Lecture 14

March 7, 21

1 Orthogonality

If $f(x)$ and $g(x)$ are two real-valued continuous functions defined on an interval $a \leq x \leq b$, we define their *inner product* to be the integral of their product:

$$(f, g) \equiv \int_a^b f(x)g(x)dx.$$

We'll call $f(x)$ and $g(x)$ *orthogonal* if $(f, g) = 0$. No function is orthogonal to itself except $f(x) \equiv 0$. The key observation in each case discussed in Sec. 5.1 is that **every eigenfunction is orthogonal to every other eigenfunction**.

We are studing the operator $A = -\frac{d^2}{dx^2}$ with some boundary conditions (either Dirichlet or Neumann or $\cdots$). Let $X_1(x)$ and $X_2(x)$ be two different eigenfunctions. Thus

$$\begin{aligned} -X_1'' = \frac{-d^2X_1}{dx^2} &= \lambda_1 X_1 \\ -X_2'' = \frac{-d^2X_2}{dx^2} &= \lambda_2 X_2, \end{aligned}$$

where both functions satisfy the boundary conditions. Let's assume that $\lambda_1 \neq \lambda_2$. We integrate to get

$$\int_a^b (-X_1''X_2 + X_1X_2'')dx = (-X_1'X_2 + X_1X_2')|_a^b. \tag{1}$$

This is sometimes called *Green's second identity.*

Case1: Dirichlet. This means that both functions vanish at both ends: $X_1(a) = X_1(b) = X_2(a) = X_2(b) = 0$. So the right side of (1) is zero.

Case2: Neumman. The first derivatves vanish at both ends $X_1'(a) = X_1'(b) = X_2'(a) = X_2'(b) = 0$. It is once again zero.

Case3: Periodic. $X_j(a) = X_j(b)$, $X_j'(a) = X_j'(b)$ for both $j = 1, 2$. Again you get zero.

Case4: Robin. $X_j'(a) = cX_j(a)$, $X_j'(b) = cX_j(b)$ for both $j = 1, 2$.

$$\begin{aligned} -X_1'(b)X_2(b) + X_1(b)X_2'(b) &= -cX_1(b)X_2(b) + cX_1(b)X_2(b) = 0 \\ -X_1'(a)X_2(a) + X_1(a)X_2'(a) &= -cX_1(a)X_2(a) + cX_1(a)X_2(a) = 0. \end{aligned}$$

On the other hand,

$$\int_a^b (-X_1''X_2 + X_1X_2'')dx = \int_a^b (\lambda_1 X_1X_2 - \lambda_2 X_1X_2)dx. \tag{2}$$

Combining Equation (2) and (1), we get in all the above four cases

$$(\lambda_1 - \lambda_2)\int_a^b X_1X_2dx = 0.$$

Therefore, X_1 and X_2 orthogonal if $\lambda_1 \neq \lambda_2$.

The right side of (1) is not always zero. For example, $X(a) = X(b)$, $X'(a) = 2X'(b)$.

The right side of (1) is $X_1'(b)X_2(b) - X_1(b)X_2'(b)$ which is not zero.

For any pair of boundary conditions

$$\begin{aligned} \alpha_1X(a) + \beta_1X(b) + \gamma_1X'(a) + \delta_1X'(b) &= 0 \\ \alpha_2X(a) + \beta_2X(b) + \gamma_2X'(a) + \delta_2X'(b) &= 0 \end{aligned} \tag{3}$$

involving eight real constants. Such a set of boundary conditions is called symmetric if

$$f'(x)g(x) - f(x)g'(x)|_{x=a}^{x=b} = 0$$

for any pair of functions $f(x)$ and $g(x)$ both of which satisfy the pair of boundary conditions (3).

Theorem 1. *If you have symmetric boundary conditions, then any two eigenfunctions that correspond to distinct eigenvalues are orthogonal. Therefore, if any functioin is expanded in a series of these eigenfunctions, the coefficients are determined.*

Proof. The first part is obvious from the above argument. If $X_n(x)$ denotes the orthogonal eigenfunctions with eigenvalue λ_n and suppose that ϕ has the following convergent series

$$\phi(x) = \sum_n A_nX_n(x).$$

Then

$$(\phi, X_m) = (\sum_n A_nX_n, X_m) = \sum_n A_n(X_n, X_m) = A_m(X_m, X_m).$$

So we have the formula for the coefficients A_m

$$A_m = \frac{(\phi, X_m)}{(X_m, X_m)}.$$

$\square$

Remark 2. We have so far avoided all questions of convergence.

Remark 3. If there are two eigenfunctions, say $X_1(x)$ and $X_2(x)$, but their eigenvalues are the same $\lambda_1 = \lambda_2$, then they do not have to be orthogonal. For example in the case of periodic boundary condtions $\sin(\frac{n\pi x}{l})$ and $\cos(\frac{n\pi x}{l}) + \sin(\frac{n\pi x}{l})$ are eigenvalues to the operator A with the same eigenvalue $\lambda = n$[illegible] They are not orthogonal. But they can be make so by the Gram-Schmidt orthogonalization procedure. The two eigenfunctions $\sin(\frac{n\pi x}{l})$ and $\cos(\frac{n\pi x}{l})$ are orthogonal on $(-l, l)$.

If $f(x)$ and $g(x)$ are two complex-valued functions, we define the inner product on (a, b) as

$$(f, g) = \int_a^b f(x)\overline{g(x)}dx.$$

The bar denotes the complex conjugate. The two functions are called *orthogonal* if $(f, g) = 0$.

Now suppose that you have the boundary conditions (3) with eight real constants. They are called symmetric (or hermitian) if

$$f'(x)\overline{g(x)} - f(x)\overline{g'(x)}|_a^b = 0$$

for all f, g satisfying the BCs.

Note that the set of functions are symmetric in the real sense implies the symmetric in the complex sense.

Theorem 4. *Under the same conditions as Theorem 1, all the eigenvalues are real numbers. Furthermore, all the eigenfunctions can be chosen to be real valued.*

Proof. If $-X'' = \lambda X$ then $-\overline{X}'' = \overline{\lambda X}$ plus BCs. Now use Green's second identity with the funtions X and $\overline{X}$. Thus

$$(\lambda - \overline{\lambda}) \int_a^b X\overline{X}dx = \int_a^b (-X''\overline{X} + X\overline{X}'')dx = (-X'\overline{X} + X\overline{X}')|_a^b = 0.$$

But $X\overline{X} = |X|^2 \geq 0$ and $X(x)$ is not allowed to be zero function. So the integral can not vanish. Therefore, $\lambda = \overline{\lambda} = 0$, which means exactly that λ is real.

Then suppose the eigenfunction $X(x)$ is complex, we can write it as $X(x) = Y(x) + iZ(x)$, where $Y(x)$ and Z(x) are real. Then $-Y'' - iZ'' = \lambda Y + i\lambda Z$.

So we get that $-Y'' = \lambda Y$ and $-Z'' = \lambda Z$. The boundary conditioins still hold for both Y and Z. It is easy to see that $\overline{X(x)}$ is also a eigenfunctions. So the linear combination of $X(x)$ and $\overline{X(x)}$ can be replaced by the linear combination of $Y(x)$ and $Z(x)$. Thus we can replace the set of complex eigenfunctions $X(x)$ and $\overline{X(x)}$ by the set of the corresponding real eigenfunctions Y and Z. □

Theorem 5. *Assume the same conditions as in Theorem 1. If*

$$f(x)f'(x)|_a^b \quad \leq \quad 0$$

for all (real-valued) functions $f(x)$ satisfying the BCs, then there is no negative eigenvalue.

Proof. Suppose there is a negative eigenvalue $\gamma < 0$ and eigenfunction such that

$$-X''(x) \quad = \quad \gamma X(x).$$

Then we have that

$$0 > \int_a^b \gamma X^2(x)dx = -\int_a^b X''(x)X(x)dx \quad = \quad -X'(b)X(b)|_a^b + \int_a^b (X')^2 dx \geq 0.$$

This is a contradiction. Thus there is no negative eigenvalue. □

From the previous computation, we have for one dimensional eigenvalue problem

$$X'' + \lambda X \quad = \quad 0$$

in (a, b) with any symmetric BC.

Theorem 6. *There are an infinite number of eigenvalues. They form a sequence $\lambda_n \to +\infty$. Moreover, we may list the eigenvalues as*

$$\lambda_1 \leq \lambda_2 \leq \lambda_3 \cdots \quad \to \quad +\infty$$

with the corresponding eigenfunctions

$$X_1, X_2, X_3 \quad \cdots \quad ,$$

which are pairwise orthogonal.

So for any function $f(x)$ on (a, b), its Fourier coefficients are defined as

$$A_n = \frac{(f, X_n)}{(X_n, X_n)} \quad = \quad \frac{\int_a^b f(x)\overline{X_n(x)}dx}{\int_a^b |X_n(x)^2|dx}.$$

Its Fourier series is the series

$$\sum_n A_n X_n(x).$$

2 Three notions of convergence.

Definition 7. We say that an infinite series $\sum_{n=1}^{\infty} f_n(x)$ converges to $f(x)$ pointwise in (a, b) if for each $a < x < b$

$$|f(x) - \sum_{n=1}^{N} f_n(x)| \to 0 \quad as \quad N \to \infty.$$

Definition 8. The series converges uniformly to $f(x)$ in $[a, b]$ if

$$\max_{a \le x \le b} |f(x) - \sum_{n=1}^{N} f_n(x)| \to 0 \quad as \quad N \to \infty.$$

Definition 9. We say the series converges in the mean-square (or L^2) sense to $f(x)$ in (a, b) if

$$\int_a^b |f(x) - \sum_{n=1}^{N} f_n(x)|^2 dx \to 0 \quad as \quad N \to \infty.$$

Example 10. Let $f_n(x) = (1 - x)x^{n-1}$ on the interval $(0, 1)$. Then the partial sums are

$$\sum_{n=1}^{N} f_n(x) = 1 - x^N \to 1 \quad as \quad N \to \infty$$

because $x < 1$. So $\sum_{n=1}^{\infty} f_n(x)$ converges pointwise to the function $f \equiv 1$. But the convergence is not uniform because

$$\max_{0 \le x \le 1} |1 - \sum_{n=1}^{N} f_n(x)| = 1 \quad as \quad N \to \infty.$$

However, it does converge in L^2 sense

$$\int_0^1 |1 - \sum_{n=1}^{N} f_n(x)|^2 = \int_0^1 |x^N|^2 dx = \frac{1}{2N + 1} \to 1 \quad as \quad N \to \infty.$$

Exercise 11. Let $f_n(x) = \frac{n}{1+n^2x^2} - \frac{n-1}{1+(n-1)^2x^2}$ in the interval $0 < x < 1$. Prove that

a) $\sum_{n=1}^{\infty} f_n(x)$ converges pointwise to $f \equiv 0$.

b) $\sum_{n=1}^{\infty} f_n(x)$ does not converges in mean-square sense to $f \equiv 0$.

c) $\sum_{n=1}^{\infty} f_n(x)$ does not uniformly converges to $f \equiv 0$.

Theorem 12. ***Least-Square Approximation.*** *Let $\{X_n\}$ be any orthogonal set of functions. Let $\int_a^b |f|^2 dx < \infty$. Let N be a fixed positive integer. Among all possible choices of N constants $c_1, c_2, \cdots, c_N$. The choices of N constants $c_1, c_2, \cdots c_N$, the choice that minimizes*

$$\int_a^b |f - \sum_{n=1}^{N} c_n X_n|^2 dx$$

is $c_n = \frac{(f, X_n)}{(X_n, X_n)}$ for $n = 1, 2, \cdots, N$.

Proof. Denote

$$E_N(c_1, \cdots, c_N) = \int_a^b |f - \sum_{n=1}^{N} c_n X_n|^2 dx \geq 0. \tag{4}$$

So we have

$$\begin{aligned} E_N(c_1, \cdots, c_N) &= \int_a^b |f(x)|^2 dx - 2 \sum_{n \leq N} c_n \int_a^b f(x) X_n(x) dx + \sum_{n \leq N} \sum_{m \leq N} c_n c_m \int_a^b X_n(x) X_m(x) dx \\ &= (f, f) - 2 \sum_{n \leq N} c_n (f, X_n) + \sum_{n \leq N} c_n^2 (X_n, X_n) \\ &= \sum_{n \leq N} \|X_n\|^2 [c_n - \frac{(f, X_n)}{(X_n, X_n)}] - \sum_{n \leq N} \frac{(f, X_n)^2}{(X_n, X_n)} + (f, f). \end{aligned} \tag{5}$$

So the minimal point of E_N is $c_n = \frac{(f, X_n)}{(X_n, X_n)}$ for $n = 1, 2, \cdots, N$. □

Denote $A_n = \frac{(f, X_n)}{(X_n, X_n)}$, we have the inequality from Equations (4) and (5),

$$(f, f) \geq \sum_{n \leq N} \frac{(f, X_n)^2}{(X_n, X_n)} \geq \sum_{n \leq N} A_n (X_n, X_n).$$

This is known as Bessel's inequality. It is valid as long as the integral of $|f|^2$ is finite.