Lecture Five

January 24, 2021

1 Causality

For wave equation on the whole line with an initial position and an initial velocity

$$\begin{cases} u_{tt} - c^2 u_{xx} = 0 & -\infty < x < +\infty \\ u(x,0) = \phi(x) & u_t(x,0) = \psi(x). \end{cases}$$

We have derived the d'Alembert formula

$$u(x,t) \quad = \quad \frac{1}{2}[\phi(x+ct) + \phi(x-ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(s)ds.$$

Note if $\phi \in C^2(\mathbb{R})$ and $\psi \in C^1(\mathbb{R})$, the solution given above is the classical solution of the wave equation on the whole line. If not, this may be some kind of "week" solution which we will learn from the graduate PDE course.

We can draw the solution's pictures from the formula:

Case one:

$$\begin{cases} \phi(x) = 1 & |x| \leq a, \\ \phi(x) = 0 & |x| > a, \\ \psi(x) \equiv 0 & -\infty < x < +\infty. \end{cases}$$

Case two:

$$\begin{cases} \phi(x) \equiv 0 & -\infty < x < +\infty, \\ \psi(x) = 1 & |x| \leq a, \\ \psi(x) = 0 & |x| > a. \end{cases}$$

The case one illustrates the effect of ϕ is a pair of waves traveling in either direction at speed c and at half the original amplitude. The effect of ψ is a wave spreading out at speed less than c in both directions. Some part of the wave may lag behind, but no part goes faster than speed c. This is called the principle of causality.

The *domain of influence* of the point $(x_0, 0)$ is a sector between $x + ct = x_0$ and $x - ct = x_0$. It means the initial values only influence the domain.

The *interval of dependence* of the point (x,t) is the values of ϕ at the two points $x \pm ct$, and the values of ψ within the interval $[x-ct, x+ct]$. From d'Alembert formula tells us the value of u at (x,t) only depends on this domain.

The *domain of dependence is bouned by the pair of characteristic lines that pass through* (x,t).

2 The law of conservation of energy

Consider the wave equations $\rho u_{tt} = Tu_{xx}$ for $-\infty < x < \infty$. The energy $E = KE + PE$ of the wave equation can be defined by the sum of kinetic energy $KE = \frac{1}{2}\rho\int_{-\infty}^{+\infty} u_t^2 dx$ and potential energy $PE = \frac{1}{2}T\int_{-\infty}^{+\infty} u_x^2 dx$. We can prove that $\frac{dE}{dt} = 0$.

Proof. We assume that ψ, ϕ vanish outside an interval $\{|x| \le R\}$, such that all the integrals converge.

$$\begin{aligned}
\frac{dE}{dt} &= \frac{1}{2}\rho\frac{d}{dt}\int_{-\infty}^{+\infty} u_t^2 dx + \frac{1}{2}T\frac{d}{dt}\int_{-\infty}^{+\infty} u_x^2 dx \\
&= \rho\int_{-\infty}^{+\infty} u_t u_{tt} dx + T\int_{-\infty}^{+\infty} u_x u_{xt} dx \\
&= \int_{-\infty}^{+\infty} u_t T u_{xx} dx + T\int_{-\infty}^{+\infty} u_x u_{xt} dx \\
&= -T\int_{-\infty}^{+\infty} u_x u_{xt} dx + T u_t u_x|_{-\infty}^{+\infty} + T\int_{-\infty}^{+\infty} u_x u_{xt} dx \\
&= 0.
\end{aligned}$$

In this case, we can also prove the uniqueness. Suppose u and $\tilde{u}$ are solutions of the wave equation, then $v = u - \tilde{u}$ satisfies

$$\begin{cases} v_{tt} - c^2 v_{xx} = 0 & -\infty < x < +\infty \\ v(x,0) = 0 & v_t(x,0) = 0. \end{cases}$$

So we have $\frac{dE}{dt} = 0$ and

$$\frac{1}{2}\rho\int_{-\infty}^{+\infty} v_t^2 dx + \frac{1}{2}T\int_{-\infty}^{+\infty} v_x^2 dx = E(t) = E(0) = 0.$$

Thus $v_t = v_x \equiv 0$. By $v(x,0) = 0$, we get $v \equiv 0$ which means the uniqueness of the wave equation. □

3 Reflections of waves

We use reflection method to get the Dirichlet problem to the wave equation on the half-line $(0,\infty)$.

$$\begin{cases} v_{tt} - c^2 v_{xx} = 0 & 0 < x < \infty \\ v(x,0) = \phi(x), v_t(x,0) = \psi(x), & 0 < x < \infty \\ v(0,t) = 0. \end{cases}$$

Consider the odd extensions of both of the initial functions to the whole line $\phi_{odd}(x)$ and $\psi_{odd}(x)$ such that

$$\phi_{odd}(x) = \begin{cases} \phi(x) & 0 < x < \infty \\ 0 & x = 0 \\ -\phi(-x) & -\infty < x < 0. \end{cases}$$

and

$$\psi_{odd}(x) = \begin{cases} \psi(x) & 0 < x < \infty \\ 0 & x = 0 \\ -\psi(-x) & -\infty < x < 0. \end{cases}$$

Let $u(x,t)$ be the solution of the following wave equation on the whole line

$$\begin{cases} u_{tt} - c^2 u_{xx} = 0 & -\infty < x < \infty \\ u(x,0) = \phi_{odd}(x), u_t(x,0) = \psi_{odd}(x), & -\infty < x < \infty. \end{cases}$$

The ϕ_{odd} and ψ_{odd} are odd functions then the solution

$$u(x,t) = \frac{1}{2}[\phi_{odd}(x+ct) + \phi_{odd}(x-ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi_{odd}(s)ds$$

is also an odd function. Because

$$\frac{1}{2}[\phi_{odd}(x+ct) + \phi_{odd}(x-ct)] = -\frac{1}{2}[\phi_{odd}(-x-ct) + \phi_{odd}(-x+ct)]$$

and

$$\frac{1}{2c}\int_{x-ct}^{x+ct} \psi_{odd}(s)ds = -\frac{1}{2c}\int_{-x-ct}^{-x+ct} \psi_{odd}(s)ds,$$

we have

$$u(x,t) = -u(-x,t).$$

Thus define $v(x,t) = u(x,t)$ for $0 < x < \infty$, the v is the precisely the solution we are looking for.

For $x > c|t|$ we have

$$v(x,t) = \frac{1}{2}[\phi(x+ct) + \phi(x-ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(y)dy. \tag{1}$$

In this case the value of v depends on the value of ϕ at the pair of points $x \pm ct$ and on the values of ψ in the interval between these points.

For $0 < x < c|t|$, we have

$$\begin{aligned} v(x,t) &= \frac{1}{2}[\phi(x+ct) - \phi(ct-x)] + \frac{1}{2c}\int_0^{x+ct} \psi(y)dy + \frac{1}{2c}\int_{x-ct}^{0} [-\psi(-y)]dy \\ &= \frac{1}{2}[\phi(x+ct) - \phi(ct-x)] + \frac{1}{2c}\int_{ct-x}^{ct+x} \psi(y)dy. \end{aligned} \tag{2}$$

In this case the value of v depends on the value of ϕ at the pair of points $ct \pm x$ and on the values of ψ in the interval between these points. Not depends on the the values of ψ in the interval $(0, ct - x)$.

The complete solution is given by the pair of formulas (1) and (2).