Lecture 11

February 24, 2021

We can also use separation of variables to solve the diffusion equation.

$$u_t = ku_{xx} \quad 0 < x < l \quad 0 < t < \infty \tag{1}$$
$$u(0,t) = u(l,t) = 0 \tag{2}$$
$$u(x,0) = \phi(x). \tag{3}$$

First, we solve equation (1) and separate the variables $u(x,t) = T(t)X(x)$ as before. This time we get

$$\frac{T'}{kT} = \frac{X''}{X} = -\lambda = constant.$$

We solve the equation for $T(t)$ to get

$$T(t) = Ae^{-\lambda kt},$$

where A is a constant.
Then we solve the equation for $X(x)$.

$$-X'' = \lambda X \quad in \quad 0 < x < l$$

with $X(0) = X(l) = 0$.
As before, the solution is

$$X(x) = \sin\sqrt{\lambda_n}x$$

where $\lambda_n = \frac{n^2\pi^2}{l^2}$.
So the solution to (1) and (2) is

$$u_n(x,t) = A_n e^{-\frac{n^2\pi^2 kt}{l^2}} \sin\frac{n\pi x}{l}.$$

The sum of u_n is also a solution to (1) and (2) due to linearity

$$u(x,t) = \sum_{n=1}^{\infty} A_n e^{-(n\pi/l)^2 kt} \sin\frac{n\pi}{l}x.$$

If

$$\phi(x) \quad = \quad \sum_{n=1}^{\infty} A_n \sin\frac{n\pi x}{l},$$

$u(x,t)$ is a solution to (1), (2) and (3).

The numbers $\lambda_n = (\frac{n\pi}{l})^2$ are called *eigenvalues* and the functions $X_n(x) = \sin(\frac{n\pi x}{l})$ are called *eigenfunctions* to the ODE

$$-X'' = \lambda_n X \quad in \quad 0 < x < l$$

with $X(0) = X(l) = 0$.

1 The Neumann condition

The same method works for both the Neumann and Robin boundary conditions. In the former case, (2) is replaced by $u_x(0,t) = u_x(l,t) = 0$. Then the eigenfunctions are the solutions $X(x)$ of

$$-X'' = \lambda X, \qquad X'(0) = X'(l) = 0,$$

other than the trivial solution $X(x) \equiv 0$.

When $\lambda = \beta^2 > 0$, as before $X(x) = c_1 \cos\beta x + c_2 \sin\beta x$. So that

$$X'(x) \quad = \quad -c_1\beta\sin\beta x + c_2\beta\cos\beta x.$$

We get $c_2 = 0$ from $X'(0) = 0$. From $X'(l) = 0$, we have

$$c_1\beta\cos\beta l \quad = \quad 0.$$

If $c_1 \neq 0$, we need to let $\beta = \frac{n\pi}{l}$.

When $\lambda = 0$, we have $X(x) = c_1 + c_2 x$. So

$$X'(x) \quad = \quad c_2.$$

From $X'(0) = X'(l) = 0$ we have $c_2 = 0$. In this case, $\lambda = 0$ is an eigenvalue of

$$-X'' = \lambda_n X \quad in \quad 0 < x < l$$

with $X'(0) = X'(l) = 0$.

For $\lambda < 0$, there is only a trival solution.

Therefore we will learn from Sec. 5.3 the list of all the eigenvalues is

$$\lambda_n = (\frac{n\pi}{l})^2 \quad for \quad n = 0, 1, 2, 3, \cdots$$

So the diffusion equation with the Neumann boundary condition

$$\begin{aligned} u_t = ku_{xx} &\quad 0 < x < l \quad 0 < t < \infty \\ u_x(0,t) = u_x(l,t) = 0 & \\ u(x,0) = \phi(x). & \end{aligned}$$

The solution is

$$u(x,t) = \frac{1}{2}A_0 + \sum_{n=1}^{\infty} A_n e^{-(n\pi/l)^2 kt} \cos\frac{n\pi x}{l},$$

provided that

$$\phi(x) = \frac{1}{2}A_0 + \sum_{n=1}^{\infty} A_n \cos\frac{n\pi x}{l}.$$

All the coefficients $A_0, A_1, A_2, \cdots$ are just constants.

Consider now the wave equation with the Neumann BCs. The eigenvalue $\lambda = 0$ then leads to $X(x) = costant$ and to the differential equation $T''(t) = \lambda c^2 T(t) = 0$, which has the solution $T(t) = A + Bt$. Therefore, the wave equation with the Neumann BCs is

$$u(x,t) = \frac{1}{2}A_0 + \frac{1}{2}B_0 t + \sum_{n=1}^{\infty}(A_n \cos\frac{n\pi ct}{l} + B_n \sin\frac{n\pi ct}{l}) \cos\frac{n\pi x}{l}. \quad (4)$$

Then the initial data must satisfy

$$\phi(x) = \frac{1}{2}A_0 + \sum_{n=1}^{\infty} A_n \cos\frac{n\pi x}{l}$$

and

$$\psi(x) = \frac{1}{2}B_0 + \sum_{n=1}^{\infty} \frac{n\pi c}{l} B_n \cos\frac{n\pi x}{l}. \quad (5)$$

Equation (5) comes from first differentiating (4) with respect to t and then setting $t = 0$.

Exercise 1. Find all the eigenvalues and eigenfunctions for the eigenvalue problem

$$\begin{cases} -X''(x) = \lambda X(x) & 0 \le x \le l, \\ X(0) = X'(l) = 0. \end{cases}$$

Exercise 2. Use separation of variables to solve the Schrödinger equation

$$\begin{cases} u_t = iu_{xx} & 0 \le x \le l, \\ u_x(0,t) = u_x(l,t) = 0 & \\ u(x,0) = \frac{1}{2}A_0 + \sum_{n=1}^{\infty} A_n \cos\frac{n\pi x}{l}. & \end{cases}$$