Lecture 22

April 19, 2021

1 Green's functions

The main goal of this chapter is to use Green's identities to study the Dirichlet problem. The following representation formula we obtained in the last lecture cannot be used directly to solve the Dirichlet problem.

$$u(x_0) = \iint_{\partial\Omega} [-u(x)\frac{\partial}{\partial\overrightarrow{n}}(\frac{1}{|x-x_0|}) + \frac{1}{|x-x_0|}\frac{\partial u}{\partial\overrightarrow{n}}]\frac{dS_x}{4\pi}.$$

Because we do not to know the value $\frac{\partial u}{\partial\overrightarrow{n}}$ on the boundary.

Definition 1. The Green's function $G(x)$ for the operator $-\triangle$ and the domain $\Omega \in \mathbb{R}^3$ at the point $x_0 \in \Omega$ is a function defined for $x \in \Omega$ such that:

i) $G(x)$ possesses continuous second derivatives and $\triangle G = 0$ in Ω, except at the point $x = x_0$.

ii) $G(x) = 0$ for $x \in \partial\Omega$.

iii) The function $G(x) + \frac{1}{4\pi|x-x_0|}$ is finite at x_0 and has continuous second derivatives everywhere and is harmonic at x_0.

We usually denote it by $G(x, x_0)$.

Theorem 2. *If $G(x, x_0)$ is the Green's function, then the solution of the Dirichlet problem ($\triangle u = 0$) is given by the formula*

$$u(x_0) = \iint_{\partial\Omega} u(x)\frac{\partial G(x,x_0)}{\partial\overrightarrow{n}}dS.$$

Proof. Let us recall the Green's second identity which is

$$\iiint_\Omega (u\triangle v - v\triangle u)dx = \iint_{\partial\Omega}(u\frac{\partial v}{\partial\overrightarrow{n}} - v\frac{\partial u}{\partial\overrightarrow{n}})dS.$$

Let $v(x) = -\frac{1}{4\pi|x-x_0|} - G(x,x_0)$ and u is a harmonic function. Then by iii) and i) we have $\triangle v = 0$ in Ω. So from the Green's second identity, we have

$$0 = \iint_{\partial\Omega}(u\frac{\partial v}{\partial\overrightarrow{n}} - v\frac{\partial u}{\partial\overrightarrow{n}})dS = \iint_{\partial\Omega}(-u\frac{\partial}{\partial\overrightarrow{n}}(\frac{1}{4\pi|x-x_0|}) - u\frac{\partial}{\partial\overrightarrow{n}}G(x,x_0) - v\frac{\partial u}{\partial\overrightarrow{n}})dS.$$

From ii) $G(x) = 0$ on the boundary, we have

$$\begin{aligned}\iint_{\partial\Omega} u(x)\frac{\partial}{\partial\overrightarrow{n}}G(x,x_0) &= \iint_{\partial\Omega} -u\frac{\partial}{\partial\overrightarrow{n}}(\frac{1}{4\pi|x-x_0|}) + \frac{1}{4\pi|x-x_0|}\frac{\partial u}{\partial\overrightarrow{n}}dS \\ &= u(x_0).\end{aligned}$$

The last equality comes from the representation formula. □

Proposition 3. *If we can show that there exists a Green's function, then it is unique. And it is always symmetric:*

$$G(z,y) = G(y,z) \quad for \quad z \neq y.$$

Proof. The uniqueness is left as an exercise. Let $u(x) = G(x,y)$ and $v(x) = G(x,z)$ and substitute them into the Green's second identity in the domain $D_\epsilon := \Omega\backslash(B_\epsilon(z)\cup B_\epsilon(y))$. Here we choose ϵ small such that $B_\epsilon(z)\cap B_\epsilon(y) = \emptyset$. We have from i) and ii) that

$$\begin{aligned}0 = \iiint_{D_\epsilon}(G(x,y)\triangle G(x,z) - G(x,z)\triangle G(x,y))dx &= \\ \iint_{\partial\Omega} + \iint_{\partial B_\epsilon(z)} + \iint_{\partial B_\epsilon(y)} &= \\ \iint_{\partial B_\epsilon(z)}[-G(x,y)\frac{\partial}{\partial\overrightarrow{n}}G(x,z) + G(x,z)\frac{\partial}{\partial\overrightarrow{n}}G(x,y)]dS & \\ + \iint_{\partial B_\epsilon(y)}[-G(x,y)\frac{\partial}{\partial\overrightarrow{n}}G(x,z) + G(x,z)\frac{\partial}{\partial\overrightarrow{n}}G(x,y)]dS. & \qquad (1)\end{aligned}$$

Because $G(x,z) = -\frac{1}{4\pi|x-z|} + H(x,z)$ where $H(\cdot,z)$ is a harmonic function in $B_\epsilon(z)$. We have

$$\begin{aligned}\iint_{\partial B_\epsilon(z)}[G(x,y)\frac{\partial}{\partial\overrightarrow{n}}G(x,z) - G(x,z)\frac{\partial}{\partial\overrightarrow{n}}G(x,y)]dx &= \\ \iint_{\partial B_\epsilon(z)}[G(x,y)\frac{\partial}{\partial\overrightarrow{n}}(-\frac{1}{4\pi|x-z|} + H(x,z))dx & \\ - \iint_{\partial B_\epsilon(z)}[(-\frac{1}{4\pi|x-z|} + H(x,z))\frac{\partial}{\partial\overrightarrow{n}}G(x,y)]dx &= \\ \iint_{\partial B_\epsilon(z)}[G(x,y)\frac{\partial}{\partial\overrightarrow{n}}(-\frac{1}{4\pi|x-z|}) + \frac{1}{4\pi|x-z|}\frac{\partial}{\partial\overrightarrow{n}}G(x,y)dx. &\end{aligned}$$

So as before, we have

$$\lim_{\epsilon\to 0}\iint_{\partial B_\epsilon(z)}(G(x,y)\frac{\partial}{\partial\overrightarrow{n}}(-\frac{1}{4\pi|x-z|}) + \frac{1}{4\pi|x-z|}\frac{\partial G(x,y)}{\partial\overrightarrow{n}})dS_x = G(z,y).$$

So we get

$$\lim_{\epsilon\to 0}\iint_{\partial B_\epsilon(z)}[G(x,y)\frac{\partial}{\partial\overrightarrow{n}}G(x,z) - G(x,z)\frac{\partial}{\partial\overrightarrow{n}}G(x,y)]dS = G(z,y).$$

Similarly,

$$\lim_{\epsilon\to 0}\iint_{\partial B_\epsilon(y)}[G(x,y)\frac{\partial}{\partial\overrightarrow{n}}G(x,z)-G(x,z)\frac{\partial}{\partial\overrightarrow{n}}G(x,y)]dS. \quad = \quad -G(y,z).$$

From (1), we have for $z\neq y$ that

$$G(z,y) \quad = \quad G(y,z).$$

□

Theorem 4. *The solution of the problem*

$$\begin{aligned}\triangle u = f \quad & in \quad \Omega\\ u = h \quad & on \quad \partial\Omega\end{aligned}$$

is given by

$$u(x_0) \quad = \quad \iint_{\partial\Omega} h(x)\frac{\partial G(x,x_0)}{\partial\overrightarrow{n}}dS+\iiint_{\Omega} f(x)G(x,x_0)dx.$$

Proof. By the Green's second identity, we have for the domain $\Omega_\epsilon := \Omega\backslash B_\epsilon(x_0)$

$$\begin{aligned}\iiint_{\Omega_\epsilon}\triangle u(x)G(x,x_0)-u\triangle G(x,x_0)dx \quad &= \quad \iint_{\partial\Omega} G(x,x_0)\frac{\partial}{\partial\overrightarrow{n}}u(x)-u(x)\frac{\partial G(x,x_0)}{\partial\overrightarrow{n}}dS\\ &\quad +\iint_{\partial B_\epsilon(x_0)} G(x,x_0)\frac{\partial}{\partial\overrightarrow{n}}u(x)-u(x)\frac{\partial G(x,x_0)}{\partial\overrightarrow{n}}dS\\ &= \quad -\iint_{\partial\Omega} u(x)\frac{\partial G(x,x_0)}{\partial\overrightarrow{n}}dS\\ &\quad +\iint_{\partial B_\epsilon(x_0)} \frac{1}{4\pi|x-z|}\frac{\partial}{\partial\overrightarrow{n}}u(x)+u(x)\frac{\partial}{\partial\overrightarrow{n}}(-\frac{1}{4\pi|x-z|})dS\\ &\quad -\iint_{\partial B_\epsilon(x_0)} H(x,x_0)\frac{\partial}{\partial\overrightarrow{n}}u(x)+u(x)\frac{\partial}{\partial\overrightarrow{n}}H(x,x_0)dS.\end{aligned}$$

Let $\epsilon\to 0$, we have

$$-\iint_{\partial B_\epsilon(x_0)} H(x,x_0)\frac{\partial}{\partial\overrightarrow{n}}u(x)+u(x)\frac{\partial}{\partial\overrightarrow{n}}H(x,x_0)dS\to 0.$$

So let $\epsilon\to 0$, we get

$$\iiint_{\Omega} f(x)G(x,x_0)dx \quad = \quad -\iint_{\partial\Omega} h(x)\frac{\partial G(x,x_0)}{\partial\overrightarrow{n}}dS+u(x_0).$$

□