Lecture 18

1 Laplace equation in rectangles and cubes

We first solve laplace equations in these particular domains by separating the variables.

Example 1. Solve the following Laplace equation in the rectangle with the boundary condition:

$$
\begin{aligned}
u_{xx}(x,y) + u_{yy}(x,y) = 0 \quad in \quad D = \{0 \le x \le a, 0 \le y \le b\} \\
u(0,y) = u_x(a,y) = u_y(x,0) + u(x,0) = 0 \\
u(x,b) = g(x).
\end{aligned}
$$

Proof. Suppose $u(x,y) = X(x)Y(y)$, then from Laplace equation we have the equation for X and Y

$$X''(x) + \lambda X(x) = 0 \quad in \quad 0 \le x \le a \tag{1}$$

and

$$Y''(y) - \lambda Y(y) = 0 \quad in \quad 0 \le y \le b. \tag{2}$$

Here λ is a constant.

From the boundary condition, we need to solve a eigenvalue problem

$$
\begin{aligned}
X''(x) + \lambda X(x) = 0 \quad in \quad 0 \le x \le a \\
X(0) = X'(a) = 0.
\end{aligned}
$$

We know from the previous lecture that there are only positive eigenvalues for this problem. So we get $\lambda_n = \beta_n^2 = (n + \frac{1}{2})^2 \frac{\pi^2}{a^2}$ for $n = 0, 1, 2, \cdots$ and $X_n(x) = \sin \frac{(n+\frac{1}{2})\pi x}{a}$.

So the corresponding solutions for Y are

$$Y_n(y) \quad = \quad A \cos h\beta_n y + B \sinh \beta_n y.$$

The boundary condition $u_y(x,0) + u(x,0) = 0$ infers that $B\beta_n + A_n = 0$. Without of generality letting $B = -1$, we have $A = \beta_n$.

So we have

$$u(x,y) = \sum_{n=0}^{\infty} A_n(\beta_n \cosh \beta_n y - \sinh \beta_n y) \sin \beta_n x. \tag{3}$$

So

$$u(x,b) = = \sum_{n=0}^{\infty} A_n(\beta_n \cosh \beta_n b - \sinh \beta_n b) \sin \beta_n x.$$

Then from the last boundary condition $u(x,b) = g(x)$. $A_n(\beta_n \cosh \beta_n b - \sinh \beta_n b)$ is the coefficients of the Fourier sine series. The solution to the problem is given by (3) with the coefficients given by

$$A_n = (\beta_n \cosh \beta_n b - \sinh \beta_n b)^{-1} \frac{2}{a} \int_0^a g(x) \sin \beta_n x dx.$$

□

Example 2. Solve the following Laplace equation in the cubes with the boundary condition:

$$u_{xx}(x,y,z) + u_{yy}(x,y,z) + u_{zz}(x,y,z) = 0 \quad in \quad D = \{0 \le x \le \pi, 0 \le y \le \pi, 0 \le z \le \pi\}$$
$$u(0,y,z) = u(x,0,z) = u(x,\pi,z) = u(x,y,0) = u(x,y,\pi) = 0$$
$$u(\pi,y,z) = g(y,z).$$

Proof. Suppose $u(x,y,z) = X(x)Y(y)Z(z)$. From Laplace equation we have

$$\frac{X''(x)}{X(x)} + \frac{Y''(y)}{Y(y)} + \frac{Z''(z)}{Z(z)} = 0.$$

Assuming $\lambda = \frac{X''(x)}{X(x)}$, $\mu = \frac{Y''(y)}{Y(y)}$ and $\gamma = \frac{Z''(z)}{Z(z)}$, we have λ, μ, γ are constants which satisfy $\lambda + \mu + \gamma = 0$.

Combining the boundary conditions$u(x,0,z) = u(x,\pi,z) = 0$ or $u(x,y,0) = u(x,y,\pi) = 0$, we get $\mu = -m^2$ and $\gamma = -n^2$ where $m = 1, 2, \cdots$ and $n = 1, 2, \cdots$. Then the solutions to $\frac{X''(x)}{X(x)} = n^2 + m^2$ are

$$X(x) = A_{mn} \sinh(\sqrt{m^2+n^2}x) + B_{mn} \cosh(\sqrt{m^2+n^2}x).$$

From the boundary condition $u(0,y,z) = 0$, we have $B_{mn} = 0$.
So the solution is

$$u(x,y,z) = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} A_{mn} \sinh(\sqrt{m^2+n^2}x) \sin my \sin nz$$

with the coefficients

$$A_{mn} = \frac{4}{\pi^2 \sinh(\sqrt{m^2+n^2}\pi)} \int_0^{\pi} \int_0^{\pi} g(y,z) \sin my \sin nz dy dz.$$

Here we used the eigenfunctions $\{\sin my \cdot \sin nz\}_{m=1,2\cdots,n=1,2,\cdots}$ are mutually orthogonal on the square $\{0 < y < \pi, 0 < z < \pi\}$ and the integral

$$\int_0^\pi \int_0^\pi (\sin my \sin nz)^2 dydz = \frac{\pi^2}{4}.$$

□

2 Rotationally invariant solutions

Proposition 3. *The Laplace equation $\triangle u = 0$ is invariant under all rigid motions (translation and rotation).*

We are going to find rotationally invariant solutions to the Laplace equation in dimension two or three.

First we write the Laplace equation in *polar coordinates*. In dimension two, letting $x = r\cos\theta$ and $y = r\sin\theta$, we have

$$\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}.$$

The harmonic functions that themselves are rotationally invariant are

$$u(r) = c_1 \log r + c_2.$$

In dimension three, letting $x = r\sin\theta\cos\phi$, $y = r\sin\theta\sin\phi$ and $z = r\cos\theta$, we have

$$\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} = \frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r} + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\frac{\partial}{\partial\theta}) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\phi^2}.$$

The harmonic functions that themselves are rotationally invariant are

$$u(r) = -c_1 r^{-1} + c_2.$$

The functions $\log r$ and r^{-1} are fundamental solutions to The Laplace equation $\triangle u = 0$ in dimension two or dimension three.

Poisson's Formula

We are going to use separation of the variable to solve the Dirichlet problem for Laplace equation in a disk. The solution is represented by the Poisson's formula.

Let's consider the problem

$$\begin{aligned} u_{xx}(x,y) + u_{yy}(x,y) = 0 \quad for \quad x^2 + y^2 < a^2 \\ u = h(\theta) \quad for \quad x^2 + y^2 = a^2. \end{aligned}$$

In polar coordinates $x = r\cos\theta$, $y = r\sin\theta$, the equation is

$$u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = u_{xx} + u_{yy} = 0. \tag{1}$$

So separating variables in polar coordinates: $u(x,y) = u(r,\theta) = R(r)\Theta(\theta)$.
From equation (1), we have ODEs for R and Θ which are

$$\Theta''(\theta) + \lambda\Theta(\theta) = 0$$

and

$$r^2R''(r) + rR'(r) - \lambda R(r) = 0 \tag{2}$$

for a constant λ.
For Θ we require periodic BCs

$$\Theta(\theta + 2\pi) = \Theta(\theta) \quad for \quad -\infty < \theta < \infty.$$

So we only have positive eigenvalues and a zero eigenvalue.
When $\lambda = 0$, we have from the periodic BCs

$$\Theta(\theta) = A.$$

In this case, the solution to the equation (2) is

$$R(r) = c_1 \log r + c_2.$$

The solution u in this case is

$$u = R\Theta = A(c_1 \log r + c_2).$$

Because of the boundedness of the solution at the origin point where $r = 0$ and $\lim_{r \to 0} \log r = -\infty$, we need to let $c_1 = 0$. So

$$u = \frac{A_0}{2}$$

in this case for another constant A_0.

When $\lambda = \beta^2 > 0$, we have from the periodic BCs $\beta = n$ and

$$\Theta(\theta) = A \cos n\theta + B \sin n\theta$$

where $n = 1, 2, 3, \cdots$.

In this case, the solution to the equation (2) is

$$R(r) = Cr^n + Dr^{-n}.$$

The solution u in this case is

$$u = R\Theta = (Cr^n + Dr^{-n})(A \cos n\theta + B \sin n\theta).$$

Because of the boundedness of the solution at the origin point where $r = 0$ and $\lim_{r \to 0} r^{-n} = \infty$, we need to let $D = 0$. So

$$u = r^n(A_n \cos n\theta + B_n \sin n\theta)$$

in this case for another constant A_n and B_n.

By linear homogeneity, we have

$$u(r, \theta) = \frac{1}{2}A_0 + \sum_{n=1}^{\infty} r^n(A_n \cos n\theta + B_n \sin n\theta). \tag{3}$$

From the boundary condition $u(a, \theta) = h(\theta)$. The coefficients are determined by Fourier coefficients of the function $h(\theta)$. For $n = 1, 2, 3, \cdots$

$$A_n = \frac{1}{\pi a^n} \int_0^{2\pi} h(\phi) \cos n\phi d\phi,$$

$$A_0 = \frac{1}{\pi} \int_0^{2\pi} h(\phi) d\phi$$

and

$$B_n = \frac{1}{\pi a^n} \int_0^{2\pi} h(\phi) \sin n\phi d\phi.$$

If we plug A_n, B_n and A_0 into u, the series (3) can be summed explicitly. In fact

$$\begin{aligned} u(r,\theta) &= \int_0^{2\pi} h(\phi)\frac{d\phi}{2\pi} + \sum_{n=1}^{\infty} \frac{r^n}{\pi a^n} \int_0^{2\pi} h(\phi)\{\cos n\phi \cos n\theta + \sin n\phi \sin n\theta\} d\phi \\ &= \int_0^{2\pi} h(\phi)\{1 + 2\sum_{n=1}^{\infty} (\frac{r}{a})^n \cos n(\theta - \phi)\}\frac{d\phi}{2\pi}. \end{aligned}$$

Because

$$\frac{e^{in(\theta-\phi)} + e^{-in(\theta-\phi)}}{2} = \cos n(\theta - \phi),$$

we have

$$\begin{aligned} 1 + 2\sum_{n=1}^{\infty}(\frac{r}{a})^n \cos n(\theta-\phi) &= 1 + \sum_{n=1}^{\infty}(\frac{r}{a})^n e^{in(\theta-\phi)} + \sum_{n=1}^{\infty}(\frac{r}{a})^n e^{-in(\theta-\phi)} \\ &= 1 + \frac{\frac{r}{a}e^{i(\theta-\phi)}}{1 - \frac{r}{a}e^{i(\theta-\phi)}} + \frac{\frac{r}{a}e^{-i(\theta-\phi)}}{1 - \frac{r}{a}e^{-i(\theta-\phi)}} \\ &= \frac{are^{i(\theta-\phi)} - 2r^2}{a^2 + r^2 - are^{i(\theta-\phi)} - are^{-i(\theta-\phi)}} + 1 \\ &= \frac{a^2 - r^2}{a^2 + r^2 - 2ar\cos(\theta-\phi)}. \end{aligned}$$

So the solution can be written into the form

$$u(r,\theta) = \frac{1}{2\pi}\int_0^{2\pi} h(\phi)\frac{a^2 - r^2}{a^2 + r^2 - 2ar\cos(\theta-\phi)} d\phi.$$

This formula is known as Poisson's formula.

In the usual (x, y) coordinate, let the point $X = (x, y) = (r\cos\theta, r\sin\theta)$ and $X' = (x', y') = (a\cos\phi, a\sin\phi)$. We have by Law of Cosine

$$|X - X'|^2 = a^2 + r^2 - 2ar\cos(\theta - \phi).$$

The arc length element on the circle is $ds' = ad\phi$. So the Poisson's formula is also wrriten as

$$u(X) = \frac{a^2 - |X|^2}{2\pi a}\int_{|X'|=a} \frac{u(X')}{|X - X'|^2} ds'.$$