Lecture 21

1 Green's first identity

We will use the following divergence theorem extensively

$$\int_D div\overrightarrow{F}dx \quad = \quad \int_{\partial D} \overrightarrow{F}\cdot\overrightarrow{n}dS,$$

where $\overrightarrow{F}$ is any vector function, D is a bounded solid region, and $\overrightarrow{n}$ is the unit outer normal on bdy D.

In three dimensions, we have the notation

$$grad\ f = \nabla f = \quad the\ vector \quad (f_1, f_2, f_3)$$

$$div\overrightarrow{F} \quad = \nabla\cdot\overrightarrow{F} = \quad \frac{\partial F^1}{\partial x^1} + \frac{\partial F^2}{\partial x^2} + \frac{\partial F^3}{\partial x^3}$$

where $\overrightarrow{F} = (F^1, F^2, F^3)$. Also,

$$\triangle_3 u = div\ grad\ u \quad = \nabla\cdot\nabla u = \quad u_{11} + u_{22} + u_{33}$$

and

$$|\nabla u|^2 \quad = |grad\ u|^2 = \quad u_1^2 + u_2^2 + u_3^2$$

here we denote $u_i = \frac{\partial u}{\partial x^i}$ and $u_{ij} = \frac{\partial^2 u}{\partial x^i \partial x^j}$.

First, we claim the identity

$$\nabla\cdot(v\nabla u) \quad = \quad \nabla v\cdot\nabla u + v\triangle u. \tag{1}$$

This is because in the local coordinate,

$$\begin{aligned}
\nabla\cdot(v\nabla u) &= \nabla\cdot(v\frac{\partial u}{\partial x^1}, v\frac{\partial u}{\partial x^2}, v\frac{\partial u}{\partial x^3}) \\
&= \frac{\partial}{\partial x^1}(v\frac{\partial u}{\partial x^1}) + \frac{\partial}{\partial x^2}(v\frac{\partial u}{\partial x^2}) + \frac{\partial}{\partial x^3}(v\frac{\partial u}{\partial x^3}) \\
&= v_1u_1 + vu_{11} + v_2u_2 + vu_{22} + v_3u_3 + vu_{33}.
\end{aligned}$$

By definition, we also have

$$\begin{aligned}\nabla v\cdot\nabla u &= (v_1,v_2,v_3)\cdot(u_1,u_2,u_3)\\ &= v_1u_1+v_2u_2+v_3u_3\end{aligned}$$

and

$$v\triangle u = vu_{11}+vu_{22}+vu_{33}.$$

So we get the identity (1).

Then we integrate and use the divergence theorem to get

$$\int_D \nabla v\cdot\nabla u dx+\int_D v\triangle u dx = \int_D \nabla\cdot(v\nabla u)dx = \int_{\partial D} v\nabla u\cdot\overrightarrow{n}dS = \int_{\partial D} v\frac{\partial u}{\partial\overrightarrow{n}}dS.$$

Thus the Green's first identity is

$$\int_D \nabla v\cdot\nabla u dx+\int_D v\triangle u dx = \int_{\partial D} v\frac{\partial u}{\partial\overrightarrow{n}}dS.$$

For example, we take $v=1$ to get

$$\int_{\partial D}\frac{\partial u}{\partial\overrightarrow{n}}dS = \int_D \triangle u dx. \tag{2}$$

For the following Neumann problem in D:

$$\begin{aligned}\triangle u = f(x) \quad &in \quad D\\ \frac{\partial u}{\partial\overrightarrow{n}} = \varphi(x) \quad &on \quad \partial D.\end{aligned}$$

The identity (2) gives us a necessary condition which is

$$\int_{\partial D}\varphi dS = \int_D f dx$$

for the existence of the Neumann problem.

2 Mean value property

In last lecture, we derived mean value property in dimension two by using Poisson's formula. In this lecture, we are going to derive three-dimensional mean value property for harmonic functions. And this method works for all dimensions.

Let $a\in(0,r)$ and apply the identity (2), we have

$$0=\int_{B_a}\triangle u dx = \int_{\partial B_a}\frac{\partial u}{\partial\overrightarrow{n}}dS = \int_{\partial B_a}\frac{\partial u}{\partial r}dS.$$

here we used $\frac{\partial u}{\partial \vec{n}} = \frac{\partial u}{\partial r}$ in the polar coordinate on the boundary of $B_a(x_0)$. In polar coordinate, $u(x) = u(r\sin\theta\cos\phi, r\sin\theta\sin\phi, r\cos\theta)$ where $r = |x - x_0|$. Thus

$$\begin{aligned} 0 = \int_{\partial B_a} \frac{\partial u}{\partial r} dS &= \int_0^{2\pi} \int_0^{\pi} \frac{\partial u}{\partial r}(a\sin\theta\cos\phi, a\sin\theta\sin\phi, a\cos\theta) a^2 \sin\theta d\theta d\phi. \\ &= a^2 \frac{\partial}{\partial r} \int_0^{2\pi} \int_0^{\pi} u(r\sin\theta\cos\phi, r\sin\theta\sin\phi, r\cos\theta)\sin\theta d\theta d\phi \end{aligned}$$

Dividing by a^2,we have

$$\begin{aligned} 0 &= \int_0^{2\pi} \int_0^{\pi} \frac{\partial u}{\partial r}(a\sin\theta\cos\phi, a\sin\theta\sin\phi, a\cos\theta)\sin\theta d\theta d\phi \\ &= \frac{\partial}{\partial r}[\int_0^{2\pi} \int_0^{\pi} u(r\sin\theta\cos\phi, r\sin\theta\sin\phi, r\cos\theta)\sin\theta d\theta d\phi]|_{r=a}. \end{aligned}$$

Because this works for all $a \in (0, r)$, we obtain

$$\frac{\partial}{\partial r}[\frac{1}{4\pi} \int_0^{2\pi} \int_0^{\pi} u(r\sin\theta\cos\phi, r\sin\theta\sin\phi, r\cos\theta)\sin\theta d\theta d\phi] = 0.$$

Thus

$$\frac{1}{4\pi} \int_0^{2\pi} \int_0^{\pi} u(r\sin\theta\cos\phi, r\sin\theta\sin\phi, r\cos\theta)\sin\theta d\theta d\phi = \frac{1}{|\partial B_r(x_0)|} \int_{\partial B_r(x_0)} u dS$$

is independent of r where $|\partial B_r(x_0)|$ is the area of the sphere $|x - x_0| = r$. In particular, if we let $r \to 0$, we get

$$\frac{1}{|\partial B_r(x_0)|} \int_{\partial B_r(x_0)} u dS = u(x_0). \tag{3}$$

This is the mean value property.

If we mutiply r^2 on both side of (3) and integrate along $r \in (0, R)$, we have

$$\begin{aligned} \frac{1}{4\pi} \int_0^R \int_0^{2\pi} \int_0^{\pi} u(a\sin\theta\cos\phi, a\sin\theta\sin\phi, a\cos\theta) r^2 \sin\theta d\theta d\phi dr &= \int_0^R r^2 u(x_0) dr \\ &= \frac{R^3 u(x_0)}{3} \end{aligned}$$

Another form of mean value property is

$$\begin{aligned} u(x_0) &= \frac{1}{\frac{4}{3}\pi R^3} \int_0^R \int_0^{2\pi} \int_0^{\pi} u(a\sin\theta\cos\phi, a\sin\theta\sin\phi, a\cos\theta) r^2 \sin\theta d\theta d\phi dr \\ &= \frac{1}{|B_R(x_0)|} \int_{B_R(x_0)} u dx. \end{aligned}$$

Lecture 22

1 Uniqueness of Dirichlet's Problem

The solution of the following Dirichlet's problem is unique

$$\begin{aligned} \triangle u = f(x) \quad & in \quad \Omega \\ u = \varphi(x) \quad & on \quad \partial\Omega. \end{aligned} \tag{1}$$

Proof. If u_1 and u_2 are solutions to (1), then $u = u_1 - u_2$ is a solution to

$$\begin{aligned} \triangle u = 0 \quad & in \quad \Omega \\ u = 0 \quad & on \quad \partial\Omega. \end{aligned}$$

Recall the Green's first identity:

$$\int_{\partial\Omega} v \frac{\partial u}{\partial \overrightarrow{n}} dS = \int_{\Omega} \nabla v \cdot \nabla u dx + \int_{\Omega} v \triangle u dx.$$

Let $v = u$ and $\triangle u = 0$, we have

$$\int_{\partial\Omega} u \frac{\partial u}{\partial \overrightarrow{n}} dS = \int_{\Omega} |\nabla u|^2 dx.$$

Then $|\nabla u| \equiv 0$ in Ω, which infers $u \equiv C$ in $\overline{\Omega}$. By the boundary condition $u = 0$ on $\partial\Omega$, we get $u \equiv 0$.

So we have uniqueness of Dirichlet's problem for Laplace equation. □

Similarly, you can prove the uniqueness of Neumann's problem.

Exercise 1. The solution of the following Neumann's problem is *unique up to a constant*

$$\begin{aligned} \triangle u = f(x) \quad & in \quad \Omega \\ \frac{\partial u}{\partial \overrightarrow{n}} = \varphi(x) \quad & on \quad \partial\Omega. \end{aligned} \tag{2}$$

2 Dirichlet's principle

Among all the functions v in Ω that satisfy the Dirichlet boundary condition

$$v(x) = \varphi(x) \quad on \quad \partial\Omega. \tag{3}$$

The lowest energy $E[v] = \frac{1}{2}\int_{\Omega} |\nabla v|^2 dx$ occurs for the harmonic function satisfying (3) is the harmonic function $\triangle u = 0$.

We need to prove for any functions $w(x)$ and the unique harmonic function $u(x)$ in Ω with same boundary data. Then

$$E[w] \quad \geq \quad E[u].$$

Proof. Let $v = u - w$. Then

$$\begin{aligned}
E[w] &= \frac{1}{2}\int_{\Omega} |\nabla(u - v)|^2 dx \\
&= E[u] - \int_{\Omega} \nabla u \cdot \nabla v dx + E[v] \\
&= E[u] + \int_{\Omega} \triangle u v dx - \int_{\partial\Omega} \frac{\partial u}{\partial \overrightarrow{n}} v dS + E[v] \\
&= E[u] + E[v] \\
&\geq E[u].
\end{aligned}$$

The equality holds if and only if $w = u$. □

If we do not apriori know when $E[v]$ will attain its minimum point in the functional space with the same Dirichlet boundary condition. We can find it from the critical points of energy.

Let $u(\epsilon) = u\text{-}\epsilon v$ for any ϵ small and for any functions $v = 0$ on the boundary. So $u(\epsilon)$ and u have the same Dirichlet boundary condition. The critical point of $E[u(\epsilon)]$ must satisfy

$$\begin{aligned}
\frac{d}{d\epsilon} E[u(\epsilon)] &= \frac{d}{d\epsilon}\{\frac{1}{2}\int_{\Omega} |\nabla(u\text{ -}\epsilon v)|^2 dx\} \\
&= \frac{d}{d\epsilon}\{E[u] - \epsilon\int_{\Omega} \nabla u \cdot \nabla v dx + \epsilon^2 E[v]\} \\
&= -\int_{\Omega} \nabla u \cdot \nabla v dx + 2\epsilon E[v] \\
\epsilon = 0 \quad &= \int_{\Omega} v \triangle u dx - \int_{\partial\Omega} v \frac{\partial u}{\partial \overrightarrow{n}} dS \\
&= \int_{\Omega} v \triangle u dx.
\end{aligned}$$

So from the critical condition $\frac{d}{d\epsilon}|_{\epsilon=0} E[u(\epsilon)] = 0$, we get

$$\int_{\Omega} v \triangle u dx = 0 \quad for\ any \quad v \in C_0^{\infty}(\Omega).$$

This implies that

$$\triangle u = 0 \quad in \quad \Omega.$$

So a harmonic function with the same boundary BC is the critical point of $E[u(\epsilon)]$. From the uniqueness of the Dirichlet problem, the harmonic function is the only function with the same Dirichlet boundary that can minimize the energy.

3 Green's second identity

Recall Green's first identity

$$\int_{\partial\Omega} v\frac{\partial u}{\partial \overrightarrow{n}}dS = \int_{\Omega} \nabla v \cdot \nabla u dx + \int_{\Omega} v\triangle u dx. \tag{4}$$

If we switch the place of v and u, we get

$$\int_{\partial\Omega} u\frac{\partial v}{\partial \overrightarrow{n}}dS = \int_{\Omega} \nabla u \cdot \nabla v dx + \int_{\Omega} u\triangle v dx. \tag{5}$$

Subtracting (4) from (5), we have the Green's second identity

$$\int_{\partial\Omega}(-v\frac{\partial u}{\partial \overrightarrow{n}} + u\frac{\partial v}{\partial \overrightarrow{n}})dS = \int_{\Omega}(u\triangle v - v\triangle u)dx.$$

We can use the Green's second identity to prove the representation formula.

Theorem 2. *If $\triangle u = 0$ in $\Omega \subseteq \mathbb{R}^3$, then*

$$u(x_0) = \frac{1}{4\pi}\iint_{\partial\Omega}[-u(x)\frac{\partial}{\partial \overrightarrow{n}}(\frac{1}{|x-x_0|}) + \frac{1}{|x-x_0|}\frac{\partial u}{\partial \overrightarrow{n}}]dS_x.$$

If $\triangle u = 0$ in $\Omega \subseteq \mathbb{R}^2$, then

$$u(x_0) = \frac{1}{2\pi}\int_{\partial\Omega}[u(x)\frac{\partial}{\partial \overrightarrow{n}}(\log|x-x_0|) - \log|x-x_0|\frac{\partial u}{\partial \overrightarrow{n}}]dS_x.$$

Proof. Let $v(x) = -\frac{1}{4\pi|x-x_0|}$. Denote $B_\epsilon(x_0)$ be the ball with radius ϵ and center x_0. In $\Omega/B_\epsilon(x_0)$, we have

$$\begin{aligned}
\triangle v &= \frac{\partial^2}{\partial x^2}v + \frac{\partial^2}{\partial y^2}v + \frac{\partial^2}{\partial z^2}v \\
&= \frac{\partial}{\partial x}(\frac{x-x_0}{4\pi|x-x_0|^3}) + \frac{\partial}{\partial y}(\frac{y-y_0}{4\pi|x-x_0|^3}) + \frac{\partial}{\partial z}(\frac{z-z_0}{4\pi|x-x_0|^3}) \\
&= \frac{3}{4\pi|x-x_0|^3} - 3\frac{(x-x_0)^2+(y-y_0)^2+(z-z_0)^2}{4\pi|x-x_0|^5} \\
&= 0.
\end{aligned}$$

So from the Green's second identity, we have in $\Omega_\epsilon = \Omega \backslash B_\epsilon(x_0)$

$$0 = \iiint_{\Omega_\epsilon} (u\triangle v - v\triangle u)dx \quad = \quad -\iint_{\partial B_\epsilon(x_0)} + \iint_{\partial\Omega} \frac{1}{4\pi|x-x_0|}\frac{\partial u}{\partial \overrightarrow{n}} - \frac{u}{4\pi}\frac{\partial}{\partial \overrightarrow{n}}(\frac{1}{|x-x_0|})dS_x.$$

On $\partial B_\epsilon(x_0)$, we have

$$\lim_{\epsilon\to 0}\frac{1}{4\pi\epsilon}\iint_{\partial B_\epsilon(x_0)}\frac{\partial u}{\partial r}dS \quad = \quad 0,$$

and

$$\lim_{\epsilon\to 0}\frac{1}{4\pi\epsilon^2}\iint_{\partial B_\epsilon(x_0)} udS \quad = \quad u(x_0).$$

Combining the above formulas, we obtain

$$\iint_{\partial\Omega} \frac{1}{4\pi|x-x_0|}\frac{\partial u}{\partial \overrightarrow{n}} - \frac{u}{4\pi}\frac{\partial}{\partial \overrightarrow{n}}(\frac{1}{|x-x_0|})dS_x \quad = \quad u(x_0).$$

The two-dimensional case is similar. □