) - xh'(t) \quad for \quad 0 < x < \infty, 0 < t < \infty \\ W_x(0,t) &= 0 \\ W(x,0) &= \phi(x) - xh(0). \end{aligned} \tag{5}$$

Exercise 4. Using the method of reflection to solve the above problem (5).