Proof that $\exp(x)\cdot\exp(y) = \exp(x+y) \quad \forall x, y \in \mathbb{R}$

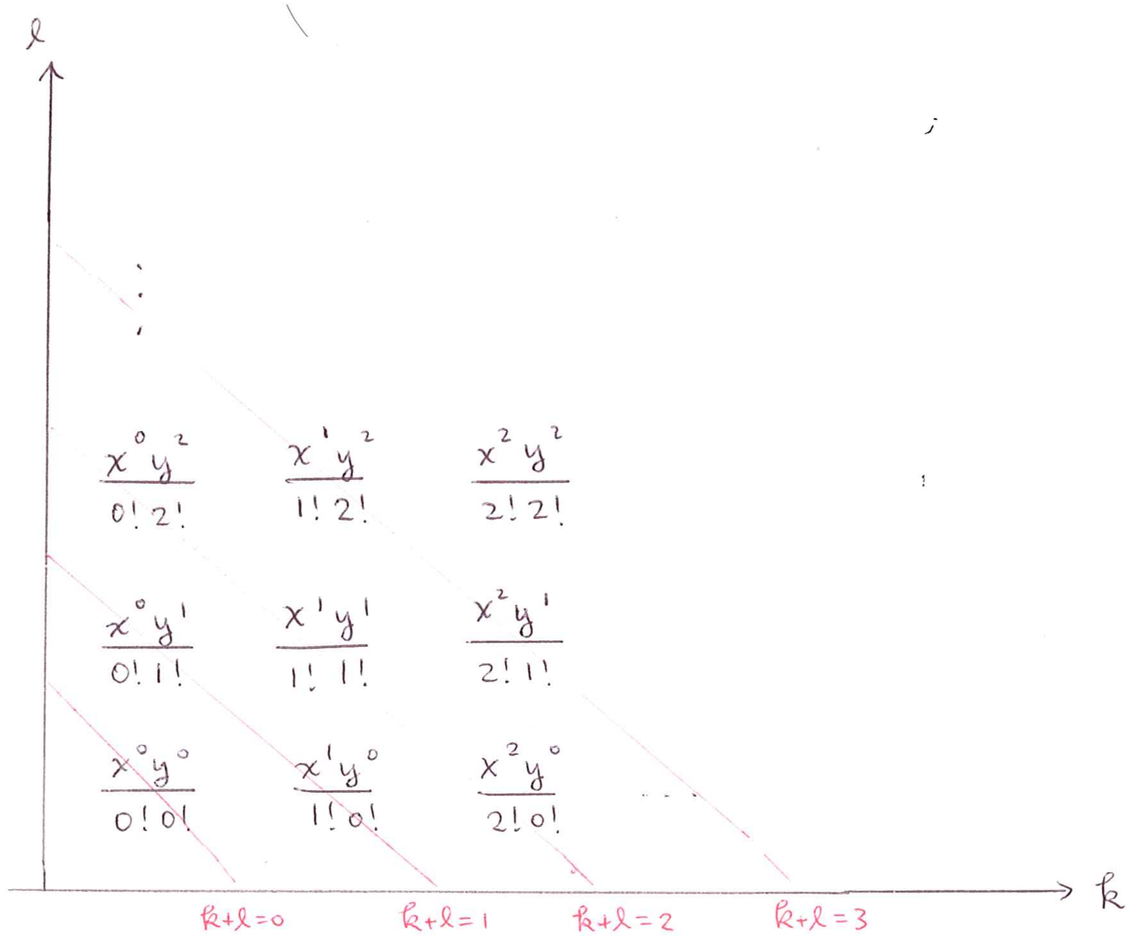
Recall $\exp(x) = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \cdots = \sum_{k=0}^{\infty} \frac{x^k}{k!}$

Hence $\forall x, y \in \mathbb{R}$,

$$\exp(x)\cdot\exp(y) = \sum_{k=0}^{\infty} \frac{x^k}{k!} \sum_{l=0}^{\infty} \frac{y^l}{l!}$$

$$= \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} \frac{x^k y^l}{k!\, l!} \quad \cdots\cdots (1)$$

This is a double sum, and pictorially we're summing



Let's first sum along a line in red, and then sum along all lines.

If $m = k+l$, then as k, l varies over $0,1,2,\dots$ m also varies over $0,1,2,\dots$

$$\therefore \quad (1) \overset{(*)}{=} \sum_{m=0}^{\infty} \sum_{l=0}^{m} \frac{x^{m-l} y^{l}}{(m-l)!\, l!}$$

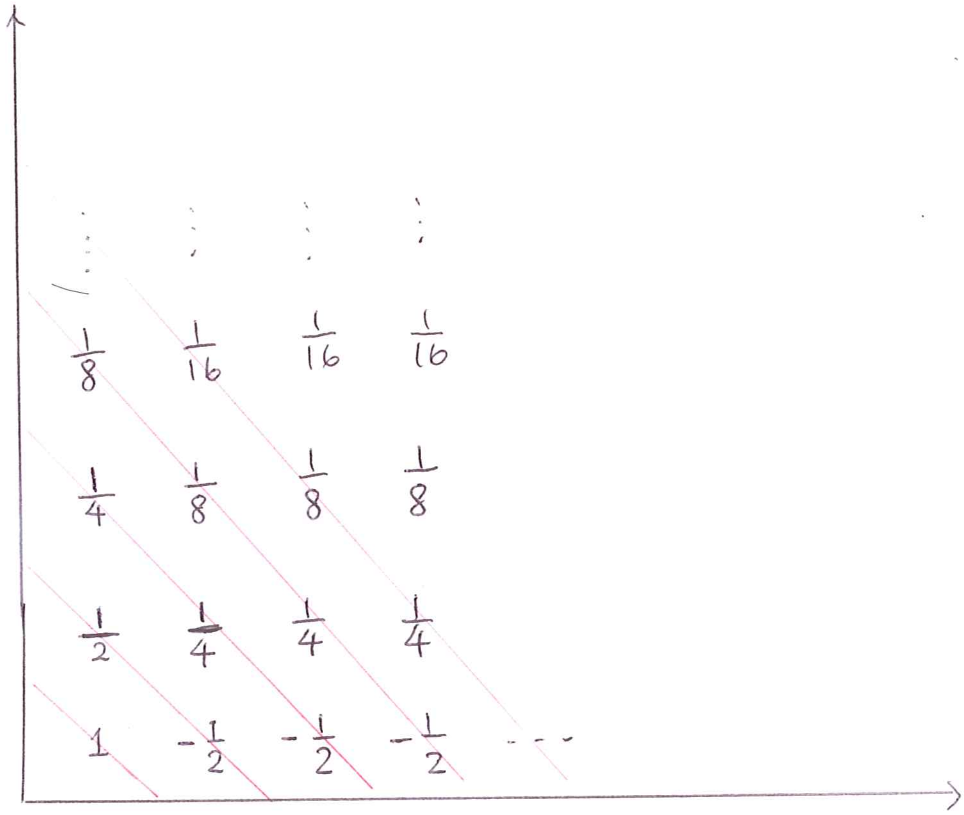
$$= \sum_{m=0}^{\infty} \frac{1}{m!} \sum_{l=0}^{m} \frac{m!}{(m-l)!\, l!} x^{m-l} y^{l}$$

$$= \sum_{m=0}^{\infty} \frac{1}{m!} (x+y)^{m} \qquad \text{(binomial theorem)}$$

$$= \exp(x+y).$$

The problem is justifying $(*)$. We won't actually give the justification of $(*)$, since that is beyond the scope of this course. But we point out that rearranging infinite series as in $(*)$ is dangerous in general:

Consider the numbers on the next page. The first column is the geometric sequence $1, \frac{1}{2}, \frac{1}{4}, \dots$ All other columns start with $-\frac{1}{2}$, and continues with the geometric series $\frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \dots$



If we first sum along columns, then

the numbers in the first column sums to 2

the numbers in the nth column sums to 0 $\forall n \geq 2$.

Hence the "sum" of all numbers above seems to be $2+0+0+\cdots = 2$

But if we first sum along the red lines,

the number in the red line in the lowest left corner = 1

the numbers in all other red lines sum to 0.

Hence the "sum" of all numbers above seems to be $1+0+0+\cdots = 1$!

But obviously $2 \neq 1$. What's wrong here?

It turns out when one deals with the sum of infinitely many numbers, the order in which the numbers are summed is very important. In general, the sum of an infinite array of numbers is not well-defined. If one sums in a different order, the answer could be different.

This example shows that the step (*) on p.2 really needs to be justified.