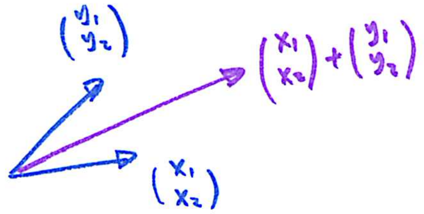
MATH 2040A Linear Algebra II 2015-16

Week 5: Inner Products / Norms (textbook § 6.1)

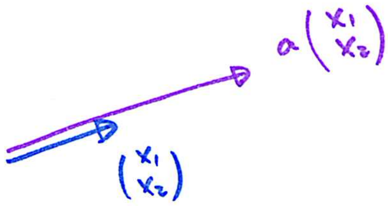
Review of Euclidean Geometry

So far, we have focused on the algebra of vectors in $\mathbb{R}^n$, i.e.

n=2: $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} x_1+y_1 \\ x_2+y_2 \end{pmatrix}$; $a\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} ax_1 \\ ax_2 \end{pmatrix}$



"vector addition"

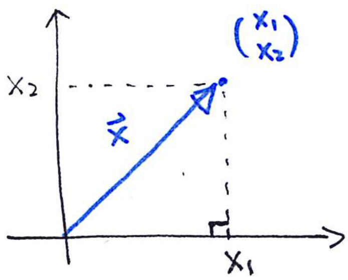


"scalar multiplication"

$\boxed{(\mathbb{R}^n, +, \cdot) \text{ forms a vector space over } \mathbb{R}.}$

But there is more... we know how to measure distances and angles in Euclidean Geometry:

n=2:



distance of the point $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ from origin

$\|$

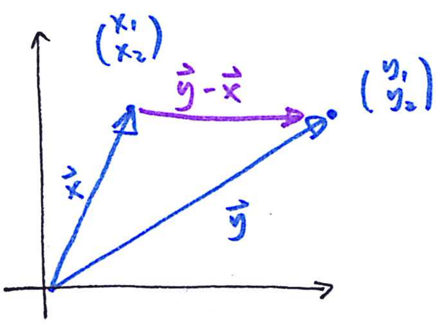
length of $\vec{x}$

$\|$

$$\boxed{\|\vec{x}\| := \sqrt{x_1^2 + x_2^2}}$$

Pythagoras' Theorem!

More generally, we can measure the distance between two points:

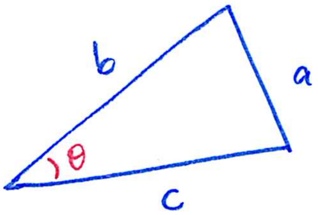


distance between $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ and $\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$

$$= \|\vec{y} - \vec{x}\|$$

$$= \sqrt{(y_1 - x_1)^2 + (y_2 - x_2)^2}.$$

Once we can measure distance, we can measure angles as well:



$$\boxed{a^2 = b^2 + c^2 - 2bc\cos\theta}$$

"cosine law"

We also learned an important operation called the dot product

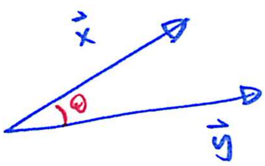
<u>n=2</u> :

$$\boxed{\vec{x}\cdot\vec{y} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \cdot \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} := x_1 y_1 + x_2 y_2}$$

It gives a formula to calculate length of a vector $\vec{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$:

$$\boxed{\|\vec{x}\| := \sqrt{x_1^2 + x_2^2} = \sqrt{\vec{x}\cdot\vec{x}}}. \quad \text{———} \quad (\#)$$

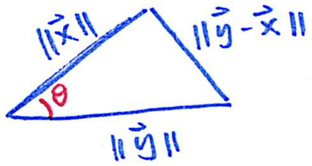
It also gives an easy formula to compute angle between two vectors (non-zero) $\vec{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ and $\vec{y} = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$:



$$\boxed{\vec{x}\cdot\vec{y} = \|\vec{x}\|\,\|\vec{y}\|\cos\theta} \qquad (*)$$

<u>Fact</u>: $(*)$ is equivalent to the usual "cosine law".

<u>Proof</u>:



$$\|\vec{y}-\vec{x}\|^2 = \|\vec{x}\|^2 + \|\vec{y}\|^2 - 2\boxed{\|\vec{x}\|\,\|\vec{y}\|\cos\theta}$$

"cosine law"

and

$$\|\vec{y}-\vec{x}\|^2 = (\vec{y}-\vec{x})\cdot(\vec{y}-\vec{x}) \qquad (\#)$$

$$= \vec{y}\cdot\vec{y} - \vec{x}\cdot\vec{y} - \vec{y}\cdot\vec{x} + \vec{x}\cdot\vec{x}$$

$$= \|\vec{x}\|^2 + \|\vec{y}\|^2 - 2\boxed{\vec{x}\cdot\vec{y}}$$

□

Important Properties of dot product:

(i) (Bilinearity): $(a_1\vec{x}+a_2\vec{y})\cdot\vec{z} = a_1(\vec{x}\cdot\vec{z})+a_2(\vec{y}\cdot\vec{z})$

$$\vec{x}\cdot(a_1\vec{y}+a_2\vec{z}) = a_1(\vec{x}\cdot\vec{y})+a_2(\vec{x}\cdot\vec{z})$$

(ii) (Symmetry): $\vec{x}\cdot\vec{y}=\vec{y}\cdot\vec{x}$

(iii) (Positivity): $\vec{x}\cdot\vec{x}\geq 0$ and "=" holds iff $\vec{x}=\vec{0}$.

Proof: Exercise!

Note: (iii) allows us to define the length / norm of $\vec{x}$ as

$$\|\vec{x}\| := \sqrt{\underbrace{\vec{x}\cdot\vec{x}}_{\geq 0}} \quad \text{and} \quad \|\vec{x}\|=0 \iff \vec{x}=\vec{0}.$$

Inner Product Space

With the help of Euclidean Geometry, we know that a "dot product" is all we need to measure length and angle, which allows us to study the geometry of vectors.

★ From now on, the field $\mathbb{F}$ will always be $\mathbb{R}$ or $\mathbb{C}$. ★

Def$^{\underline{n}}$: Let V be a vector space over $\mathbb{F}=\mathbb{R}$ or $\mathbb{C}$.

An inner product on V is a "function":

$$\langle\cdot,\cdot\rangle : V\times V \longrightarrow \mathbb{F}$$

s.t. (i) (Linearity in 1^{st} slot)

$$\langle a_1\vec{x}+a_2\vec{y},\vec{z}\rangle = a_1\langle\vec{x},\vec{z}\rangle + a_2\langle\vec{y},\vec{z}\rangle$$

(ii) (conjugate symmetry)

complex conjugate. $\rightarrow$ $\overline{\langle\vec{x},\vec{y}\rangle} = \langle\vec{y},\vec{x}\rangle$

(iii) (Positivity)

$$\underbrace{\langle\vec{x},\vec{x}\rangle}_{\in\mathbb{R}} \geq 0 \quad \text{and "=" holds iff } \vec{x}=\vec{0}.$$

<u>Names</u>: If $\mathbb{F} = \mathbb{R}$, $(V, \langle , \rangle)$ is a <u>real</u> inner product space.

If $\mathbb{F} = \mathbb{C}$, $(V, \langle , \rangle)$ is a <u>complex</u> inner product space.

<u>Examples</u>:

(1) $V = \mathbb{R}^n$; $\boxed{\langle \vec{x}, \vec{y} \rangle := \sum_{i=1}^{n} x_i y_i}$ where $\vec{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$, $\vec{y} = \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix}$.

"dot product" = standard inner product on $\mathbb{R}^n$.

(2) $V = \mathbb{C}^n$, $\mathbb{F} = \mathbb{C}$; $\boxed{\langle \vec{x}, \vec{y} \rangle := \sum_{i=1}^{n} x_i \overline{y_i}}$ standard inner product on $\mathbb{C}^n$

<u>E.g.</u> $\left\langle \begin{pmatrix} 1 \\ i \end{pmatrix}, \begin{pmatrix} i \\ 1 \end{pmatrix} \right\rangle = 1 \cdot \overline{i} + i \cdot \overline{1} = -i + i = 0$.

(3) $V = \mathbb{R}^2$, $\mathbb{F} = \mathbb{R}$, then

$$\left\langle \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \right\rangle := 2 x_1 y_1 + 3 x_2 y_2$$

defines an inner product on $\mathbb{R}^2$, which is different from the standard inner product.

(4) $V = M_{n\times n}(\mathbb{R})$, $\mathbb{F} = \mathbb{R}$; $\boxed{\langle A, B \rangle := \operatorname{tr}(B^t A)}$ Frobenius inner product

defines an inner product.

<u>E.g.</u>: $\left\langle \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \right\rangle = \operatorname{tr}\left(\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}^t \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \right) = \operatorname{tr}\begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix} = 2$.

(5) $V = M_{n\times n}(\mathbb{C})$, $\mathbb{F} = \mathbb{C}$; $\boxed{\langle A, B \rangle := \operatorname{tr}(B^* A)}$ is an inner product

where B^* is the conjugate transpose/adjoint of B defined by

$\boxed{B^* := \overline{B}^t}$ e.g: $\begin{pmatrix} i & 1+2i \\ 2 & 3+4i \end{pmatrix}^* = \begin{pmatrix} -i & 2 \\ 1-2i & 3-4i \end{pmatrix}$.

<u>Note</u>: If B has real entries, then $B^* = B^t$.

(thus, B^* is just the complex version of transpose.)

An infinite dimensional example

(1) $V = C([0,1])$ space of continuous function on $[0,1]$.
$(\mathbb{F} = \mathbb{R})$ (real-valued)

(Exercise: Prove this)

$$\boxed{\langle f, g \rangle := \int_0^1 f(t)\, g(t)\, dt}$$

defines an inner product (L^2- inner product)

(2) $V = C([0,2\pi])$, $(\mathbb{F} = \mathbb{C})$, space of continuous complex-valued function on $[0,2\pi]$.

$$\boxed{\langle f, g \rangle := \underbrace{\frac{1}{2\pi}}_{\text{normalization constant}} \int_0^{2\pi} f(t)\, \overline{g(t)}\, dt}$$

defines an inner product.

integral of $\mathbb{C}$-valued function:

$$\int f := \int f_1 + i \int f_2 \quad \text{where } f = f_1 + i f_2$$

real-valued functions

E.g.: $\langle \sin t, \cos t \rangle = \frac{1}{2\pi} \int_0^{2\pi} \sin t \cos t \, dt$

$$= \frac{1}{2\pi} \int_0^{2\pi} \frac{1}{2} \sin 2t \, dt$$

$$= \frac{1}{2\pi} \left[-\frac{1}{4} \cos 2t \right] \Bigg|_{t=0}^{2\pi} = 0 .$$

FACT: The inner product defined above is very useful in Engineering through Fourier analysis and Physics through Quantum Mechanics.

Prop: Let $(V, \langle , \rangle)$ be an inner product space. Then,

(a) (conjugate linear in 2^{nd} slot)

$$\langle \vec{x}, a_1\vec{y} + a_2\vec{z} \rangle = \overline{a_1}\langle \vec{x}, \vec{y} \rangle + \overline{a_2}\langle \vec{x}, \vec{z} \rangle$$

(b) (non-degenerate) If $\langle \vec{x}, \vec{y} \rangle = \langle \vec{x}, \vec{z} \rangle$ for $\underline{\underline{\text{all}}}$ $\vec{x} \in V$,

then $\vec{y} = \vec{z}$.

Proof: (a)

$$\begin{aligned}
\langle \vec{x}, a_1\vec{y} + a_2\vec{z} \rangle &= \overline{\langle a_1\vec{y} + a_2\vec{z}, \vec{x} \rangle} && \text{(conjugate symmetry)} \\
&= \overline{a_1\langle \vec{y}, \vec{x} \rangle + a_2\langle \vec{z}, \vec{x} \rangle} && \text{(linear in } 1^{st} \text{ slot)} \\
&= \overline{a_1}\,\overline{\langle \vec{y}, \vec{x} \rangle} + \overline{a_2}\,\overline{\langle \vec{z}, \vec{x} \rangle} \\
&= \overline{a_1}\langle \vec{x}, \vec{y} \rangle + \overline{a_2}\langle \vec{x}, \vec{z} \rangle && \text{(conjugate symmetry)}
\end{aligned}$$

(b) $\langle \vec{x}, \vec{y} \rangle = \langle \vec{x}, \vec{z} \rangle$ for $\underline{\underline{\text{all}}}$ $\vec{x} \in V$

$\Rightarrow$ $\langle \vec{x}, \vec{y} - \vec{z} \rangle = 0$ for $\underline{\underline{\text{all}}}$ $\vec{x} \in V$

Take $\vec{x} = \vec{y} - \vec{z}$ in particular, we have

$$\langle \vec{y} - \vec{z}, \vec{y} - \vec{z} \rangle = 0$$

By positivity, this implies $\vec{y} - \vec{z} = \vec{0}$, hence $\vec{y} = \vec{z}$.

$\square$