Week 2: Change of coordinates/basis, Eigenvalues and Eigenvectors

(textbook §2.5 and §5.1)

Change of basis

Recall: Once we picked ordered bases β and γ for V and W respectively, we have a 1-1 correspondence:

$$\left(\begin{array}{c}\text{matrix representation}\\ \text{of } T\\ \text{w.r.t. } \beta \text{ and } \gamma\end{array}\right): \quad \begin{array}{ccc} \mathcal{L}(V,W) & \longleftrightarrow & M_{m\times n}(\mathbb{F}) \\ \Downarrow & & \Downarrow \\ T & \longleftrightarrow & [T]_\beta^\gamma \end{array}$$

To actually compute the matrix $[T]_\beta^\gamma$:

let $\beta = \{v_1, v_2, \ldots, v_n\} \subset V$ be the ordered bases.
$\gamma = \{w_1, w_2, \ldots, w_m\} \subset W$

there exist unique $a_{ij} \in \mathbb{F}$, $i=1,\ldots,m$, $j=1,\ldots,n$ s.t. (Q: why?)

$$(*)\quad \begin{cases} T(v_1) = a_{11}w_1 + a_{21}w_2 + \cdots + a_{m1}w_m \\ T(v_2) = a_{12}w_1 + a_{22}w_2 + \cdots + a_{m2}w_m \\ \quad\vdots \qquad\qquad\qquad \vdots \\ T(v_n) = a_{1n}w_1 + a_{2n}w_2 + \cdots + a_{mn}w_m \end{cases}$$

then,

$$[T]_\beta^\gamma = \underbrace{\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}}_{m\times n \text{ matrix}} = \begin{pmatrix} | & | & & | \\ [T(v_1)]_\gamma & [T(v_2)]_\gamma & \cdots & [T(v_n)]_\gamma \\ | & | & & | \end{pmatrix}$$

[Caution: This is somewhat the "transpose" of the ordering in (*)]

Notation: When $V=W$ and $\beta=\gamma$, we simply write $[T]_\beta$.

Example: Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the projection onto x-axis :

$$T(a_1, a_2) = (a_1, 0).$$

Let $\beta = \{e_1, e_2\} = \left\{ \begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix} \right\}$ be the standard basis.

$$\left\{ \begin{array}{l} T(e_1) = \begin{pmatrix}1\\0\end{pmatrix} = 1\cdot e_1 + 0\cdot e_2 \\ T(e_2) = \begin{pmatrix}0\\0\end{pmatrix} = 0\cdot e_1 + 0\cdot e_2 \end{array} \right. \Rightarrow [T]_\beta = \begin{array}{cc} T(e_1) & T(e_2) \\ \multicolumn{2}{c}{\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}} \end{array}.$$

Now, let $\beta' = \left\{ \begin{pmatrix}1\\1\end{pmatrix}, \begin{pmatrix}-1\\1\end{pmatrix} \right\}$ be another ordered basis.

$$\left\{ \begin{array}{l} T\begin{pmatrix}1\\1\end{pmatrix} = \begin{pmatrix}1\\0\end{pmatrix} = \frac{1}{2}\begin{pmatrix}1\\1\end{pmatrix} - \frac{1}{2}\begin{pmatrix}-1\\1\end{pmatrix} \\ T\begin{pmatrix}-1\\1\end{pmatrix} = \begin{pmatrix}-1\\0\end{pmatrix} = -\frac{1}{2}\begin{pmatrix}1\\1\end{pmatrix} + \frac{1}{2}\begin{pmatrix}-1\\1\end{pmatrix} \end{array} \right. \Rightarrow [T]_{\beta'} = \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{pmatrix}.$$

- Since both matrices $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ and $\begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{pmatrix}$ represent the <u>Same</u> linear transformation T, they should have some properties in common and be "similar" in some sense.

Def$^{\underline{n}}$: Let $A, B \in M_{n\times n}(\mathbb{F})$. We say that B is similar to A if $\exists$ invertible $Q \in M_{n\times n}(\mathbb{F})$ s.t. $\boxed{B = Q^{-1}AQ}$.

Ex: "similar" is an equivalence relation.

Prop: If A and B are similar matrices, then $\boxed{\operatorname{tr} A = \operatorname{tr} B}$ and $\boxed{\det A = \det B}$.

Proof: Exercise! Recall that $\operatorname{tr}(AB) = \operatorname{tr}(BA)$ and

$$\underbrace{\det(AB) = \det(BA)}_{\det(A)\cdot\det(B)}.$$

The reason why we define similar matrices as above is that they should represent the $\underline{\underline{\text{same}}}$ linear transformation (just w.r.t. different basis). It follows from the following theorem.

<u>Thm A</u>: Let $T: V \to V$ be a linear transformation ($\dim V < +\infty$). Suppose β and β' are two ordered basis for V. Then,

$$\boxed{[T]_{\beta'} = Q^{-1}[T]_{\beta}\, Q}$$

where $Q := [I_V]_{\beta'}^{\beta}$ is the change of coordinate matrix (from β' to β). Hence, $[T]_\beta$ and $[T]_{\beta'}$ are similar matrices.

<u>Example</u>: As before, we take $V = \mathbb{R}^2$ and,

$$\beta = \left\{ \begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix} \right\} \quad \text{and} \quad \beta' = \left\{ \begin{pmatrix}1\\1\end{pmatrix}, \begin{pmatrix}-1\\1\end{pmatrix} \right\}.$$

to compute $Q := [I_V]_{\beta'}^{\beta}$, note that

$$\left.\begin{aligned} I_V\begin{pmatrix}1\\1\end{pmatrix} &= \begin{pmatrix}1\\1\end{pmatrix} = 1\cdot\begin{pmatrix}1\\0\end{pmatrix} + 1\cdot\begin{pmatrix}0\\1\end{pmatrix} \\ I_V\begin{pmatrix}-1\\1\end{pmatrix} &= \begin{pmatrix}-1\\1\end{pmatrix} = -1\cdot\begin{pmatrix}1\\0\end{pmatrix} + 1\cdot\begin{pmatrix}0\\1\end{pmatrix} \end{aligned}\right\} \Rightarrow Q = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$$

One can check by direct calculation that

$$[T]_{\beta'} = \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}^{-1} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} = Q^{-1}[T]_\beta Q$$

How to find Q^{-1}? $[Q \mid I] \to [I \mid Q^{-1}]$

$$\left(\begin{array}{cc|cc} 1 & -1 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{array}\right) \to \left(\begin{array}{cc|cc} 1 & -1 & 1 & 0 \\ 0 & 2 & -1 & 1 \end{array}\right) \to \left(\begin{array}{cc|cc} 1 & -1 & 1 & 0 \\ 0 & 1 & -\frac{1}{2} & \frac{1}{2} \end{array}\right) \to$$

$$\to \left(\begin{array}{cc|cc} 1 & 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & 1 & -\frac{1}{2} & \frac{1}{2} \end{array}\right) \quad \text{so } \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{pmatrix}.$$

Properties of $Q = [I_V]_{\beta'}^{\beta}$:

(a) Q is invertible

(b) $\boxed{[v]_\beta = Q\,[v]_{\beta'}}$ $\quad \forall\ v \in V$

(i.e. the coordinate vectors of v relative to β and β' are related by the change of coordinate matrix Q)

By def$^{\underline{n}}$, if $T : \underset{\beta}{V} \longrightarrow \underset{\gamma}{W}$, then $[T(v)]_\gamma = [T]_\beta^\gamma [v]_\beta$

Moreover, if

$$\underset{\alpha}{V} \xrightarrow{T} \underset{\beta}{W} \xrightarrow{U} \underset{\gamma}{Z} \quad \text{then} \quad [U \circ T]_\alpha^\gamma = [U]_\beta^\gamma [T]_\alpha^\beta$$

Proof: (a)

$$\underset{\beta}{V} \xrightarrow{I_V} \underset{\beta'}{V} \xrightarrow{I_V} \underset{\beta}{V}$$

$I_V = I_V \circ I_V \implies I = [I_V]_\beta^\beta = [I_V]_{\beta'}^{\beta} [I_V]_\beta^{\beta'}$

Therefore, $Q = [I_V]_{\beta'}^{\beta}$ is invertible with $Q^{-1} = [I_V]_\beta^{\beta'}$.

(b) By def$^{\underline{n}}$ with $I_V : \underset{\beta'}{V} \longrightarrow \underset{\beta}{V}$. Exercise!

$\square$

Proof of Thm A :

Consider the compositions :

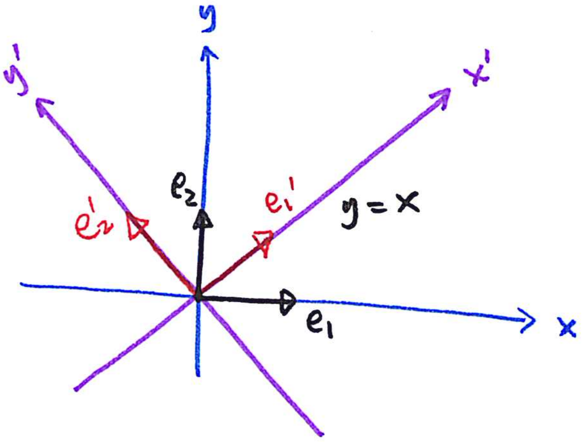
$$\underset{\beta'}{V} \xrightarrow{I_V} \underset{\beta}{V} \xrightarrow{T} \underset{\beta}{V} \xrightarrow{I_V} \underset{\beta'}{V}$$

$$\implies [T]_{\beta'} = [I_V]_\beta^{\beta'} [T]_\beta [I_V]_{\beta'}^{\beta} = Q^{-1} [T]_\beta Q$$

$\square$

Thm A can often help us compute $[T]_\beta$ by a "clever" choice of a new basis β' so that $[T]_{\beta'}$ is easier to compute (this is basically just doing a change of variables!)

Example: Let $T : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ be the linear transformation which is the projection onto the line $\{y = x\} \subseteq \mathbb{R}^2$. Find $[T]_\beta$ where $\beta = \{e_1, e_2\}$ is the standard basis for $\mathbb{R}^2$.



Consider the basis

$$\beta' = \{e_1', e_2'\} = \left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \end{pmatrix} \right\}.$$

in the basis β'.

$$[T]_{\beta'} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

Change of coordinate matrix:

$$Q = [I_{\mathbb{R}^2}]_{\beta'}^{\beta} = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$$

Apply Thm A:

$$[T]_\beta = Q\,[T]_{\beta'}\,Q^{-1}$$

$$= \underbrace{\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}}_{\begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}}\underbrace{\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}^{-1}}_{\begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{pmatrix}} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}.$$

$\square$

Diagrams

$$\begin{tikzcd}
 & v \in V \arrow[dl, "\cong_{\beta'}"'] \arrow[dr, "\cong_{\beta}"] & \\
\mathbb{F}^n \arrow[rr, "Q"] & & \mathbb{F}^n \\
{[v]_{\beta'}} & & {[v]_\beta = Q\,[v]_{\beta'}}
\end{tikzcd}$$

&

$$\begin{tikzcd}
\mathbb{F}^n \arrow[r, "{[T]_{\beta'}}"] \arrow[dd, dashed, bend right, "Q"'] & \mathbb{F}^n \\
V \arrow[u, "\cong_{\beta'}"] \arrow[r, "T"] \arrow[d, "\cong_{\beta}"'] & V \arrow[u, "\cong_{\beta'}"'] \arrow[d, "\cong_{\beta}"] \\
\mathbb{F}^n \arrow[r, "{[T]_\beta}"] & \mathbb{F}^n \arrow[uu, dashed, bend right, "Q^{-1}"']
\end{tikzcd}$$

A (slightly) more complicated example:

Let $T = L_A : \mathbb{R}^3 \to \mathbb{R}^3$ be the linear map given by the left multiplication by the matrix

$$A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 2 & 2 \\ 0 & 0 & 3 \end{pmatrix},$$

find $[T]_{\beta'}$ where $\beta' = \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} \right\}$.

Solution: let $\beta = \{e_1, e_2, e_3\}$ be the standard basis of $\mathbb{R}^3$.

change of coordinate matrix : $Q = \left(\boxed{\begin{matrix} 1 \\ 0 \\ 0 \end{matrix}} \; \boxed{\begin{matrix} 1 \\ 1 \\ 0 \end{matrix}} \; \boxed{\begin{matrix} 1 \\ 2 \\ 1 \end{matrix}} \right)$

(Just putting the vectors in β' as column vectors. Why is it ?)

It is easy to see that $[L_A]_\beta = A$. Therefore, by Thm A.

$$[T]_{\beta'} = Q^{-1} [T]_\beta Q$$

$$= \underbrace{\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix}^{-1}} \underbrace{\begin{pmatrix} 1 & 1 & 0 \\ 0 & 2 & 2 \\ 0 & 0 & 3 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix}}$$

$$= \begin{pmatrix} 1 & -1 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 \\ 0 & 2 & 6 \\ 0 & 0 & 3 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}$$ ♡ diagonal matrix!

A Natural Question: Given A, a square matrix, can we find a "good" basis β' s.t. $[L_A]_{\beta'}$ is diagonal ?

Ans: look for eigenvalues / eigenvectors!

Eigenvalues and Eigenvectors

<u>Def$^{\underline{n}}$</u> : (i) A linear operator $T: V \to V$, where $\dim V < +\infty$, is diagonalizable if $\exists$ ordered basis β for V s.t.

$$[T]_\beta \text{ is } \underline{\text{diagonal}}.$$

(ii) A square matrix A is diagonalizable if the linear operator $L_A : \mathbb{F}^n \to \mathbb{F}^n$ is diagonalizable.

<u>Note</u>: By Thm. A, A is diagonalizable $\iff$ $\exists$ invertible matrix Q s.t. $Q^{-1}AQ$ is diagonal.

Let $\beta = \{v_1, v_2, \dots, v_n\}$ be an ordered basis s.t. $[T]_\beta$ is diagonal, i.e.

$$[T]_\beta = \begin{pmatrix} \lambda_1 & & & 0 \\ & \lambda_2 & & \\ & & \ddots & \\ 0 & & & \lambda_n \end{pmatrix} \iff \boxed{T v_i = \lambda_i v_i}$$

The vector v_i just gets rescaled by the scalar λ_i after transformed by the linear map T.

<u>Def$^{\underline{n}}$</u> : Let $T: V \to V$ be linear. If $v \in V$ is a <u>nonzero</u> vector s.t.

$$\boxed{Tv = \lambda v}$$

then v is an eigenvector of T with eigenvalue λ.

Similarly, we say that $v \in \mathbb{F}^n$ is an eigenvector of a matrix $A \in M_{n\times n}(\mathbb{F})$ with eigenvalue λ if $v \neq \vec{0}$ and

$$\boxed{Av = \lambda v}$$

<u>Note</u>: Since $T(0_V) = 0_V$ for <u>any</u> $T: V \to V$ linear, we do not consider 0_V as eigenvector. However, 0 can be an eigenvalue!

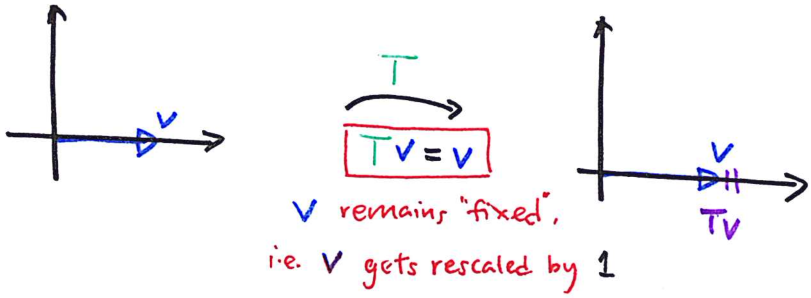
Using the geometric interpretation that an eigenvector simply gets rescaled by a factor under the linear transformation, one can sometimes "see" the eigenvectors / eigenvalues directly from geometric.

<u>Example</u>: Let $T: \mathbb{R}^2 \to \mathbb{R}^2$ be the projection onto the x-axis.

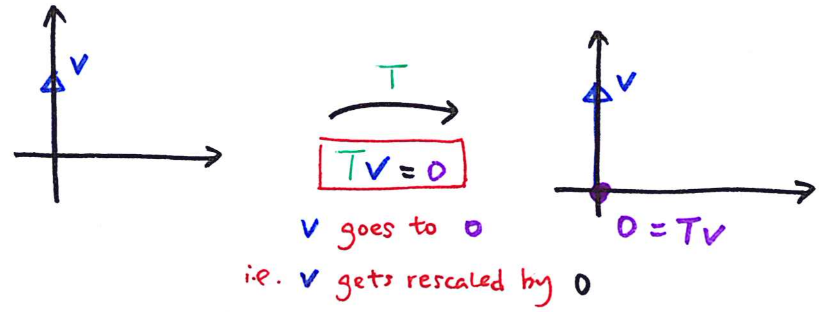
for a general vector v :

However, if v lies in the x-axis / y-axis :

$v \parallel e_1$:



$v \parallel e_2$:



Therefore, for any $a \neq 0$,

$\begin{pmatrix} a \\ 0 \end{pmatrix}$ is an eigenvector of T with eigenvalue $\lambda = 1$.

$\begin{pmatrix} 0 \\ a \end{pmatrix}$ is an eigenvector of T with eigenvalue $\lambda = 0$. (allowed to have zero eigenvalue, but not zero eigenvector!)

[Q: Is T diagonalizable?] Yes! $[T]_\beta = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$

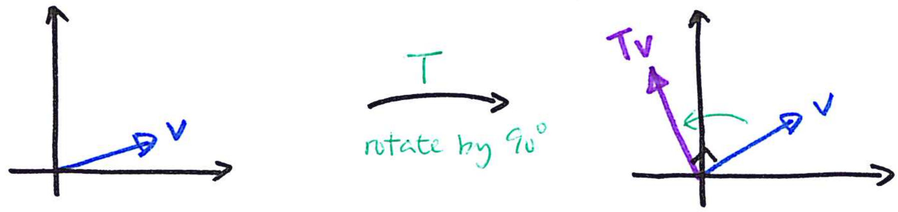
<u>Two easy observations</u> :

(1) $0 \neq v \in N(T) \iff v$ is an eigenvector of T with eigenvalue 0.

(2) v is an eigenvector of $T \iff a \cdot v$ is an eigenvector $\forall\, 0 \neq a \in \mathbb{F}$.
(with the <u>same</u> eigenvalue)

- Eigenvalues may not exist (over $\mathbb{R}$)

For example, let $T: \mathbb{R}^2 \to \mathbb{R}^2$ be rotation (counterclockwise) by 90°.



Obviously, no vector v stays in the same direction under T.

Therefore, T has no eigenvalues / eigenvectors. (over $\mathbb{R}$)

Equivalently, the matrix

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \text{ has no real eigenvalues}$$

BUT, it has complex eigenvalues:

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ -i \end{pmatrix} = \begin{pmatrix} i \\ 1 \end{pmatrix} = i \cdot \begin{pmatrix} 1 \\ -i \end{pmatrix}$$

So, i is a complex eigenvalue.

[Q: Can you find another?]

- Infinite dimensional case:

Let $C^\infty(\mathbb{R}) = \{ f: \mathbb{R} \to \mathbb{R} \mid f \text{ infinitely differentiable} \}$. Consider the linear operator: (Note: $\dim C^\infty(\mathbb{R}) = +\infty$).

$$T: C^\infty(\mathbb{R}) \longrightarrow C^\infty(\mathbb{R}) \qquad T(f) := f'$$

If $0 \neq f \in C^\infty(\mathbb{R})$ is an eigenvector of T with eigenvalue $\lambda \in \mathbb{R}$, then $T(f) = \lambda f$, i.e. f satisfies the ODE

$$f' = \lambda f$$

which we know the solutions are $f(t) = Ce^{\lambda t}$ for any $0 \neq C \in \mathbb{R}$.

Therefore, any $\lambda \in \mathbb{R}$ is an eigenvalue of T.

Q: How to find eigenvalues / eigenvectors?

Thm: Let $A \in M_{n\times n}(\mathbb{F})$. Why? $\exists\, v \neq 0$ s.t. $Av = \lambda v = \lambda I v \iff (A-\lambda I)v = 0$

(i) $\lambda \in \mathbb{F}$ is an eigenvalue of $A \iff \boxed{\det(A - \lambda I_n) = 0}$

characteristic equation.

I_n = identity matrix

Def$^{\underline{n}}$: $f(\lambda) := \det(A - \lambda I_n)$

characteristic polynomial of A

(ii) $0 \neq v \in \mathbb{F}^n$ is an eigenvector with eigenvalue λ of A

$\iff 0 \neq v \in N(A - \lambda I_n)$.

Steps to compute eigenvalues / eigenvectors of a matrix $A \in M_{n\times n}(\mathbb{F})$

(1) Solve the characteristic equation:

$$f(\lambda) = \det(A - \lambda I) = 0$$ ← degree n polynomial equation. in λ

(2) For each solution $\lambda \in \mathbb{F}$ to the characteristic equation, find all the eigenvectors with eigenvalue λ by computing the null space $N(A - \lambda I)$.

Example: Find all the eigenvalues / eigenvectors of

$$A = \begin{pmatrix} 1 & 2 \\ 3 & 2 \end{pmatrix} \qquad (\mathbb{F} = \mathbb{R}.)$$

Is A diagonalizable?

Solution: Find characteristic polynomial of A

$$f(\lambda) = \det(A - \lambda I) = \det\begin{pmatrix} 1-\lambda & 2 \\ 3 & 2-\lambda \end{pmatrix}$$

$$= (1-\lambda)(2-\lambda) - 6 = \lambda^2 - 3\lambda - 4$$

Solve characteristic equation

$$f(\lambda) = \lambda^2 - 3\lambda - 4 = 0 \quad \Rightarrow \quad \lambda_1 = 4, \quad \lambda_2 = -1$$

(2 distinct real eigenvalues)

Find eigenvectors for $\lambda_1 = 4$: i.e. $\vec{0} \neq v \in N(A-4I)$

$$N(A-4I) = N\begin{pmatrix} -3 & 2 \\ 3 & -2 \end{pmatrix} = \text{span}\left\{\begin{pmatrix} 2 \\ 3 \end{pmatrix}\right\}, \text{ i.e. } v = a\begin{pmatrix} 2 \\ 3 \end{pmatrix}, \ a \neq 0.$$

For $\lambda_2 = -1$,

$$N(A+I) = N\begin{pmatrix} 2 & 2 \\ 3 & 3 \end{pmatrix} = \text{span}\left\{\begin{pmatrix} 1 \\ -1 \end{pmatrix}\right\}, \text{ i.e. } v = a\begin{pmatrix} 1 \\ -1 \end{pmatrix}, \ a \neq 0.$$

A is diagonalizable since if we take $\beta = \left\{\begin{pmatrix} 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \end{pmatrix}\right\}$, then for

$Q = \begin{pmatrix} 2 & 1 \\ 3 & -1 \end{pmatrix}$, we have $Q^{-1}AQ = \begin{pmatrix} 4 & 0 \\ 0 & -1 \end{pmatrix}$.

$\square$

Example: Find all the eigenvalues / eigenvectors of

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

Is A diagonalizable?

Solution: Solve characteristic equation:

$$0 = f(\lambda) = \det(A - \lambda I) = \det\begin{pmatrix} 1-\lambda & 1 \\ 0 & 1-\lambda \end{pmatrix} = (1-\lambda)^2$$

$\Rightarrow \lambda = 1$ (only 1 eigenvalue – "multiplicity 2")

Find eigenvectors for $\lambda = 1$, i.e. $\vec{0} \neq v \in N(A-I)$.

$$N(A-I) = N\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \text{span}\left\{\begin{pmatrix} 1 \\ 0 \end{pmatrix}\right\}$$ 1 - dimensional !!

Therefore, A is NOT diagonalizable because if it were, then $\exists$ basis $\beta = \{\vec{v}_1, \vec{v}_2\}$ consisting of eigenvectors. but $\exists$ only 1 eigenvalue $\lambda = 1$ so both $\vec{v}_1, \vec{v}_2$ must have eigenvalue 1, ie $\vec{v}_1, \vec{v}_2 \in N(A-I)$ and thus

$\beta = \{\vec{v}_1, \vec{v}_2\}$ must be linearly dependent

as $\dim N(A-I) = 1$.

This contradicts that β is a basis.

$\square$

Sometimes, eigenvalues / eigenvectors exist or not depends on the field $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$.

Example: Find all the eigenvalues of

$$A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

Is A diagonalizable?

Solution: The characteristic equation is

$$f(\lambda) = \det(A - \lambda I) = \det\begin{pmatrix} -\lambda & -1 \\ 1 & -\lambda \end{pmatrix} = \lambda^2 + 1 = 0$$

$\exists$ no real solutions $\Rightarrow$ NO $\mathbb{R}$ eigenvalues

so, NOT diagonalizable over $\mathbb{R}$.

BUT, $\exists$ two complex eigenvalues $\lambda_1 = i$, $\lambda_2 = -i$.

Eigenvectors for $\lambda_1 = i$:

$$N(A - iI) = N\begin{pmatrix} -i & -1 \\ 1 & -i \end{pmatrix} = \operatorname{span}_{\mathbb{C}}\left\{\begin{pmatrix} i \\ 1 \end{pmatrix}\right\}$$

Eigenvectors for $\lambda_2 = -i$:

$$N(A + iI) = N\begin{pmatrix} i & -1 \\ 1 & i \end{pmatrix} = \operatorname{span}_{\mathbb{C}}\left\{\begin{pmatrix} i \\ -1 \end{pmatrix}\right\}$$

So, A is diagonalizable over $\mathbb{C}$, ie if we take

$$\beta = \left\{\begin{pmatrix} i \\ 1 \end{pmatrix}, \begin{pmatrix} i \\ -1 \end{pmatrix}\right\} \text{ or equivalently, } Q = \begin{pmatrix} i & i \\ 1 & -1 \end{pmatrix}$$

then $Q^{-1} A Q = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$.

$$\boxed{\begin{array}{l} \textbf{Fundamental Theorem of Algebra}: \mathbb{C} \text{ is "algebraically closed", i.e.} \\ \text{any degree } n \text{ polynomials over } \mathbb{C},\ P(t), \text{ splits:} \\ P(t) = \underset{\mathbb{C}}{\underset{\cap}{c}}\,(t - \underset{\mathbb{C}}{\underset{\cap}{a_1}})(t - \underset{\mathbb{C}}{\underset{\cap}{a_2}})(t - \underset{\mathbb{C}}{\underset{\cap}{a_3}}) \cdots (t - \underset{\mathbb{C}}{\underset{\cap}{a_n}}) \quad (\text{Some } a_i\text{'s could be the same}) \end{array}}$$

$\Rightarrow$ $\mathbb{C}$ eigenvalues always exist (BUT $\not\Rightarrow$ always diagonalizable over $\mathbb{C}$) e.g.?

How to find eigenvalues / eigenvectors of $T: V \to V$? ($\dim V = n$)

(1) Fix <u>ANY</u> basis β to obtain $A = [T]_\beta \in M_{n\times n}(\mathbb{F})$.

(2) Find eigenvalues / eigenvectors for the matrix A.

$\Rightarrow$ (3) These are the eigenvalues and the coordinate representation of the eigenvectors (w.r.t. β) for T.

Exercise: Why does it work?

Example: Let $T: M_{2\times 2}(\mathbb{R}) \longrightarrow M_{2\times 2}(\mathbb{R})$ be $T(A) = A^t$.

Find all eigenvalues / eigenvectors of T.

Is T diagonalizable?

Solution: Take a standard basis for $M_{2\times 2}(\mathbb{R})$:

$$\beta = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}$$

$$\begin{matrix} \downarrow T & \downarrow T & \downarrow T & \downarrow T \end{matrix}$$

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \quad \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

Therefore,

$$A = [T]_\beta = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Char. eq$^{\underline{n}}$: $\det(A - \lambda I) = \det \begin{pmatrix} 1-\lambda & 0 & 0 & 0 \\ 0 & -\lambda & 1 & 0 \\ 0 & 1 & -\lambda & 0 \\ 0 & 0 & 0 & 1-\lambda \end{pmatrix}$

$$= (1-\lambda)^2(\lambda^2 - 1) = (\lambda - 1)^3(\lambda + 1) = 0$$

eigenvalues $\lambda_1 = 1$, $\lambda_2 = -1$.

Eigenvectors: $\boxed{\lambda_1 = 1}$ $T(A) = 1 \cdot A \iff A^t = A$ "symmetric matrices"

$\boxed{\lambda_2 = -1}$ $T(A) = -1 \cdot A \iff A^t = -A$ "skew-symmetric matrices"

$$\left\{\begin{matrix}\text{symmetric}\\ \text{matrices}\end{matrix}\right\} = \{A \in M_{2\times 2}(\mathbb{R}) \mid A^t = A\} = \operatorname{span}\left\{\begin{pmatrix}1&0\\0&0\end{pmatrix}, \begin{pmatrix}0&0\\0&1\end{pmatrix}, \begin{pmatrix}0&1\\1&0\end{pmatrix}\right\}.$$

$$\left\{\begin{matrix}\text{skew-symmetric}\\ \text{matrices}\end{matrix}\right\} = \{A \in M_{2\times 2}(\mathbb{R}) \mid A^t = -A\} = \operatorname{span}\left\{\begin{pmatrix}0&1\\-1&0\end{pmatrix}\right\}.$$

Hence, T is diagonalizable: if $\beta' = \left\{\begin{pmatrix}1&0\\0&0\end{pmatrix}, \begin{pmatrix}0&0\\0&1\end{pmatrix}, \begin{pmatrix}0&1\\1&0\end{pmatrix}, \begin{pmatrix}0&1\\-1&0\end{pmatrix}\right\}$

then
$$[T]_{\beta'} = \begin{pmatrix}1&&&\\&1&&\\&&1&\\&&&-1\end{pmatrix}.$$

$\square$