MATH 4900E: SEMINAR

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Topic: All about inequalities

Description: We will cover selected topics from the book "A view from the top" by Alex Iosevich. The point of departure is inequalities; possible topics include Fourier analysis, geometry, combinatorics, and number theory. The unity across different fields of mathematics will be emphasized, as material will often be drawn from more than one area of mathematics. Students are expected to supplement the text by materials they gather or develop on their own.

Some sample problems to be considered are as follows:

1. How many integer points are there in a ball of radius R in $\mathbb{R}^n$? For geometers, this is the number of eigenvalues of the standard Laplacian on the n-dimensional flat torus that are smaller than R, and for number theorists, this is basically $\sum_{0\leq m\leq R} r_n(m)$ if $r_n(m)$ is the number of ways to write the integer m as the sum of n squares. Along the way we will come across some simple and yet beautiful applications of Fourier analysis.

2. For $j = 1, 2, \ldots, n$, let π_j be the j-th coordinate projection on $\mathbb{R}^n$. For a measurable set E in $\mathbb{R}^n$, we can consider $\frac{|E|}{|\pi_j(E)|}$ as the width of E in the j-th coordinate projection, where $|E|$ denotes the Lebesgue measure of E; similarly for $|\pi_j(E)|$. It turns out that one can give a lower bound of $|E|$ in terms of the average widths of E in all n coordinate directions. This is an application of the Loomis-Whitney inequality, which can be deduced easily from the Cauchy-Schwarz in 3-dimensions. The Loomis-Whitney inequality is in turn a special case of the Brascamp-Lieb inequality, which can be deduced, among other ways, using monotonicity formulas in the study of heat equations; along the way we will recover a proof of the Hölder's inequality by heat flows.

3. Let $\mathbb{F}_n^d$ be the d-dimensional vector space over the finite field $\mathbb{F}_n$. How small can a subset of $\mathbb{F}_n^d$ be, if it contains a line in every possible direction? This is the famous Kakeya problem over finite fields, and is only resolved relatively recently by Zeev Dvir; we will examine its proof (using algebra), and discuss possible connections to an analogue on $\mathbb{R}^d$, which stands squarely at the intersection of harmonic analysis, additive combinatorics, PDEs, and algebraic topology.