Lecture 10

February 21, 2021

We are going to use separation of variables to solve some second order PDEs. Consider the homogeneous Dirichlet condtions for the wave equation:

$$u_{tt} - c^2 u_{xx} = 0 \quad for \quad 0 < x < l \tag{1}$$
$$u(0,t) = 0 = u(l,t) \tag{2}$$

with some initial conditions

$$u(x,0) = \phi(x) \tag{3}$$
$$u_t(x,0) = \psi(x). \tag{4}$$

A *separated solution* is a solution of (1) and (2) of the form

$$u(x,t) = X(x)T(t).$$

The equation (1) gives us

$$X(x)T''(t) = c^2 X''(x)T(t).$$

Or we write into the form

$$-\frac{T''(t)}{c^2 T(t)} = -\frac{X''(x)}{X(x)} = \lambda.$$

Note that

$$\lambda = -\frac{X''(x)}{X(x)} \tag{5}$$

is independent with t, and

$$\lambda = -\frac{T''(t)}{c^2 T(t)} \tag{6}$$

does not depend on x. So λ is a constant.

Thus it reduced to solve two second order ODEs.

Theorem. *Given a second order linear equation with constant coefficients*

$$y''(x) + by'(x) + cy(x) = 0, \tag{7}$$

Solve its characteristic equation $r^2+br+c=0$. The general solution depends on the type of roots obtained.

Case 1. When $b^2-4c>0$, there are two distinct real roots r_1, r_2. The general solution to (7) will be

$$y(x) = c_1e^{r_1x}+c_2e^{r_2x}.$$

Case 2. When $b^2-4c<0$, there are two complex conjugate roots $r_1=\lambda+\mu i$, $r_2=\lambda-\mu i$. The general solution to (7) will be

$$y(x) = c_1e^{\lambda x}\cos\mu x+c_2e^{\lambda x}\sin\mu x.$$

Case 3. When $b^2-4c=0$, there is one repeated real root r. Then

$$y(x) = c_1e^{rx}+c_2xe^{rx}.$$

From the ODE results, if $\lambda=0$, we have from Case 3 that

$$X(x) = c_1+c_2x.$$

But $X(0)=X(l)=0$ infers only $c_1=c_2=0$. It only gives us a trivial solution.

If $\lambda=-\beta^2<0$, we have from Case 1 that

$$X(x) = c_1e^{\beta x}+c_2e^{-\beta x}.$$

Similarly, $X(0)=X(l)=0$ will gives us $c_1=c_2=0$ which is also a trivial solution.

If we assume $\lambda=\beta^2>0$ we have from Case 2

$$\begin{aligned} X(x) &= C\cos\beta x+D\sin\beta x \\ T(t) &= A\cos\beta ct+B\sin\beta ct. \end{aligned}$$

The second step is to impose the boundary condition (2) on the separated solution.

$$\begin{aligned} u(0,t)=X(0)T(t) &= 0 \\ u(l,t)=X(l)T(t) &= 0. \end{aligned}$$

So we have

$$\begin{aligned} X(0) &= C=0 \\ X(l) &= D\sin\beta l=0. \end{aligned}$$

If $C=D=0$, this is a trivial solution which we are not interested. So we must have $\beta=\frac{n\pi}{l}$.

That is $\lambda_n=(\frac{n\pi}{l})^2$, $X_n(x)=\sin\frac{n\pi x}{l}$.

So we have many solutions

$$u_n(x,t) = (A_n \cos \frac{n\pi ct}{l} + B_n \sin \frac{n\pi ct}{l}) \sin \frac{n\pi x}{l}$$

where A_n and B_n are arbitrary constants (which can be determined by ϕ and ψ).

The sum of solutions is again a solution to (1) and (2)

$$u(x,t) = \sum_n (A_n \cos \frac{n\pi ct}{l} + B_n \sin \frac{n\pi ct}{l}) \sin \frac{n\pi x}{l}. \tag{8}$$

If formula (8) also solves (3) and (4), it has to provide that

$$\phi(x) = \sum_n A_n \sin \frac{n\pi x}{l}$$

and

$$\psi(x) = \sum_n \frac{n\pi c}{l} B_n \sin \frac{n\pi x}{l}.$$

1 Review second order linear ODE.

We are going to review how to get the ODE Theorem 1. Let us consider the constant coefficient second linear ODE

$$y''(t) + py'(t) + qy(t) = 0 \tag{9}$$

There is an existence and uniqueness theorem due to Picard–Lindelof.

Theorem 2. *Consider the initial value problem*

$$\begin{aligned} y''(t) + p(t)y'(t) + q(t)y = g(t) \qquad & a \leq t \leq b \\ y(t_0) = y_0 \qquad & \\ y'(t_0) = \widetilde{y}_0. \qquad & \end{aligned} \tag{10}$$

If the functions p, q and g are continuous on the interval $[a,b]$ which containing the point t_0. Then there exists a unique solution $y(t)$ to the initial value problem.

This theorem guarantee us there are solutions to the equation (9). We begin to find the solution with some simple examples.

Example. $y(t) = C \sin t$ is a solution to the equation $y''(t) + y(t) = 0$. From the solution $y(t) = C \sin t$ we know that if we impose initial or boundary conditions on different points there may have no uniqueness. For instance, for any C the solutions $y(t) = C \sin t$ will satisfy the following condition 1: $y(0) = 0$ and $y'(\frac{\pi}{2}) = 0$; or condition 2: $y(0) = 0$ and $y(\pi) = 0$.

Example. And $y(t) = Ce^t$ is a solution to the equation $y''(t) - y(t) = 0$.

So we may guess solutions to the equation (9) will be in the form $y(t) = e^{rt}$. From the equation (9) we have

$$r^2 e^{rt} + pre^{rt} + qe^{rt} = 0.$$

Thus we need solve the *characteristic equation* $r^2 + pr + q = 0$ of Equation (9).

Case 1. $p^2 - 4q > 0$, we have two distinct real roots $r_1 = \frac{-p+\sqrt{p^2-4q}}{2}$, $r_2 = \frac{-p-\sqrt{p^2-4q}}{2}$. Then we get two solutions

$$X_1(t) = e^{r_1 t}$$

and

$$X_2(t) = e^{r_2 t}.$$

Claim. In this case, the general solution to (9) must be in the form

$$y(t) = c_1 X_1(t) + c_2 X_2(t) = c_1 e^{r_1 t} + c_2 e^{r_2 t}.$$

Proof. In order to prove this claim, we need to introduce the Wronskian which is defined

$$W(t) = X_1(t)X_2'(t) - X_2(t)X_1'(t).$$

If $W(t) \neq 0$, we say that $X_1$and X_2 are *linearly independent* solutions to the equation (9). In particularly $W(t_0) \neq 0$. One advantage is that suppose the solution is in the form of $y(t) = c_1X_1(t)+c_2X_2(t)$, we can uniquely determine the constant c_1 and c_2 from the initial value conditions $y(t_0) = y_0$ and $y'(t_0) = \widetilde{y}_0$. So we have only two parametric freedoms.

One important property of the Wronskian is that

$$\begin{aligned} W'(t) &= [X_1(t)X_2'(t) - X_2(t)X_1'(t)]' \\ &= X_1'(t)X_2'(t) + X_1(t)X_2''(t) - X_2'(t)X_1'(t) - X_2(t)X_1''(t) \\ &= X_1(t)X_2''(t) - X_2(t)X_1''(t) \\ from\ (9) &= X_1(t)[-pX_1'(t) - qX_1(t)] - X_2(t)[-pX_2'(t) - qX_2(t)] \\ &= -p[X_1(t)X_1'(t) - X_2(t)X_1'(t)] \\ &= -pW(t). \end{aligned} \tag{11}$$

From this we can prove uniquess of Equation (9) with given initial conditions $y(t_0) = y_0$ and $y'(t_0) = \widetilde{y}_0$. Suppose that $y(t)$ and $\widetilde{y}(t)$ are the solutions satisfy the initial value problem (10) then

$$W(t_0) = y(t_0)\widetilde{y}'(t_0) - y'(t_0)\widetilde{y}(t_0) = y_0\widetilde{y}_0 - \widetilde{y}_0 y_0 = 0.$$

So from the equation (11) for W, we have $W(t) \equiv 0$. Which means $y(t) = c\widetilde{y}(t)$ then from the same initial condition $y(t_0) = y_0 = \widetilde{y}(t_0)$ we have $y(t) \equiv \widetilde{y}(t)$. This gives another proof of the uniqueness of Theorem 2.

Combining the uniqueness and the linear independent solutions X_1 and X_2, we have proved the general solution to Equation (9) must be in the form of

$$y(t) = c_1 X_1(t) + c_2 X_2(t).$$

$\square$

Case 3. $p^2 - 4q = 0$, there is one repeated real root $r = -\frac{p}{2}$. So one independent solution $X_1(t) = e^{rt}$. We find another independent solution $X_2(t)$ from Equation (11). From Equation (11), we have

$$W(t) = e^{\int -p dt} = e^{\int 2r dt}.$$

On the other hand by the definition

$$W(t) = X_1(t)X_2'(t) - X_2(t)X_1'(t).$$

Let $X_1(t) = e^{rt}$, we have

$$e^{rt}X_2'(t) - re^{rt}X_2(t) = e^{2rt}.$$

Solving this first order linear inhomogeneous ODE, we get the other independent solution

$$X_2(t) = te^{rt}.$$

By a similar reason as before, the general solution to Equation (9) is

$$y(t) = c_1 e^{rt} + c_2 te^{rt}.$$

Case 2. $p^2 - 4q < 0$, there are two complex conjugate roots $r_1 = \lambda + \mu i$, $r_2 = \lambda - \mu i$. So the fundamental solutions may be in the form

$$\widetilde{X}_1(t) = e^{\lambda t + \mu t i} = e^{\lambda t}(\cos \mu t + i \sin \mu t)$$

and

$$\widetilde{X}_2(t) = e^{\lambda t - \mu t i} = e^{\lambda t}(\cos \mu t - i \sin \mu t).$$

So in the real form the fundamental solution may be

$$X_1(t) = e^{\lambda t} \cos \mu t$$

and

$$X_2(t) = e^{\lambda t} \sin \mu t.$$

We can check the Wronskian $W = \mu e^{2\lambda t}$ is nonzero. So X_1 and X_2 are independent solutions.

By a similar reason as before, the general solution to Equation (9) in this case is

$$y(t) = c_1 e^{\lambda t} \cos \mu t + c_2 e^{\lambda t} \sin \mu t.$$