SOME CONSEQUENCES OF THE MEAN-VALUE THEOREM

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Below we establish some important theoretical consequences of the mean-value theorem. First recall the mean value theorem:

Theorem 1 (Mean value theorem)**.** *Suppose* $f\colon [a,b] \to \mathbb{R}$ *is a function defined on a closed interval* $[a,b]$ *where* $a, b \in \mathbb{R}$*. If* f *is continuous on the closed interval* $[a,b]$*, and* f *is differentiable on the open interval* (a,b)*, then there exists* $c \in (a,b)$ *such that*

$$f(b) - f(a) = f'(c)(b-a).$$

This has some important corollaries. To proceed further, we need the following definitions:

Definition 1. Suppose $f\colon I \to \mathbb{R}$ is defined on some interval I.

(a) f is said to be *constant* on I, if and only if $f(x) = f(y)$ for any $x, y \in I$.
(b) f is said to be *increasing* on I, if and only if for any $x, y \in I$ with $x < y$, we have $f(x) \leq f(y)$.
(c) f is said to be *strictly increasing* on I, if and only if for any $x, y \in I$ with $x < y$, we have $f(x) < f(y)$.
(d) f is said to be *decreasing* on I, if and only if for any $x, y \in I$ with $x < y$, we have $f(x) \geq f(y)$.
(e) f is said to be *strictly decreasing* on I, if and only if for any $x, y \in I$ with $x < y$, we have $f(x) > f(y)$.

(Can you draw some examples of such functions?)

Corollary 2. *Suppose* $f\colon [a,b] \to \mathbb{R}$ *is a function defined on a closed interval* $[a,b]$ *where* $a, b \in \mathbb{R}$*. Suppose also that* f *is continuous on the closed interval* $[a,b]$*, and that* f *is differentiable on the open interval* (a,b)*.*

(i) If $f'(t) = 0$ *for all* $t \in (a,b)$*, then* f *is constant on* $[a,b]$*.*
(ii) If $f'(t) \geq 0$ *for all* $t \in (a,b)$*, then* f *is increasing on* $[a,b]$*.*
(iii) If $f'(t) > 0$ *for all* $t \in (a,b)$*, then* f *is strictly increasing on* $[a,b]$*.*
(iv) If $f'(t) \leq 0$ *for all* $t \in (a,b)$*, then* f *is decreasing on* $[a,b]$*.*
(v) If $f'(t) < 0$ *for all* $t \in (a,b)$*, then* f *is strictly decreasing on* $[a,b]$*.*

Proof. Suppose f is continuous on $[a,b]$, and differentiable on (a,b). Fix two points $x, y \in [a,b]$, with say $x < y$. Then f is continuous on the closed interval $[x,y]$, and differentiable on the open interval (x,y), so the mean value theorem applies, and there exists some $c \in (x,y) \subset (a,b)$ such that

$$f(y) - f(x) = f'(c)(y-x).$$

(i) If $f'(t) = 0$ for all $t \in (a,b)$, then in particular $f'(c) = 0$. So
$$f(y) - f(x) = f'(c)(y-x) = 0,$$
i.e. $f(x) = f(y)$. Since x, y are arbitrary in $[a,b]$, this shows f is a constant on $[a,b]$.

(ii) If $f'(t) \geq 0$ for all $t \in (a,b)$, then in particular $f'(c) \geq 0$. So using also $y - x > 0$, we see that
$$f(y) - f(x) = f'(c)(y-x) \geq 0,$$
i.e. $f(x) \leq f(y)$. Since x, y are arbitrary in $[a,b]$, this shows f is increasing on $[a,b]$.

(iii) If $f'(t) > 0$ for all $t \in (a,b)$, then in particular $f'(c) > 0$. So using also $y - x > 0$, we see that
$$f(y) - f(x) = f'(c)(y-x) > 0,$$
i.e. $f(x) < f(y)$. Since x, y are arbitrary in $[a,b]$, this shows f is strictly increasing on $[a,b]$.

(iv) Similar to (ii) (or apply (ii) to $-f$)

(v) Similar to (iii) (or apply (iii) to $-f$)

□

Note: The converse of (i), (ii) and (iv) are true, but the converse of (iii), (v) are not! Verify yourself.

Next, we recall the definition of second derivatives: If f is a differentiable function on an open interval (a,b), and if f' is also differentiable on (a,b), then the derivative of f' is denoted f'', and is called the second derivative of f. Such functions are said to be *twice differentiable* on (a,b).

The sign of the second derivative is related to the concept of *convexity*, which we defined below.

Definition 2. Suppose $f\colon (a,b) \to \mathbb{R}$ is a continuous function on some open interval (a,b).

(a) f is said to be *convex* on (a,b), if and only if

$$f\left(\frac{x+y}{2}\right) \leq \frac{f(x)+f(y)}{2}$$

for any $x,y \in (a,b)$.

(b) f is said to be *concave* on (a,b), if and only if

$$f\left(\frac{x+y}{2}\right) \geq \frac{f(x)+f(y)}{2}$$

for any $x,y \in (a,b)$.

(Can you draw some examples of such functions?)

(Some people call convex functions "convex up", and concave functions "convex down". We will not use these terminologies.)

Corollary 3. *Suppose $f\colon (a,b) \to \mathbb{R}$ is twice differentiable on an open interval (a,b).*

(i) If $f''(t) \geq 0$ for all $t \in (a,b)$, then f is convex on (a,b).

(ii) If $f''(t) \leq 0$ for all $t \in (a,b)$, then f is concave on (a,b).

Proof. Fix two points $x,y \in (a,b)$, with say $x<y$. For brevity, let $c = \frac{x+y}{2}$. Consider the function $F\colon (a,b) \to \mathbb{R}$, defined by

$$F(t) = f(t) - [f'(c)(t-c) + f(c)] \quad \text{for all } t \in (a,b).$$

Then F is differentiable on (a,b), and for all $t \in (a,b)$, we have

$$F'(t) = f'(t) - f'(c).$$

(i) If $f''(t) \geq 0$ for all $t \in (a,b)$, then f' is increasing on (a,b), so

$$F'(t) = f'(t) - f'(c) \begin{cases} \leq 0 & \text{if } t \in (a,c) \\ \geq 0 & \text{if } t \in (c,b) \end{cases}$$

Hence F is decreasing on $[x,c]$, and increasing on $[c,y]$. It follows that

$$F(x) \geq F(c), \quad \text{and} \quad F(y) \geq F(c).$$

Now we add the two inequalities. Recall $F(c) = 0$,

$$\begin{cases} F(x) = f(x) - f'(c)(x-c) - f(c) \\ F(y) = f(y) - f'(c)(y-c) - f(c) \end{cases}.$$

Note that $x - c = -(y-c)$, since $c = \frac{x+y}{2}$ is the mid-point of x and y. Hence the terms involving $f'(c)$ cancels out when we add $F(x)$ and $F(y)$. The result is then

$$f(x) + f(y) - 2f(c) \geq 0, \quad \text{i.e.} \quad f(c) \leq \frac{f(x)+f(y)}{2}.$$

This proves what we want.

(ii) Just reverse the signs of f'' in the proof above, or apply (i) to $-f$.

□

One can easily show now that e.g. $\exp(x)$ is strictly increasing and convex on $\mathbb{R}$.

The above can help us determine whether a critical point is a local maxima and minima. We recall the following definitions:

Definition 3. Suppose $f\colon (a,b) \to \mathbb{R}$ is defined on an open interval (a,b).

(i) If $c \in (a,b)$ is a point, such that $f(x) \geq f(c)$ in a neighborhood of c (i.e. there exists a (possibly tiny) open interval (α,β), such that $c \in (\alpha,\beta) \subset (a,b)$, and $f(x) \geq f(c)$ for all $x \in (\alpha,\beta)$), then c is called a local minimum of the function f.

(ii) If $c \in (a,b)$ is a point, such that $f(x) \leq f(c)$ in a neighborhood of c (i.e. there exists a (possibly tiny) open interval (α,β), such that $c \in (\alpha,\beta) \subset (a,b)$, and $f(x) \leq f(c)$ for all $x \in (\alpha,\beta)$), then c is called a local maximum of the function f.

(Sometimes a local maximum is called a relative maximum, and a local minimum is called a relative minimum. Sometimes we also say a point is a local (or relative) extremum, if it is a local maximum or a local minimum.)

Corollary 4 (First derivative test)**.** *Suppose $f\colon (a,b) \to \mathbb{R}$ is defined on an open interval (a,b), and $c \in (a,b)$. Suppose also that f is differentiable on $(a,b) \setminus \{c\}$, and f is continuous at c.*

(i) If there exists α, $\beta \in (a,b)$, with $\alpha < c < \beta$, such that

$$\begin{cases} f'(t) \leq 0 & \text{for all } t \in (\alpha,c), \\ f'(t) \geq 0 & \text{for all } t \in (c,\beta), \end{cases}$$

then c is a local minimum of f.

(ii) If there exists α, $\beta \in (a,b)$, with $\alpha < c < \beta$, such that

$$\begin{cases} f'(t) \geq 0 & \text{for all } t \in (\alpha,c), \\ f'(t) \leq 0 & \text{for all } t \in (c,\beta), \end{cases}$$

then c is a local maximum of f.

Proof. (i) If f is as in (i), then f is decreasing on $[\alpha,c]$, and increasing on $[c,\beta]$. So $f(x) \geq f(c)$ on (α,β). It follows that c is a local minimum of f.

(ii) If f is as in (ii), then f is increasing on $[\alpha,c]$, and decreasing on $[c,\beta]$. So $f(x) \leq f(c)$ on (α,β). It follows that c is a local maximum of f.

□

Corollary 5 (Second derivative test)**.** *Suppose $f\colon (a,b) \to \mathbb{R}$ is differentiable on an open interval (a,b), and $c \in (a,b)$. Suppose also that $f'(c) = 0$, and that f' is differentiable at c.*

(i) If $f''(c) > 0$, then c is a local minimum of f.

(ii) If $f''(c) < 0$, then c is a local maximum of f.

Proof. Since $f'(c) = 0$ and $f''(c)$ exists, we have

$$\lim_{x\to c} \frac{f'(x)}{x-c} = \lim_{x\to c} \frac{f'(x) - f'(c)}{x-c}$$

exists and equals $f''(c)$.

(i) If $f''(c) > 0$, then

$$\lim_{x\to c} \frac{f'(x)}{x-c} > 0.$$

It follows that

$$\frac{f'(x)}{x-c} > 0$$

for all x in a deleted neighborhood of c, i.e. there exists $\alpha,\beta \in (a,b)$ with $c \in (\alpha,\beta)$ such that $\frac{f'(x)}{x-c} > 0$ holds for $x \in (\alpha,\beta) \setminus \{c\}$. Since $x - c > 0$ when $x > c$, and $x - c < 0$ when $x < c$, it follows that

$$f'(x) \begin{cases} < 0 & \text{if } x \in (\alpha,c) \\ > 0 & \text{if } x \in (c,\beta) \end{cases}$$

Using the first derivative test, it follows that c is a local minimum of f.

(ii) Similar to (i).

□

It is now easy to sketch the graph of a function using its first (and possibly second) derivatives. This in turn allows us to determine (sometimes) the global maximum or minimum of a function over unbounded intervals, and establish some identities / inequalities.

Later on we will use some of these corollaries of the mean value theorem, to derive the familiar properties of the trigonometric functions (like sine and cosine).