Lecture 8

February 3, 2021

Let us review the formula for the initial value problem for wave equation and heat equation on the whole line.

For the wave equation:

$$\begin{cases} u_{tt}(x,t) - c^2 u_{xx}(x,t) = 0 \quad -\infty < x < \infty, \\ u(x,0) = \phi(x) \\ u_t(x,0) = \psi(x). \end{cases}$$

By d'Alembert formula,

$$u(x,t) = \frac{1}{2}[\phi(x+ct) + \phi(x-ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(s)ds. \tag{1}$$

For the diffusion equation:

$$\begin{cases} u_t(x,t) - u_{xx}(x,t) = 0 \quad -\infty < x < \infty \\ u(x,0) = \phi(x). \end{cases}$$

We have the formula

$$u(x,t) = \frac{1}{\sqrt{4\pi t}}\int_{-\infty}^{\infty} e^{-(x-y)^2/4t}\phi(y)dy. \tag{2}$$

Exercise. If $\phi(x)$ is a bounded piecewise-continuous function for $-\infty < x < \infty$, then prove that

$$\lim_{t\to 0^+} u(x,t) \quad = \lim_{t\to 0^+} \tfrac{1}{\sqrt{4\pi t}} \textstyle\int_{-\infty}^{\infty} e^{-(x-y)^2/4t}\phi(y)dy = \quad \frac{1}{2}[\phi(x+) + \phi(x-)]$$

for all $x \in \mathbb{R}$, where $\phi(x+)$ and $\phi(x-)$ stand for the right hand side and left hand side limits of ϕ at x.

We are going to compare the properties of these two equations.

i) The PDE exhibits finite propagation speed if the following holds:

If the initial data consists of functions with compact support, then for every $t > 0$ the solution $u(\cdot, t)$ has compact support.

Otherwise, we say that it has infinite speed.

One can make this quantitative: the speed of propagation is $\leq c$ provided that the following holds:

If the initial data consists of functions with support contained in a ball $B(a, R)$, then for every $t > 0$ the solution $u(\cdot, t)$ has support contained in $B(a, R + ct)$.

The waves have *finite* speed. This can be easily seen from the d'Alembert formula (1).

But the diffusions have *infinite* speed of propagation. It was seen in the example of the heat kernel, which is strictly positive for all $x \in \mathbb{R}$ for $t > 0$. The initial data is compact support in 0.

ii) Wave equation transported singularities along characteristics for $t > 0$. We saw from the "hammer blow" and "box wave" that singularities are preserved and are transported along the characteristics.

Sigularities for $t > 0$ lost immediately to the diffusions. For piecewise-continuous initial data ϕ or some weaker conditions on ϕ, the solution to the diffusions equation will immediately (for any $t > 0$) become infinitely differentiable immediately.

iii) *Well-posed* for wave equation for any t. It can be seen from d'Alembert formula (1) and the law of conservation of energy

$$E(t) = \quad \tfrac{1}{2}\int_{-\infty}^{+\infty} u_t^2 dx + \tfrac{1}{2}\int_{-\infty}^{+\infty} u_x^2 dx \quad = E(0).$$

Diffusion equations are well-posed for $t > 0$ (at least for bounded solutions) but ill-posed for $t < 0$. The first part can be proved by Formula (2) and the maximum principle in the previous lecture. The second part can be seen as following:

$$u_n(x,t) \quad = \quad \frac{1}{n} e^{-n^2 t} \sin nx$$

satisfies the diffusion equation for all x, t. And $u_n(x, 0) = \frac{1}{n}\sin nx \to 0$ uniformly as $n \to \infty$. But consider $t < 0$, say $t = -1$. Then $u_n(x, -1) = \frac{1}{n} e^{n^2} \sin nx \to \pm\infty$ uniformly as $n \to \infty$ except at a few x. This violates the stability in the uniform sense at least.

iv) Maximum principle holds for diffusions but does not hold for waves. This can be seen from the "hammer blow".

v) Energy is constant for waves but decays to zero (if ϕ is integrable) for diffusions. Notice $\phi(x) = e^{-x}$, the solution to the diffusion equation is

$$\begin{aligned} u(x,t) &= \frac{1}{\sqrt{4\pi t}} \int_{-\infty}^{+\infty} e^{-(x-y)^2/4t} e^{-y} dy \\ &= \frac{1}{\sqrt{4\pi t}} \int_{-\infty}^{+\infty} e^{-\frac{(y+2t-x)^2}{4t} + t - x} dy \\ &= \frac{e^{t-x}}{\sqrt{\pi}} \int_{-\infty}^{+\infty} e^{-p^2} dp \\ &= e^{t-x}. \end{aligned}$$

It did not decay, but rather "traveled" from right to left. This was due to ϕ being non-integrable.

vi) The fact that information is *transported* by the solutions of the wave equation is seen from the fact that the initial data is propagated along the characteristics. So the information will travel along the characteristics as well.

In the case of the heat equation, the information is *gradually lost*, which can be seen from the graph of a typical solution (think of the heat kernel). The heat from the higher temperatures gets dissipated and after a while it is not clear what the original temperatures were.