SIAM J. NUMER. ANAL.
Vol. 58, No. 5, pp. 2515–2543

DOMAIN DECOMPOSITION WITH LOCAL IMPEDANCE CONDITIONS FOR THE HELMHOLTZ EQUATION WITH ABSORPTION*

IVAN G. GRAHAM†, EUAN A. SPENCE†, AND JUN ZOU‡

Abstract. We consider one-level additive Schwarz preconditioners for a family of Helmholtz problems with increasing wavenumber k. These problems are discretized using the Galerkin method with nodal conforming finite elements of any (fixed) order on meshes with diameter $h = h(k)$, chosen to maintain accuracy as k increases. The action of the preconditioner requires the solution of independent (parallel) subproblems (with impedance boundary conditions) on overlapping subdomains of diameter H and overlap $\delta \leq H$. The solutions of these subproblems are linked together using prolongation/restriction operators defined using a partition of unity; this formulation was previously proposed in [J.-H. Kimn and M. Sarkis, *Comput. Methods Appl. Mech. Engrg.*, 196 (2007), pp. 1507–1514]. In numerical experiments (with $\delta \sim H$) for a model interior impedance problem, we observe robust (i.e., k-independent) GMRES convergence as k increases, with $H \sim k^{-\alpha}$ and $\alpha \in [0, 0.4]$ as k increases. This provides a highly parallel, k-robust one-level domain decomposition method. We provide a supporting theory by studying the preconditioner applied to a range of absorptive problems, $k^2 \mapsto k^2 + \mathrm{i}\epsilon$, with absorption parameter ϵ. Working in the Helmholtz "energy" inner product, and using the underlying theory of Helmholtz boundary-value problems, we prove a k-independent upper bound on the norm of the preconditioned matrix, valid for all $|\epsilon| \lesssim k^2$. We also prove a strictly positive lower bound on the distance of the field of values of the preconditioned matrix from the origin which holds when ϵ/k is constant or growing arbitrarily slowly with k. These results imply robustness of the preconditioner for the corresponding absorptive problem as k increases (given an appropriate choice of H). Since it is known that the absorptive problem provides a good preconditioner for the pure Helmholtz problem when $\epsilon \sim k$, our results provide some theoretical support for the observed robustness of the preconditioner for the pure Helmholtz problem. Since the subdomains used in our preconditioner shrink only slowly (relative to the fine grid size) as k increases, cheaper approximate (two- or multilevel) versions of the preconditioner analyzed here are important in practice and are reviewed here.

Key words. Helmholtz equation, high frequency, preconditioning, GMRES, domain decomposition, subproblems with impedance conditions, robustness

AMS subject classifications. 65F08, 65F10, 65N55

DOI. 10.1137/19M1272512

1. Introduction.

The efficient solution of the wave equation is of intense current interest because of the equation's many applications (in, e.g., computational medicine, underwater acoustics, earthquake modelling, and seismic imaging). This paper concerns efficient iterative methods for computing conforming finite-element approximations of any fixed order of the Helmholtz equation (i.e., the wave equation in the frequency domain) in two dimensions (2-d) or three dimensions (3-d). We formulate and analyze parallel preconditioners for use with GMRES and provide theory indicating that our preconditioners should remain effective as the wavenumber k

*Received by the editors July 5, 2019; accepted for publication (in revised form) May 18, 2020; published electronically September 16, 2020.

https://doi.org/10.1137/19M1272512

Funding: The work of the first author was supported by the EPSRC grant EP/S003975/1. The work of the second author was supported by the EPSRC grant EP/R005591/1. The work of the third author was supported by the Hong Kong RGC General Research Fund project 14306718 and the NSFC/Hong Kong RGC Joint Research Scheme 2016/17 grant N_CUHK437/16.

†Department of Mathematical Sciences, University of Bath, Bath, BA2 7AY, UK (I.G.Graham@bath.ac.uk, e.a.spence@bath.ac.uk).

‡Department of Mathematics, Chinese University of Hong Kong, Shatin, N.T., Hong Kong (zou@math.cuhk.edu.hk).

increases.

As k increases, there are several difficulties that make the Helmholtz problem hard, both mathematically and numerically: (i) the solution becomes more oscillatory and, in general, meshes need to be increasingly refined, leading to huge linear systems with dimension growing at least with $\mathcal{O}(k^d)$; (ii) the linear systems become more indefinite; (iii) many "standard" preconditioning techniques that are motivated by positive-definite problems become unusable in practice; (iv) there is relatively little rigorous theory for justifying effective preconditioning of such large and indefinite problems. Regarding (i), we recall that [35, Chapter 4] shows that in the linear finite element method for a one-dimensional (1-d) Helmholtz problem, $h \sim k^{-3/2}$ is necessary to ensure a bounded relative error as k increases; the extension of this result to higher dimensions is given in [39].

Our analysis is carried out for the model Helmholtz problem with absorption:

$$-\Delta u - (k^2 + \mathrm{i}\varepsilon)u = f, \tag{1.1}$$

on an open bounded polygonal (for $d = 2$) or Lipschitz polyhedral (for $d = 3$) domain $\Omega \subset \mathbb{R}^d$, with mixed boundary conditions

$$\frac{\partial u}{\partial n} - \mathrm{i}\eta u = g \quad \text{on} \quad \Gamma_I, \quad \text{and} \quad u = 0 \quad \text{on} \quad \Gamma_D, \tag{1.2}$$

where the wavenumber is $k > 0$, and $\Gamma = \Gamma_I \cup \Gamma_D$ is the boundary of Ω, partitioned into Γ_I and Γ_D, where Γ_I has positive surface measure. In applications, $k = \omega/c$, with ω the angular frequency and c the wave speed. Here we restrict ourselves to the case when c is a positive constant. We allow the *absorption parameter* ε to be negative, zero, or positive (with $\varepsilon = 0$ corresponding to the "pure Helmholtz" case); more details on ε and η are given in section 2.

In practical wave scattering problems, the PDE (1.1) is commonly posed on the infinite domain exterior to a bounded scatterer, which is then truncated using an artificial boundary. The significance of the impedance boundary condition in (1.2) is that (with $\eta = \sqrt{k^2 + \mathrm{i}\varepsilon}$) it is the simplest possible approximation to the Sommerfeld radiation condition. The problem (1.1), (1.2) can therefore model acoustic scattering by a sound-soft scatterer. Also included in (1.1), (1.2) is the *interior impedance problem*, where $\Gamma_D = \emptyset$, and Γ_I is the boundary of Ω. We assume that if $\Gamma_D \neq \emptyset$, then the surface measure of Γ_D is positive.

The standard variational formulation for (1.1), (1.2) is as follows: Given $f \in L^2(\Omega)$, $g \in L^2(\Gamma_I)$, find $u \in H^1_{0,D}(\Omega) := \{v \in H^1(\Omega) : v = 0 \text{ on } \Gamma_D\}$, such that

$$a_\varepsilon(u, v) = F(v) \quad \text{for all } v \in H^1_{0,D}(\Omega), \tag{1.3}$$

where

$$a_\varepsilon(u, v) := \int_\Omega \nabla u \cdot \overline{\nabla v} - (k^2 + \mathrm{i}\varepsilon) \int_\Omega u\overline{v} - \mathrm{i}\eta \int_{\Gamma_I} u\overline{v} \quad \text{and} \quad F(v) := \int_\Omega f\overline{v} + \int_{\Gamma_I} g\overline{v}; \tag{1.4}$$

when $\varepsilon = 0$ and $\eta = k$ we write a instead of a_ε. We approximate (1.3) using the Galerkin method in a conforming finite-element space $\mathcal{V}^h \subset H^1_{0,D}(\Omega)$ (consisting of continuous piecewise polynomials of arbitrary fixed order), on a shape-regular mesh $\mathcal{T}^h$ with mesh diameter h (assumed to resolve the interface $\overline{\Gamma_I} \cap \overline{\Gamma_D}$ when this is nonempty and points on the interface are treated as Dirichlet points). This yields the linear system

$$A_\varepsilon \mathbf{U} \;:=\; (S - (k^2 + \mathrm{i}\varepsilon)M - \mathrm{i}\eta N)\mathbf{U} \;=\; \mathbf{F}, \tag{1.5}$$

where $\mathbf{U}$ is the vector of nodal values of the finite-element approximation $u_h \approx u$, S is the stiffness matrix for the negative Laplace operator, M is the domain mass matrix, and N is the boundary mass matrix (corresponding, respectively, to each of the terms in $a_\varepsilon(u,v)$ in (1.4), and described in more detail in section 2.2). A_ε is large, sparse, and indefinite. When $\varepsilon = 0$ and $\eta = k$, we write A instead of A_ε. In common with many other investigations in the literature, we consider the situation where, following point (i) on the previous page, h is chosen as a function of k to maintain accuracy as k increases (see Remark 2.9 for more details).

One way to understand the essential difficulty in preconditioning A (as k increases) is to recall that the fundamental solution of the operator in (1.1) with $\varepsilon = 0$ (in three dimensions) is $G(x,y) = \exp(\mathrm{i}kr)/r$, where $r = |x-y|$, with $|\cdot|$ denoting the Euclidean norm, and so a good preconditioner for (1.1) with $\varepsilon = 0$ should, roughly speaking, approximate the integral operator with kernel G. When $k = 0$ this operator is "data-sparse," since the jth derivative of G decays with order $\mathcal{O}(r^{-(j+1)})$, when x and y are well-separated. Thus, a source in a given region is only felt weakly far away, a fact that underlies many successful preconditioners for Laplace-like problems (e.g., multigrid, domain decomposition, or $\mathcal{H}$-matrices). However, when k is large, the jth derivative of G decays with the much slower rate $\mathcal{O}(k^j r^{-1})$, and the application of Laplace-like preconditioning strategies becomes problematic. While directional clustering methods (see, e.g., [16], [7], and the references therein) have been developed for homogeneous Helmholtz problems, formulated using boundary integral equations, domain-based methods such as those considered here remain of great importance, due to their applicability to general problems with sources and heterogeneities.

Introducing absorption, $\varepsilon \neq 0$, has the effect of improving the decay of the Green's function. While absorptive problems do appear in applications (and our results here cover these), our deeper motivation for including ε is that it has proved useful for both constructing and providing the theory for preconditioners for the case $\varepsilon = 0$. In [27] it was proved (subject to certain natural conditions on Ω, h, and ε) that there is a constant K, independent of h, k, and ε, such that

$$\|I - A_\varepsilon^{-1} A\|_2 \;\leq\; K\frac{|\varepsilon|}{k}\,. \tag{1.6}$$

Thus the left-hand side of (1.6) can then be made small by choosing ε to be a small-enough multiple of k. However A_ε^{-1} is not a practical preconditioner for A, and we therefore replace it by an approximation $B_\varepsilon^{-1} \approx A_\varepsilon^{-1}$. Using the classical results about GMRES in [15], we say that B_ε^{-1} is a *good preconditioner* for A if both (i) the matrix $B_\varepsilon^{-1}A$ has Euclidean norm bounded above, and (ii) the field of values (in the Euclidean norm) of $B_\varepsilon^{-1}A$ is bounded away from the origin, with both bounds independent of k and ε. If both (i) and (ii) are satisfied then, by [15], GMRES for $B_\varepsilon^{-1}A$ will converge in a number of iterations independent of k and ε.

In order to characterize good choices of B_ε^{-1}, we can write

$$B_\varepsilon^{-1}A \;=\; B_\varepsilon^{-1}A_\varepsilon \;-\; B_\varepsilon^{-1}A_\varepsilon(I - A_\varepsilon^{-1}A). \tag{1.7}$$

Then (1.6) combined with (1.7) suggests that B_ε^{-1} will be a good preconditioner for A provided that

(1.8) B_ε^{-1} is a good preconditioner for A_ε when $|\varepsilon| = ck$, with c sufficiently small.

(An argument making this statement rigorous is given in Appendix A.)

1.1. The novel results of this paper. We give a new and rigorous proof that B_ε^{-1} is a good preconditioner for A_ε when B_ε^{-1} is a simple additive Schwarz preconditoner, constructed by solving independent (local) Helmholtz impedance subproblems on overlapping subdomains of Ω, linked by prolongation/restriction operators defined via a partition of unity (see section 1.2).

Theorem 3.12 gives general estimates for the norm and the distance of the field of values from the origin of $B_\varepsilon^{-1}A_\varepsilon$, under the general assumption that the local solvers are sufficiently good approximate inverses for the localized global problem (assumption (3.43)). The estimates are explicit in the wavenumber k, the fine mesh diameter h, the number of overlaps Λ, the subdomain diameter H, the overlap size δ, and the absorption parameter ε. Corollaries 3.15 and 3.16 then provide more concrete estimates under additional conditions on H, δ and ε. In particular, we present the following:

(1) Corollary 3.15 provides conditions under which the norm of $B_\varepsilon^{-1}A_\varepsilon$ is uniformly bounded from above for all $0 \le |\varepsilon| \le k^2$.
(2) Corollary 3.16 provides conditions under which the field of values of $B_\varepsilon^{-1}A_\varepsilon$ is uniformly bounded away from the origin. These hold (for appropriate $H = H(k)$) when $|\varepsilon| \sim k^{1+\beta}$, with β arbitrarily close to 0, or when $|\varepsilon| = Ck$ for some large enough constant C.

Although both the latter requirement in (2) and the requirement in (1.8) suggest we should take ε proportional to k, the required constants C, c are not explicitly known and so a rigorous lower bound on the field of values cannot be deduced in the pure Helmholtz case. Nevertheless, numerical experiments in section 4 still suggest that B_0^{-1} is a good preconditioner for the pure Helmholtz problem, for certain choices of $H(k)$, decreasing as $k \to \infty$.

Important features of the results of Theorem 3.12 and Corollaries 3.15 and 3.16 are that (a) they hold for bounded polygonal or Lipschitz polyhedral domains and cover sound-soft scattering problems, truncated using first order absorbing boundary conditions; (b) the theory allows finite element methods of any fixed order on shape-regular meshes; and general shape-regular subdomains; (c) the proof constitutes a substantial extension of classical Schwarz theory to the non-self-adjoint case; (d) via a duality argument, the theory covers both left- and right-preconditioning simultaneously.

To achieve the goal of a highly parallel and provably $\mathcal{O}(n)$ solver for the Helmholtz equation as k increases, one would need the following:

(i) a k-independent (i.e., $\mathcal{O}(1)$) number of iterations,
(ii) the action of the preconditioner to be $\mathcal{O}(n)$, and
(iii) (roughly speaking) the preconditioner to be as parallel as possible.

Since we propose here a one-level additive Schwarz method, (iii) is achieved. The main achievement of our paper is fundamental theory obtaining conditions under which (i) is achieved for Schwarz methods even without global coarse solver. As can be seen from the experiments in section 4, the subdomain size H needed to ensure robustness can shrink to zero as k increases, but remains large relative to the fine grid size h. Further work is needed to achieve requirement (ii). However, in section 1.4 we briefly discuss the cost of the subdomain problems, together with ways of reducing this cost (some of which have been recently implemented and tested [30, 31, 5, 4, 6]). Relatively large subdomain problems are also encountered in sweeping-style preconditioners, although in this case they are typically posed on "quasi $(d-1)$-dimensional" slices of an original d-dimensional domain, and efficient direct solvers have been developed for these (e.g., [51]).

Finally, we note that it is perhaps remarkable that this one-level additive Schwarz method can be robust when the subdomain size $H \to 0$. This conflicts with standard intuition and existing understanding, even for self-adjoint coercive PDEs; there, if $H \to 0$, the condition number of the one-level preconditioned problem grows like $\mathcal{O}((\delta H)^{-1})$. In the Helmholtz case, however, we are solving a family of problems parametrized by k. Even though the problem itself becomes "harder" as k increases, the results of this paper show that the one-level preconditioner can still remain robust.

1.2. The preconditioner. Our algorithm is a variation of the simple one-level additive Schwarz method and is based on a set of open polyhedral subdomains $\{\Omega_\ell\}_{\ell=1}^N$, forming an overlapping cover of Ω. We assume that each $\overline{\Omega_\ell}$ is nonempty and is a union of elements of the mesh $\mathcal{T}^h$. The key component of the preconditioner for (1.5) is the solution of discrete "local" versions of (1.1):

$$-\Delta u - (k^2 + \mathrm{i}\varepsilon)u = f \quad \text{on} \quad \Omega_\ell, \tag{1.9}$$

subject to boundary conditions

$$\frac{\partial u}{\partial n} - \mathrm{i}\eta u = 0 \quad \text{on} \quad \partial\Omega_\ell \backslash \Gamma_D \quad \text{(assumed nonempty)}, \quad \text{and} \quad u = 0 \quad \text{on} \quad \partial\Omega_\ell \cap \Gamma_D. \tag{1.10}$$

We assume that if $\partial\Omega_\ell \cap \Gamma_D \neq \emptyset$, then it has positive surface measure. Because Ω_ℓ consists of a union of fine grid elements, $\partial\Omega_\ell \cap \Gamma_D$ then contains at least one fine grid element.

To connect these local problems, we use a partition of unity $\{\chi_\ell\}_{\ell=1}^N$ with properties

$$\text{for each } \ell: \ \chi_\ell : \overline{\Omega} \to \mathbb{R}, \quad \operatorname{supp}\chi_\ell \subseteq \overline{\Omega}_\ell \quad \text{and} \quad 0 \le \chi_\ell(\boldsymbol{x}) \le 1, \quad \text{when } \boldsymbol{x} \in \overline{\Omega}, \tag{1.11}$$

and such that

$$\sum_\ell \chi_\ell(\boldsymbol{x}) \ = \ 1 \quad \text{for all} \ \ \boldsymbol{x} \in \overline{\Omega}.$$

(Here we define $\operatorname{supp}\chi_\ell := \{\boldsymbol{x} \in \overline{\Omega} : \chi_\ell(\boldsymbol{x}) \neq 0\}$.)

The finite-element space $\mathcal{V}^h \subset H^1_{0,D}(\Omega)$ underlying (1.5) is assumed to have a nodal basis so that each $v_h \in \mathcal{V}^h$ is uniquely determined by its values $\{V_p := v_h(\boldsymbol{x}_p),\ p \in \mathcal{I}^h\}$, at nodes $\{\boldsymbol{x}_p : p \in \mathcal{I}^h\} \subset \overline{\Omega}$ (where $\mathcal{I}^h$ is a suitable index set). Nodes on the subdomain $\overline{\Omega_\ell}$ are denoted $\{\boldsymbol{x}_p : p \in \mathcal{I}^h(\overline{\Omega_\ell})\}$. Using this notation, we can define a restriction matrix R_ℓ that uses χ_ℓ to map a nodal vector defined on $\overline{\Omega}$ to a nodal vector on $\overline{\Omega_\ell}$:

$$(R_\ell \mathbf{V})_p \ = \ \chi_\ell(\boldsymbol{x}_p) V_p, \quad p \in \mathcal{I}^h(\overline{\Omega_\ell}). \tag{1.12}$$

We denote by $A_{\varepsilon,\ell}$ the matrix obtained by approximating (1.9) and (1.10) in $\mathcal{V}^h$ (restricted to $\overline{\Omega_\ell}$); this matrix is a local analogue of the matrix A_ε in (1.5). Our preconditioner for A_ε is then simply

$$B_\varepsilon^{-1} := \sum_{\ell=1}^N R_\ell^\top (A_{\varepsilon,\ell})^{-1} R_\ell, \tag{1.13}$$

where $R_\ell^\top$ is the transpose of R_ℓ. Hence the action of B_ε^{-1} consists of N parallel "local impedance solves" added up with the aid of appropriate restrictions/prolongations.

B_ε^{-1} coincides with the "OBDD-H" preconditioner proposed (without theory) by Kimn and Sarkis in [37] (also called "SORAS"; see, e.g., [12, section 7.7.2]).

In the analysis we use the k-dependent inner product and norm:

$$\langle \mathbf{V}, \mathbf{W} \rangle_{D_k} := \mathbf{W}^* D_k \mathbf{V}, \quad \|\mathbf{V}\|_{D_k} = \langle \mathbf{V}, \mathbf{V} \rangle_{D_k}^{1/2}, \quad \text{where} \quad D_k = (S + k^2 M). \tag{1.14}$$

In fact, D_k is the stiffness matrix arising from approximating via the Galerkin method in $\mathcal{V}^h$ the Helmholtz energy norm

$$\|v\|_{1,k} := (v, v)_{1,k}^{1/2}, \quad \text{where} \quad (v, w)_{1,k} := (\nabla v, \nabla w)_{L^2(\Omega)} + k^2 (v, w)_{L^2(\Omega)}. \tag{1.15}$$

When $\widetilde{\Omega}$ is any subdomain of Ω we write $(\cdot, \cdot)_{1,k,\widetilde{\Omega}}$ and $\|\cdot\|_{1,k,\widetilde{\Omega}}$ for the corresponding inner product and norm on $\widetilde{\Omega}$.

1.3. Related literature. There have been two important recent ideas that have had a large effect on the field of iterative solvers for the Helmholtz equation. The first is the "shifted Laplace" preconditioner, arising from initial ideas in [1, 41], and then developed and advocated in [23, 21, 57]. Since the fundamental solution of (1.1) enjoys "Laplace-like" decay when ε is large enough, the "shifted Laplace" preconditioner uses a multigrid approximation of the absorptive problem to precondition the "pure Helmholtz" problem $\varepsilon = 0$.

The second concerns a class of multiplicative domain decomposition methods that fall under the general heading of "sweeping," e.g., [17, 18, 19, 54, 9, 51, 58, 26]. Restricting to a simple context, suppose (1.1) is discretized on a tensor product grid on the unit square and the unknowns are ordered lexicographically, yielding a block tridiagonal system matrix, each block corresponding to a row of nodes. Sweeping methods can be thought of as approximate block-elimination methods for this system. The Schur complement that arises in the block-elimination at a given line corresponds to the solution of a Helmholtz problem in the domain below that line, and these problems can be suitably truncated, to thinner strips either by "moving perfectly matched layer (PML)" or $\mathcal{H}$-matrix approximation. The polarized trace algorithm [58] takes this idea a step further by precomputing and compressing the solution operators on each strip, expediting the online process.

Both of these ideas have led to computation of challenging industrial strength applications, but neither of them has a rigorous theory. For "sweeping," the underpinning physical principle applies only to rectangular two-dimensional (2-d) domains and tensor-product discretizations (since the relevant low-rank result [45] does not hold for general domains and discretizations [20]), and to the elimination of nodes in blocks, each consisting of a small number of rows. Although the overarching principle of sweeping methods is serial, there have been considerable innovations to enhance parallel efficiency. For example, [51, 58] propose recursive subdivision of the inner solves in each multiplicative sweeping step. Very recently, [55] proposed the "L-sweeps" algorithm in which information propagates in a 90-degree cone, allowing checkerboard domain decomposition. In [42] an overlapping domain decomposition solver is proposed, with independent subdomain solves at each step.

On the other hand, the "shifted Laplace" algorithm is not, in general, robust with respect to k, since the choice $|\varepsilon| \sim k^2$, which is needed to make multigrid work [11], turns out to be too large a perturbation of the pure Helmholtz problem to remain robust as $k \to \infty$. Although recent enhancements based on deflation [53, 52, 22, 14] have greatly improved the shifted Laplace preconditioner, a full theory is still missing. A recent survey of shifted Laplace and related preconditioners is given in [40].

Domain decomposition methods offer the attractive feature that their coarse grid and local problems can be adapted to allow for "wave-like" behavior. There is a large literature (mostly empirical) on this (see, e.g., [3, 24, 25, 37, 38, 33, 34, 28]). A recent example is [44], which proposes a multiplicative overlapping domain decompositon method as a smoother in a multigrid algorithm for Helmholtz problems discretized by the continuous interior penalty method. However, there is no rigorous theory when k is large for methods with *either* many subdomains of general shape *or* coarse grids. The paper [30] provided the first such rigorous analysis for the problem with absorption, but the bounds for $|\varepsilon| \ll k^2$ in [30] were very pessimistic. The current paper extends this line of research to the case when wave-like components are inserted into the domain decomposition method. The results we obtain for the one-level method (i.e., with no coarse solver) with impedance boundary conditions on the subdomains give practical bounds for much lower levels of absorption than in [30].

Finally, we remark that domain decomposition methods (with and without global coarse solver) for the case when k is fixed and $h \to 0$ are, in principle, analyzed in Cai and Widlund [8], since for small enough h the Laplacian becomes the dominant term in the discrete Helmholtz equation. However, the current paper concentrates instead on analysis for the more challenging case that allows $k \to \infty$.

1.4. Cost of the preconditioner. Here we discuss the cost of the preconditioner (1.13), along with its possible approximations. We also give a brief comparison with other preconditioners. The action of (1.13) requires the solution of $\mathcal{O}(H^{-d})$ subproblems each of size $\mathcal{O}((H/h)^d)$, with d being the physical dimension. If $h \sim k^{-\gamma}$ with $\gamma \geq 1$ and $H \sim k^{-\alpha}$, with $0 < \alpha < \gamma$, then the dimension of the global system grows quickly with k, having dimension $n \sim k^{\gamma d}$. The action of the preconditioner then requires the solution of $\mathcal{O}(k^{\alpha d})$ subproblems, each of size $\mathcal{O}(k^{(\gamma-\alpha)d})$. Since $k \sim n^{1/\gamma d}$, this is equivalent to

$$\mathcal{O}(n^{\alpha/\gamma}) \quad \text{independent subproblems, each of size} \quad \mathcal{O}(n^{(1-\alpha/\gamma)}). \tag{1.16}$$

In the case $\alpha = 0.5$ (seen in Table 2 in section 4 to have an iteration count growing slowly with k) and the case $\gamma = 1.5$ (needed for accuracy of linear elements), the preconditioner has

$$\mathcal{O}(n^{1/3}) \quad \text{independent subproblems, each of size} \quad \mathcal{O}(n^{2/3}). \tag{1.17}$$

In the cases $\alpha = 0.5$ and $\gamma = 1$ (e.g., a fixed number of grid points per wavelength, commonly used in practice and reasonable for higher order methods), the preconditioner has

$$\mathcal{O}(n^{1/2}) \quad \text{independent subproblems, each of size} \quad \mathcal{O}(n^{1/2}). \tag{1.18}$$

These subproblem sizes are comparable to those arising from the (very successful) sweeping preconditioners, although (as pointed out above) the systems arising in sweeping are on thin rectangular subdomains and hence have beneficial special structure. In sweeping methods, an approximate inverse of A is computed by an approximate $LDL^\top$ factorization. In the moving PML variant (formulated for a cubic domain with tensor product grid and appropriate boundary conditions) one solves sequentially $\mathcal{O}(n^{1/d})$ subproblems (on slices of the domain), each of dimension $\mathcal{O}(n^{(1-1/d)})$. When $d = 3$ this coincides with (1.17) and when $d = 2$ it coincides with (1.18). The sweeping method in its basic format [18] is multiplicative, whereas our preconditioner is fundamentally additive. On the other hand, sweeping provides an approximate inverse of

the Helmholtz operator, while our aim here is only to provide a good preconditioner (a somewhat weaker requirement). As a result, our method is applicable in much more general geometrical situations.

Several other practical implementations of the preconditioner analyzed here have been tested. For example, [5] reduced the subproblem size and added a coarse grid solver to reduce iteration count. Although, now not completely robust as k (and hence n) increases, a slow growth of iteration count of about $\mathcal{O}(n^{0.1})$ for three-dimensional (3-d) Helmholtz problems of size up to $n = 10^7$ was observed. A similar method was used for 3-d Maxwell systems in [4, 6], where good parallel performance was reported on systems of size up to 1 billion. Here, the fact that absorption is added into the preconditioner turns out to be advantageous in practice, since the absorptive coarse grid problem can be quickly solved with an inner iterative method and does not dominate the overall cost.

Another approach to reduce the cost of the preconditioner is to observe that the local impedance solves are local copies of the original problem (but on smaller domains and hence with smaller effective wavenumber). This allows them to be quickly resolved by an (inner) preconditioned GMRES combined with the same preconditioner. A preliminary (serial) implementation of this method is given in [31, section 5.2.2] where, on a 2-d domain of size $\mathcal{O}(1)$, the outer preconditioner was formulated on subdomains of size $\mathcal{O}(k^{-0.4})$ and the inner preconditioner on subdomains of size $\mathcal{O}(k^{-0.8})$. This was implemented on a fine discretization with ten grid points per wavelength, in which only very small problems of size $\mathcal{O}((k^{-0/8}/k^{-1})^2) = \mathcal{O}(k^{0.4})$ had to be solved directly. Results for k up to 300 are given in [31, section 5.2.2], showing very low inner iteration counts and outer iteration counts growing slowly ($\sim \mathcal{O}(n^{0.2})$). For $k = 300$, direct solvers were needed for systems of a size of only a few hundred. The idea of recursive subdivision of subdomains also features heavily in efficient versions of sweeping [43], and also in the polarized trace algorithm [58].

1.5. Structure of the paper. In subsections 2.1 and 2.2 we provide key estimates for the local impedance solution operator at the continuous PDE level, and its discretization. The properties of the preconditioner are established via its interpretation as a sum of projections; this is set up in section 2.3. We prove the main results in section 3 and present numerical experiments in section 4. In Appendix A we give a rigorous basis for the discussion around (1.8).

2. Preliminaries. Throughout we write $a \lesssim b$ when there exists a $C > 0$, independent of all parameters of interest (here $\varepsilon, k, h, H, \delta$, Λ, and ℓ, with some of these defined later), such that $a \leq Cb$. We write $a \sim b$ if $a \lesssim b$ and $b \lesssim a$. We make the following basic assumptions on k, ε, and η throughout this paper.

ASSUMPTION 2.1. *The parameters k, ε, and η satisfy*

$$k \gtrsim 1, \quad 0 \leq |\varepsilon| \leq k^2, \quad \text{and} \quad |\eta| \sim k. \tag{2.1}$$

We recall the inequalities (valid for all $a, b > 0$ and $\epsilon > 0$)

$$2ab \leq \frac{a^2}{\epsilon} + \epsilon b^2, \quad \text{and} \quad \frac{1}{\sqrt{2}}(a+b) \leq \sqrt{a^2+b^2} \leq a+b. \tag{2.2}$$

2.1. A priori estimates. The basic well-posedness of (1.3) is classical.

PROPOSITION 2.2. *If either* (i) $\varepsilon > 0$ *and* $\Re(\eta) > 0$, *or* (ii) $\varepsilon < 0$ *and* $\Re(\eta) < 0$, *or* (iii) $\varepsilon = 0$ *and* $\Re(\eta) \neq 0$, *the problem* (1.4) *has a unique solution.*

Sketch proof. For cases (i) and (ii), uniqueness can be established by taking $v = u$ and $F = 0$ in the weak form (1.3) and then taking the imaginary part to show that $u = 0$. Case (iii) is the standard "pure Helmholtz" case; uniqueness can be obtained by the unique continuation principle (e.g., [46, Remark 8.1.1], [29, section 3]). Existence then follows for all cases via the Fredholm alternative, since a_ε satisfies a Gårding inequality. ∎

In the domain decomposition method below we will be interested in local impedance solves on subdomains that may shrink in diameter as $k \to \infty$. For this reason we introduce the following.

DEFINITION 2.3 (characteristic length scale). *A domain has* characteristic length scale L *if its diameter* $\sim L$*, its surface area* $\sim L^{d-1}$*, and its volume* $\sim L^d$*.*

LEMMA 2.4 (continuity and coercivity of the sesquilinear form a_ε).

(i) *Assume that* Ω *has characteristic length scale* L *and that* ε *and* η *satisfy* (2.1)*. Then the sesquilinear form* a_ε *is continuous, i.e.,*

$$|a_\varepsilon(u,v)| \le C_{\rm cont}\|u\|_{1,k}\|v\|_{1,k}, \quad \text{with} \quad C_{\rm cont} \lesssim \left(1+(kL)^{-1}\right) \quad \text{for all } u,v \in H^1(\Omega).$$

(ii) *Let* $\sqrt{k^2+\mathrm{i}\varepsilon}$ *be defined via the square root with the branch cut on the positive real axis. If* η *satisfies*

$$\Re\big(\eta\overline{\sqrt{k^2+\mathrm{i}\varepsilon}}\big) \ge 0, \tag{2.3}$$

then a_ε *is coercive, i.e.,*

$$|a_\varepsilon(v,v)| \;\gtrsim\; C_{\rm coer}\|v\|^2_{1,k}, \quad \text{with} \quad C_{\rm coer} \sim \frac{|\varepsilon|}{k^2} \quad \text{for all } v \in H^1(\Omega).$$

Proof. The assertion (ii) is Lemma 2.4 in [30] (note that the omitted constants in that result do not depend on L). The assertion (i) follows from the Cauchy–Schwarz inequality and the multiplicative trace inequality,

$$\|v\|^2_{L^2(\Gamma)} \lesssim \left(\frac{1}{L}\,\|v\|^2_{L^2(\Omega)} + \|\nabla v\|_{L^2(\Omega)}\,\|v\|_{L^2(\Omega)}\right)$$

(see, e.g., [32, last equation on p. 41]), and the inequalities (2.2). ∎

Remark 2.5 (adjoint coercivity). The definition of $\sqrt{k^2+\mathrm{i}\varepsilon}$ implies that when η is chosen to satisfy (2.3), the coercivity constant for a_ε is exactly the same as the coercivity constant for the sesquilinear form for the adjoint problem obtained by replacing ε by $-\varepsilon$ and η by $-\eta$.

DEFINITION 2.6. *A Lipschitz open set* D *is called* starshaped with respect to a ball *if there exist a point* $\boldsymbol{x}_0 \in D$ *and a* $\gamma > 0$ *such that the position vector of any point* $\boldsymbol{x} \in D$ *satisfies* $(\boldsymbol{x}-\boldsymbol{x}_0)\cdot\mathbf{n}(\boldsymbol{x}) \ge \gamma$ *when the normal vector* $\mathbf{n}(\boldsymbol{x})$ *is defined; see, e.g.,* [49, Lemma 5.4.1].

THEOREM 2.7 (a priori bound on solution of (1.3)). *Let* Ω *be starshaped with respect to a ball and have characteristic length scale* L*, and recall that we have assumed that* Γ_I *has positive measure. Let* u *be* either *the solution to* (1.3) *with* $f \in L^2(D)$ *and* $g = 0$*,* or *the solution to the adjoint problem under the same assumptions on* f *and* g*. Then, there exist* C_1, C_2 *(independent of* k, ε, η*, and* L*) such that*

$$\|u\|_{1,k} \le C_1 L\,\|f\|_{L^2(\Omega)}, \quad \text{provided that} \quad \frac{|\varepsilon|L}{k} \le C_2. \tag{2.4}$$

Proof. This result is essentially given by [27, Theorem 2.9 and Remark 2.5], except the dependence of the constants on L is not kept track of there. To see that the condition $|\varepsilon|/k \le c$ in [27, Theorem 2.9] is really the right-hand inequality in (2.4), one needs to examine the argument near the end of the proof of [27, Theorem 2.9] (just before Remark 2.16) and observe that $R\,(:= \sup_{\boldsymbol{x}\in\Omega} |\boldsymbol{x}|) \sim L$. To see why the bound (2.4) has the factor of L on the right-hand side, observe that choosing $\delta_3 = 1/(2R)$ and $\delta_4 \sim k^2$ in the proof of [27, Theorem 2.9] means that, in [27, eq. (2.29)], the factor multiplying $\|f\|^2_{L^2(\Omega)}$ is $\sim L^2$. (The L-explicit bound (2.4) in the case $\varepsilon = 0$ is also obtained in [50, Remark 3.6].) ∎

For simplicity, in the rest of this paper we assume that either $\eta = \mathrm{sign}(\varepsilon)k$ or $\eta = \sqrt{k^2 + \mathrm{i}\varepsilon}$; observe that both of these choices satisfy the requirements on η in (2.1), the conditions for uniqueness of the solution of (1.3) in Proposition 2.2, and the more-restrictive condition for coercivity (2.3) (see [30, Remark 2.5]).

2.2. Finite element method and subproblems. Let $\mathcal{T}^h$ be a family of conforming simplicial meshes that are shape regular as the mesh diameter $h \to 0$. A typical element of $\mathcal{T}^h$ is written $\tau \in \mathcal{T}^h$ and is considered as a closed subset of $\overline{\Omega}$. Our approximation space $\mathcal{V}^h$ is then the space of all continuous functions on Ω that are polynomial of (total) degree $r-1$ with $r \ge 2$ (when restricted to any τ) and vanish on Γ_D. We assume we have a nodal basis for this space (for example, the standard Lagrange basis), i.e., with nodes $\mathcal{N}^h = \{\boldsymbol{x}_q : q \in \mathcal{I}^h\}$, where $\mathcal{I}^h$ is a suitable index set and corresponding basis $\{\phi_p : p \in \mathcal{I}^h\}$ with $\phi_p(\boldsymbol{x}_q) = \delta_{p,q}$. For any continuous function g on $\overline{\Omega}$, we introduce the standard nodal interpolation operator $\Pi^h g = \sum_{p\in\mathcal{I}^h} g(\boldsymbol{x}_p)\phi_p$, and we assume the standard error estimate (e.g., [10, section 3.1]):

$$\|(I-\Pi^h)v\|_{L^2(\tau)} + h|(I-\Pi^h)v|_{H^1(\tau)} \;\le\; Ch^r|v|_{H^r(\tau)} \quad \text{for all } v \in H^r(\Omega), \tag{2.5}$$

for each $\tau \in \mathcal{T}^h$, with C independent of τ, provided $v \in H^r(\tau)$. The Galerkin approximation of (1.3) in the space $\mathcal{V}^h$ is equivalent to the linear system (1.5) where $F_\ell := \int_\Omega f\phi_\ell + \int_{\Gamma_I} g\phi_\ell$, and

$$S_{\ell,m} = \int_\Omega \nabla\phi_\ell \cdot \nabla\phi_m, \quad M_{\ell,m} = \int_\Omega \phi_\ell\phi_m, \quad N_{\ell,m} = \int_\Gamma \phi_\ell\phi_m, \quad \ell, m \in \mathcal{I}^h\ . \tag{2.6}$$

We assume that the subdomains Ω_ℓ introduced in section 1.2 are Lipschitz polyhedra (polygons in 2-d) that are *shape regular with parameter* H_ℓ in the sense that each Ω_ℓ has characteristic length scale H_ℓ, and we set $H = \max_\ell H_\ell$. In our analysis we allow H to depend on k in such a way that H could approach 0 as $k \to \infty$. Some of the results below require that each Ω_ℓ is starshaped with respect to a ball, with the corresponding parameters $\gamma = \gamma_\ell$ in Definition 2.6 satisfying $\gamma_\ell \ge \gamma_* > 0$ for all ℓ. We describe this property by saying that the Ω_ℓ are *starshaped with respect to a ball, uniformly in* ℓ.

Concerning the overlap, for each $\ell = 1, \dots, N$, let $\mathring{\Omega}_\ell$ denote the part of Ω_ℓ that is not overlapped by any other subdomains. (Note that $\mathring{\Omega}_\ell = \emptyset$ is possible.) For $\mu > 0$ let $\Omega_{\ell,\mu}$ denote the set of points in Ω_ℓ, every element of which is a distance no more than μ from the interior boundary $\partial\Omega_\ell\backslash\Gamma$. Then we assume that there exist constants $0 < \delta_\ell \lesssim H$ and $0 < b < 1$ such that, for each $\ell = 1, \dots, N$,

$$\Omega_{\ell,b\delta_\ell} \subset \Omega_\ell\backslash\mathring{\Omega}_\ell \subset \Omega_{\ell,\delta_\ell}. \tag{2.7}$$

The case when $\delta_\ell \geq cH_\ell$ for some constant c independent of ℓ is called *generous* overlap. In Figure 1 we depict a typical subdomain, with its parts which are overlapped by its neighbors and its (possibly) nonoverlapped part.

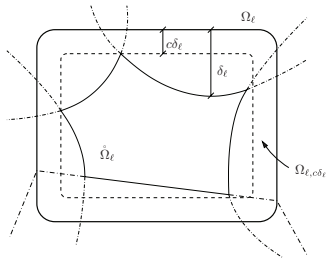

FIG. 1. *The overlap parameter δ_ℓ, the "interior" $\mathring{\Omega}_\ell$, and the "near-boundary subset" $\Omega_{\ell,c\delta_\ell}$ for a particular example of a subdomain Ω_ℓ being overlapped by its neighbors.*

We introduce the parameter

$$\delta := \min_{\ell=1,\ldots,N} \delta_\ell. \tag{2.8}$$

We make the *finite-overlap assumption*. There exists a finite $\Lambda > 1$ independent of N such that

$$\Lambda \;=\; \max\big\{\#\Lambda(\ell) : \ell = 1,\ldots,N\big\}, \quad \text{where} \quad \Lambda(\ell) = \big\{\ell' : \overline{\Omega_\ell} \cap \overline{\Omega_{\ell'}} \neq \emptyset\big\}\ . \tag{2.9}$$

It follows immediately from (2.9) that, for all $v \in L^2(\Omega)$,

$$\sum_{\ell=1}^{N} \|v\|^2_{L^2(\Omega_\ell)} \leq \Lambda \|v\|^2_{L^2(\Omega)} \quad \text{and} \quad \sum_{\ell=1}^{N} \|v\|^2_{1,k,\Omega_\ell} \leq \Lambda \|v\|^2_{1,k}\,, \text{ when } \ v \in H^1(\Omega). \tag{2.10}$$

For each ℓ, we introduce the space of finite-element functions on $\overline{\Omega_\ell}$ given by $\mathcal{V}_\ell^h := \{v_h|_{\overline{\Omega_\ell}} : v_h \in \mathcal{V}^h\}$. Recalling that functions in $\mathcal{V}^h$ vanish on the (outer) Dirichlet boundary Γ_D, functions in $\mathcal{V}_\ell^h$ also vanish on $\partial\Omega_\ell \cap \Gamma_D$ (which contains at least one element if it is nonempty), but are otherwise unconstrained. The local impedance sesquilinear form on Ω_ℓ is

$$a_{\varepsilon,\ell}(v,w) \;:=\; \int_{\Omega_\ell} \Big(\nabla v \cdot \nabla \overline{w} - (k^2 + \mathrm{i}\varepsilon) v\overline{w}\Big) \;-\; \mathrm{i}\eta \int_{\partial\Omega_\ell \setminus \Gamma_D} v\overline{w} \tag{2.11}$$

for $v, w \in H^1_D(\Omega_\ell) := \{z \in H^1(\Omega_\ell) : z = 0 \text{ on } \partial\Omega_\ell \cap \Gamma_D\}$. For general $F_\ell \in (H^1(\Omega_\ell))'$, the continuous local impedance problem is as follows: find $u_\ell \in H^1_D(\Omega_\ell)$ such that

$$a_{\varepsilon,\ell}(u_\ell, v_\ell) \;=\; F_\ell(v_\ell) \quad \text{for all} \quad v_\ell \in H^1_D(\Omega_\ell); \tag{2.12}$$

this problem is well-posed by Proposition 2.2 and its finite-element approximation is as follows: find $u_{h,\ell} \in \mathcal{V}_\ell^h$ such that

$$a_{\varepsilon,\ell}(u_{h,\ell}, v_{h,\ell}) \;=\; F_\ell(v_{h,\ell}) \quad \text{for all} \quad v_{h,\ell} \in \mathcal{V}_\ell^h. \tag{2.13}$$

The system matrix arising from (2.13) is $\big(A_{\varepsilon,\ell}\big)_{i,j} := a_{\varepsilon,\ell}(\phi_j, \phi_i)$ for $i, j \in \mathcal{I}^h(\overline{\Omega_\ell})$.

THEOREM 2.8 (bounds on the solutions of the local problems (2.13)).

(i) *For all* $|\varepsilon| > 0$*, and for any mesh size* h*,* (2.13) *has a unique solution* $u_{h,\ell}$ *which satisfies*

$$\|u_{h,\ell}\|_{1,k,\Omega_\ell} \lesssim \Theta(\varepsilon, H_\ell, k) \max_{v_h \in \mathcal{V}_\ell^h} \left(\frac{|F(v_h)|}{\|v_h\|_{1,k,\Omega_\ell}} \right), \tag{2.14}$$

with

$$\Theta(\varepsilon, H_\ell, k) = k^2/|\varepsilon|. \tag{2.15}$$

(ii) *If each* Ω_ℓ *is starshaped with respect to a ball uniformly in* ℓ*, then for all* $|\varepsilon| \geq 0$*, there exists a mesh threshold function* $\overline{h}(k,r)$ *such that when* $h \leq \overline{h}(k,r)$*,* (2.13) *has a unique solution* $u_{h,\ell}$ *which satisfies* (2.14) *with*

$$\Theta(\varepsilon, H_\ell, k) = \min\left\{(1 + kH_\ell), k^2/|\varepsilon|\right\}, \tag{2.16}$$

where we adopt the convention that $\Theta(0, H, k) = 1 + Hk$*.*

Proof. The result (i) is a consequence of Lemma 2.4 and the Lax–Milgram lemma. The result (ii) follows from the fact (used in the case of Helmholtz problems by the authors of [47, 48] and their associated work) that when a sesquilinear form satisfies a Gårding inequality and the solution of the variational problem is unique, a "Schatz-type" argument obtains quasi-optimality under conditions on the approximability of the adjoint problem, and then the Gårding inequality can be used to verify a discrete inf-sup condition. Indeed, following the proof of [47, Theorem 4.2] and using the bound (2.4) and the fact that Ω_ℓ has characteristic length scale H_ℓ, we find that, when $|\varepsilon|H_\ell/k \leq C_2$,

$$\inf_{0 \neq v_h \in \mathcal{V}_\ell^h} \sup_{0 \neq w_h \in \mathcal{V}_\ell^h} \frac{|a_{\varepsilon,\ell}(v_h, w_h)|}{\|v_h\|_{1,k}\|w_h\|_{1,k}} \geq \frac{1}{2 + C_{\rm cont}^{-1} + C_1 kH_\ell}. \tag{2.17}$$

Then, from (2.13),

$$\|u_{h,\ell}\|_{1,k,\Omega_\ell} \lesssim (1 + kH_\ell) \sup_{0 \neq v_h \in \mathcal{V}_\ell^h} \frac{|F(v_h)|}{\|v_h\|_{1,k,\Omega_\ell}}, \tag{2.18}$$

when $|\varepsilon|H_\ell/k \leq C_2$. If $|\varepsilon|H_\ell/k > C_2$, then $1 + H_\ell k > C_2 k^2/|\varepsilon|$ and (2.16) follows from (2.14). ∎

Remark 2.9 (the mesh-threshold function $\overline{h}(k,r)$). Bounds on $\overline{h}(k,r)$ are discussed in detail in [48, subsections 5.1.2 and 5.2]. For 2-d polygonal domains, $k(hk/(r-1))^{r-1}$ is required to be sufficiently small (see [48, eq. (5.13)]), equivalently h is a sufficiently small multiple of $(r-1)k^{-(r/(r-1))}$. Therefore, when $r = 2$ we require hk^2 small, but the requirement relaxes as r increases. In one dimension (1-d), numerical experiments indicate that the requirement hk^2 sufficiently small is necessary for quasi-optimality [36, Figures 7–9], [35, section 4.5.4 and Figure 4.12]. The theoretical benefit of requiring $h \leq \overline{h}(k,r)$ is that the estimate (2.14) holds uniformly over all choices of overlapping star-shaped subdomains Ω_ℓ, each of which has characteristic length H_ℓ. However, to the best of our knowledge, the requirement $h \sim k^{-2}$ is never imposed in practical computations.

If one is only concerned with ensuring solvability, a weaker requirement on h arises. In 1-d, the relative error in both the H^1-seminorm and the L^2-norm is bounded

independently of k if $hk^{3/2}$ is sufficiently small [36, eq. (3.25)], [35, eq. (4.5.15)], with numerical experiments indicating that this is sharp [36, Figure 11], [35, Figure 4.13]. Numerical experiments in [2, section 3] showed that, at least for certain 2-d problems, the relative error in the L^2-norm is bounded independently of k if $hk^{3/2}$ is sufficiently small; this fact has recently been proved in [39]. In [13], under certain regularity assumptions, it has been proved (in 2-d and 3-d) that, if $h^{2(r-1)}k^{2r-1}$ is small enough, then the H^1 error is of order $h^{2(r-1)}k^{2r-1}$. Thus, e.g., when $r = 2$, taking $h \lesssim k^{-3/2}$ ensures that the problem is solvable and the error remains bounded as k increases. This discussion is all for domains of diameter $\mathcal{O}(1)$; for subdomains of decreasing diameter $\mathcal{O}(H_\ell)$, the effective wavenumber is reduced to $\mathcal{O}(kH_\ell)$, and so the requirement on h is even weaker.

2.3. Projection operators. We now give more detail about the partition of unity $\{\chi_\ell\}$ and the restiction and prolongation matrices $R_\ell, R_\ell^\top$ discussed in section 1.2. Note that since the subdomains are assumed to be unions of fine grid elements, their boundaries (and the boundaries of their supports) are fine-grid dependent. This is standard for domain decomposition methods (e.g., [56, p. 57]). We choose the functions χ_ℓ to be continuous piecewise linear on the mesh $\mathcal{T}^h$, satisfying

$$\|\nabla\chi_\ell\|_{\infty,\tau} \lesssim \delta_\ell^{-1} \quad \text{for all} \quad \tau \in \mathcal{T}_h, \tag{2.19}$$

where the hidden constant is also required to be independent of the element τ. A partition of unity satisfying this condition is explicitly constructed in [56, section 3.2].

We will use the operator $\Pi^h \circ \chi_\ell$. In fact, if $w_{h,\ell} \in \mathcal{V}_\ell^h$ with nodal values $\mathbf{W}$, then

$$\Pi^h\big(\chi_\ell w_{h,\ell}\big) = \sum_{p\in\mathcal{I}^h} \big(R_\ell^T\mathbf{W}\big)_p \phi_p,$$

where R_ℓ is defined by (1.12), and thus $\Pi^h \circ \chi_\ell$ defines a prolongation from $\mathcal{V}_\ell^h$ to $\mathcal{V}^h$.

To analyze the preconditioner (1.13), we define the projections $Q_{\varepsilon,\ell}^h : H^1(\Omega) \to \mathcal{V}_\ell^h$, by requiring that, given $v \in H^1(\Omega)$, $Q_{\varepsilon,\ell}^h v \in \mathcal{V}_\ell^h$ satisfies

$$a_{\varepsilon,\ell}(Q_{\varepsilon,\ell}^h v, w_{h,\ell}) = a_\varepsilon(v, \Pi^h(\chi_\ell w_{h,\ell})) \quad \text{for all} \quad w_{h,\ell} \in \mathcal{V}_\ell^h. \tag{2.20}$$

For $|\varepsilon| > 0$, $Q_{\varepsilon,\ell}$ is well-defined by part (i) of Theorem 2.8. For $\varepsilon = 0$, $Q_{\varepsilon,\ell}$ is well-defined for all $h \le \overline{h}(k,r)$ by part (ii) of Theorem 2.8. To combine the actions of these local projections additively, we define the global projection by

$$Q_\varepsilon^h := \sum_{\ell=1}^N \Pi^h(\chi_\ell Q_{\varepsilon,\ell}^h), \tag{2.21}$$

where again each term in the sum can be interpreted as an element of $H^1(\Omega)$. The following theorem shows that the matrix representation of Q_ε^h restricted to $\mathcal{V}^h$ coincides with the preconditioned matrix $B_\varepsilon^{-1}A_\varepsilon$. This result uses the weighted inner product defined in (1.14).

THEOREM 2.10 (from projection operators to matrices). *Let $v_h \in \mathcal{V}^h$, with nodal values given in the vector $\mathbf{V}$. Then, for any ℓ, when the function $Q_{\varepsilon,\ell}^h v_h \in \mathcal{V}_\ell^h$ is well-defined it has nodal vector*

$$\mathbf{W} = A_{\varepsilon,\ell}^{-1} R_\ell A_\varepsilon \mathbf{V}. \tag{2.22}$$

Consequently, for any $u_h, v_h \in \mathcal{V}^h$,

$$(u_h, Q_\varepsilon^h v_h)_{1,k} = \langle \mathbf{U}, B_\varepsilon^{-1}A_\varepsilon\mathbf{V}\rangle_{D_k}. \tag{2.23}$$

Proof. With $\mathbf{W}$ as given in (2.22), we have $(A_{\varepsilon,\ell}\mathbf{W})_q = (R_\ell A\mathbf{V})_q$, for all $q \in \mathcal{I}^h(\overline{\Omega_\ell})$, and so (recalling the definition of R_ℓ in (1.12)),

$$\sum_{p\in\mathcal{I}^h(\overline{\Omega_\ell})} a_{\varepsilon,\ell}(\phi_p,\phi_q)W_p = \chi_\ell(x_q)\sum_{p\in\mathcal{I}^h(\overline{\Omega})} a_\varepsilon(\phi_p,\phi_q)V_p \quad \text{for each} \quad q\in\mathcal{I}^h(\overline{\Omega_\ell}).$$

Then, letting $w_h \in \mathcal{V}_\ell^h, v_h \in \mathcal{V}^h$ be defined by the nodal values $\mathbf{W}$, $\mathbf{V}$, we have

$$a_{\varepsilon,\ell}(w_h,\phi_q) = a_\varepsilon(v_h,\chi_\ell(\boldsymbol{x}_q)\phi_q) \quad \text{for each} \quad q\in\mathcal{I}^h(\overline{\Omega_\ell}).$$

By multiplying by $v_h(\boldsymbol{x}_q)$ and using the definition of Π^h and summing over q, we then have that

$$a_{\varepsilon,\ell}(w_h,v_h) = a_\varepsilon(v_h,\Pi^h(\chi_\ell v_h)) \quad \text{for all} \quad v_h\in\mathcal{V}^h\ .$$

The definition of $Q^h_{\varepsilon,\ell}$ (2.20) and uniqueness then imply that $w_h = Q^h_{\varepsilon,\ell}v_h$ which proves (2.22). Recalling (1.14) and (2.21), we obtain as a consequence of (2.22) that

$$\begin{aligned}(u_h,Q^h_\varepsilon v_h)_{1,k} &= \sum_\ell (u_h,\Pi^h(\chi_\ell Q^h_{\varepsilon,\ell}v_h))_{1,k}\\ &= \sum_\ell \langle \mathbf{U}, R_\ell^\top A^{-1}_{\varepsilon,\ell}R_\ell A_\varepsilon\mathbf{V}\rangle_{D_k} = \langle \mathbf{U}, B_\varepsilon^{-1}A_\varepsilon\mathbf{V}\rangle_{D_k}.\end{aligned}$$ ∎

3. The main results.

3.1. Estimates involving the overlapping decomposition.

LEMMA 3.1 (estimates on norms involving χ_ℓ). *With δ_ℓ as defined in* (2.7),

$$\|\chi_\ell v\|^2_{1,k,\Omega_\ell} - 2\|v\|^2_{1,k,\Omega_\ell} \lesssim \frac{1}{(k\delta_\ell)^2}\|v\|^2_{1,k,\Omega_\ell} \quad \textit{for all } v\in H^1(\Omega_\ell), \tag{3.1}$$

$$\sum_{\ell=1}^N \|\chi_\ell v\|^2_{1,k,\Omega_\ell} \lesssim \Lambda\left(1+\frac{1}{(k\delta)^2}\right)\|v\|^2_{1,k} \quad \textit{for all } v\in H^1(\Omega), \tag{3.2}$$

$$\sum_{\ell=1}^N \|\chi_\ell^2 v\|^2_{1,k,\Omega_\ell} \lesssim \Lambda\left(1+\frac{1}{(k\delta)^2}\right)^2\|v\|^2_{1,k} \quad \textit{for all } v\in H^1(\Omega). \tag{3.3}$$

$$\sum_{\ell=1}^N \|\chi_\ell f\|^2_{L^2(\Omega_\ell)} \geq \frac{1}{\Lambda}\|f\|^2_{L^2(\Omega)} \quad \textit{for all } f\in L^2(\Omega), \tag{3.4}$$

$$\sum_{\ell=1}^N \|\chi_\ell f\|^2_{1,k,\Omega_\ell} \geq \frac{1}{\Lambda}\|f\|^2_{1,k} - C\frac{\Lambda}{k\delta}\|f\|^2_{1,k} \quad \textit{for all } f\in H^1(\Omega), \tag{3.5}$$

where C denotes a parameter-independent constant.

Proof. Using $\nabla(\chi_\ell v) = (\nabla\chi_\ell)v + \chi_\ell\nabla v$, (1.11), and (2.19), we have that, for some constant C,

$$|\nabla(\chi_\ell v)(\boldsymbol{x})|^2 \leq 2\left(\frac{C}{\delta_\ell^2}|v(\boldsymbol{x})|^2 + |\nabla v(\boldsymbol{x})|^2\right)$$

for all $\boldsymbol{x}\in\Omega_\ell$. Then

$$\|\chi_\ell v\|^2_{1,k,\Omega_\ell} \leq \frac{2C}{\delta_\ell^2}\|v\|^2_{L^2(\Omega_\ell)} + 2|v|^2_{H^1(\Omega_\ell)} + k^2\|v\|^2_{L^2(\Omega_\ell)} \leq 2\left(1+\frac{C}{(k\delta_\ell)^2}\right)\|v\|^2_{1,k,\Omega_\ell},$$

which yields the estimate (3.1).

From (2.10) and (2.8), we see that (3.2) follows from (3.1). The estimate (3.3) follows from two successive applications of (3.1), summing both sides of the resulting estimate over ℓ, and then using (2.10).

To prove (3.4), first define, for each $\boldsymbol{x} \in \overline{\Omega}$, a positive integer $m = m(\boldsymbol{x})$ by

$$m(\boldsymbol{x}) := \#\left\{\ell \in \{1,\dots,N\} : \boldsymbol{x} \in \operatorname{supp}\chi_\ell\right\}. \tag{3.6}$$

Note that, because $\operatorname{supp}(\chi_\ell) \subseteq \overline{\Omega}_\ell$, the assumption (2.9) ensures $1 \le m(\boldsymbol{x}) \le \Lambda$ for all $\boldsymbol{x} \in \Omega$. Then, for any integer $j \in \{1,\dots,\Lambda\}$, we define the subset of Ω: $D_j := \{\boldsymbol{x} \in \overline{\Omega} : m(\boldsymbol{x}) = j\}$, so that $\boldsymbol{x} \in D_j$ if and only if $\boldsymbol{x}$ lies in the supports of exactly j of the partition of unity functions $\{\chi_\ell\}$. Corresponding to these we also define the index sets:

$$\mathcal{D}(j) \;=\; \left\{\ell \in \{1,\dots,N\} : \operatorname{supp}\chi_\ell \cap D_j \neq \emptyset\right\}. \tag{3.7}$$

This notation is illustrated in section 4 in the context of the particular overlapping cover used there. As that example shows, some of the sets D_j can have zero Lebesgue measure as subsets of Ω.

Then, we have

$$\overline{\Omega} = \bigcup_{j=1}^{\Lambda} D_j \quad \text{and} \quad D_i \cap D_j = \emptyset \;\text{ if } i \neq j\,. \tag{3.8}$$

Moreover, for all $j = 1,\dots,\Lambda$,

$$\sum_{\ell\in\mathcal{D}(j)} \chi_\ell(\boldsymbol{x}) = 1 \quad \text{when} \;\; \boldsymbol{x} \in D_j\,. \tag{3.9}$$

Then, noting that $\#\{\ell \in \mathcal{D}(j) : \chi_\ell(\boldsymbol{x}) \neq 0\} = j \le \Lambda$ and using (3.9) and the Cauchy–Schwarz inequality we obtain, for all $\boldsymbol{x} \in D_j$,

$$1 \;=\; \left(\sum_{\ell\in\mathcal{D}(j)} \chi_\ell(\boldsymbol{x})\right)^2 \le j \sum_{\ell\in\mathcal{D}(j)} \chi_\ell^2(\boldsymbol{x}) \le \Lambda \sum_{\ell\in\mathcal{D}(j)} \chi_\ell^2(\boldsymbol{x})\,. \tag{3.10}$$

Using (3.8), (3.10), and (3.7), we find

$$\begin{aligned}
\sum_{\ell=1}^{N} \int_{\Omega_\ell} \chi_\ell^2(\boldsymbol{x})|f(\boldsymbol{x})|^2 \mathrm{d}\boldsymbol{x} \;&=\; \sum_{j=1}^{\Lambda}\sum_{\ell=1}^{N} \int_{\Omega_\ell\cap D_j} \chi_\ell^2(\boldsymbol{x})|f(\boldsymbol{x})|^2 \mathrm{d}\boldsymbol{x} \\
&=\; \sum_{j=1}^{\Lambda} \sum_{\ell\in\mathcal{D}(j)} \int_{\Omega_\ell\cap D_j} \chi_\ell^2(\boldsymbol{x})|f(\boldsymbol{x})|^2 \mathrm{d}\boldsymbol{x} \\
&=\; \sum_{j=1}^{\Lambda} \int_{D_j} \Big(\sum_{\ell\in\mathcal{D}(j)} \chi_\ell^2(\boldsymbol{x})\Big)|f(\boldsymbol{x})|^2 \mathrm{d}\boldsymbol{x} \;\ge\; \frac{1}{\Lambda}\sum_{j=1}^{\Lambda} \int_{D_j} |f(\boldsymbol{x})|^2 \mathrm{d}\boldsymbol{x} \;=\; \frac{1}{\Lambda}\int_{\Omega} |f(\boldsymbol{x})|^2 \mathrm{d}\boldsymbol{x},
\end{aligned}$$

which is (3.4). Finally, for (3.5), we use (1.11) and (2.19) to obtain

$$\|\chi_\ell f\|_{1,k,\Omega_\ell}^2 \;=\; k^2\|\chi_\ell f\|_{L^2(\Omega_\ell)}^2 + \|\chi_\ell|\nabla f|\|_{L^2(\Omega_\ell)}^2$$

$$+ 2\mathrm{Re}\int_{\Omega_\ell} \chi_\ell f \nabla\chi_\ell.\nabla\overline{f} + \||f|\nabla\chi_\ell|\|^2_{L^2(\Omega_\ell)}$$
$$\geq k^2\|\chi_\ell f\|^2_{L^2(\Omega_\ell)} + \||\chi_\ell|\nabla f|\|^2_{L^2(\Omega_\ell)} - \frac{C}{k\delta_\ell}\|f\|^2_{1,k,\Omega_\ell},$$

and the result is obtained by summing, and using (3.4), (2.10), and (2.8). ∎

Remark 3.2. The estimate (3.2) provides a "stable splitting," i.e., any $v \in H^1(\Omega)$ has a decomposition into components $\chi_\ell v \in H^1(\Omega_\ell)$, with $v = \sum_\ell \chi_\ell v$, so that sum of the squares of the energies of the components is bounded in terms of the square of the energy of v, with a constant that is independent of k, h, H, and δ, provided only that $k\delta \gtrsim 1$. Corollary 3.5 provides an analogous stable splitting for finite-element functions. This result is perhaps a little surprising since, for positive-definite elliptic problems, families of subdomains with decreasing diameter do not enjoy this property (and a coarse space is needed to restore it) [56]. Here the stable splitting holds without coarse space as $k \to \infty$ (i.e., for a family of Helmholtz problems of increasing difficulty). This includes, for example, subdomains of diameter $H \sim k^{-\alpha}$ with $\alpha \in [0, 1]$ and overlap $k^{-1} \lesssim \delta \leq H$.

LEMMA 3.3 (error in interpolation of $\chi_\ell w_h$). *Given $\ell \in \{1, \dots, N\}$, suppose $v_h \in \mathcal{V}^h_\ell$. Then*

$$\|(\mathrm{I} - \Pi^h)(\chi_l v_h)\|_{1,k,\Omega_l} \lesssim (1 + kh_\ell)\left(\frac{h_\ell}{\delta_\ell}\right)\|v_h\|_{H^1(\Omega_l)}, \tag{3.11}$$

where $h_\ell := \max_{\tau\subset\overline{\Omega_\ell}} h_\tau$, and the hidden constant is independent of ℓ.

Proof. For each element $\tau \in \mathcal{T}^h$ with $\tau \subset \overline{\Omega_\ell}$, from (2.5) we have

$$\|(\mathrm{I} - \Pi^h)(\chi_l v_h)\|_{L^2(\tau)} + h_\tau|(I - \Pi^h)(\chi_l v_h)|_{H^1(\tau)} \lesssim h^r_\tau|\chi_l v_h|_{H^r(\tau)}. \tag{3.12}$$

Let α be any multi-index of order $|\alpha| = r$. Since, on τ, χ_ℓ is of degree 1 and v_h is of degree $r-1$, the Leibnitz formula tells us that $D^\alpha(\chi_\ell v_h)$ consists of only a linear combination of functions of the form $(D^\beta\chi_\ell)(D^{\alpha-\beta}v_h)$, for all multi-indices with $|\beta| = 1$ (with coefficients independent of τ). Combining this with (2.19) leads to

$$|\chi_\ell v_h|_{H^r(\tau)} \lesssim \delta_\ell^{-1}|v_h|_{H^{r-1}(\tau)}. \tag{3.13}$$

Then, using (3.12) and an elementwise inverse estimate for shape regular elements,

$$k\|(\mathrm{I} - \Pi^h)(\chi_l v_h)\|_{L^2(\tau)} \lesssim kh_\tau\frac{h_\tau}{\delta_\ell}h^{r-2}_\tau|v_h|_{H^{r-1}(\tau)} \lesssim kh_\tau\frac{h_\tau}{\delta_\ell}\|v_h\|_{H^1(\tau)}. \tag{3.14}$$

Similarly,

$$|(\mathrm{I} - \Pi^h)(\chi_l v_h)|_{H^1(\tau)} \lesssim \frac{h_\tau}{\delta_\ell}h^{r-2}_\tau|v_h|_{H^{r-1}(\tau)} \lesssim \frac{h_\tau}{\delta_\ell}\|v_h\|_{H^1(\tau)}. \tag{3.15}$$

Combining (3.14) and (3.15) yields the result. ∎

We now specify a simplifying assumption on h, k, and δ.

ASSUMPTION 3.4. *We have*

$$kh \lesssim 1 \quad \text{and} \quad k\delta \gtrsim 1. \tag{3.16}$$

The left-hand inequality in (3.16) simply says that the fine mesh resolves the oscillatory solution (which is always needed for an accuracy anyway; see Remark 2.9), while the right-hand inequality requires that the overlap contains at least one oscillation. Clearly, (3.16) is equivalent to

$$kh_\ell \lesssim 1 \quad \text{and} \quad k\delta_\ell \gtrsim 1 \quad \text{for each } \ell, \tag{3.17}$$

provided the hidden constants are independent of ℓ, and this, in turn, implies that $h_\ell/\delta_\ell \lesssim 1$. This latter inequality requires that the overlapped part of any subdomain need only be large enough with respect to the *local* fine mesh diameter h_ℓ. We retain the ratio h_ℓ/δ_ℓ in the error estimate (3.11), since in many situations this can approach 0 as $k \to \infty$.

COROLLARY 3.5. *Under Assumption* 3.4*, for* $v_h \in \mathcal{V}^h$*,*

$$v_h = \sum_{\ell=1}^N \Pi^h(\chi_\ell v_h) \quad \textit{and} \quad \sum_{\ell=1}^N \|\Pi^h(\chi_\ell v_h)\|^2_{1,k,\Omega_\ell} \lesssim \Lambda \|v_h\|^2_{1,k,\Omega}.$$

Proof. Using the triangle inequality, and then (3.1), (3.11), and (3.17), we have

$$\begin{aligned} \|\Pi^h(\chi_\ell v_h)\|_{1,k,\Omega_\ell} &\le \|\chi_\ell v_h\|_{1,k,\Omega_\ell} + \|(I-\Pi^h)(\chi_\ell v_h)\|_{1,k,\Omega_\ell} \\ &\lesssim \|v_h\|_{1,k,\Omega_\ell} + \|v_h\|_{H^1(\Omega_\ell)} \lesssim \|v_h\|_{1,k,\Omega_\ell}, \end{aligned} \tag{3.18}$$

and the result follows by squaring, summing, and applying (2.10). ∎

The next result is a kind of converse to the stable splitting result discussed in Remark 3.2.

LEMMA 3.6. *For each* $\ell = 1, \dots, N$*, choose any functions* $v_\ell \in H^1(\Omega)$*, with* $\operatorname{supp} v_\ell \subset \overline{\Omega_\ell}$*. Then*

$$\left\| \sum_{\ell=1}^N v_\ell \right\|^2_{1,k} \le \Lambda \sum_{\ell=1}^N \|v_\ell\|^2_{1,k,\Omega_\ell} . \tag{3.19}$$

Proof. The proof follows almost verbatim from [30, Lemma 4.2], with a little extra care needed to obtain the explicit constant Λ on the right-hand side. ∎

3.2. Results about the projection operators. In this subsection, we study the projection operators $Q^h_{\varepsilon,\ell}$ which were defined in (2.20). Our goal is a bound on the operator $Q^h_{\varepsilon,\ell} - \Pi^h \chi_\ell$ with respect to the Helmholtz energy norm $\|\cdot\|_{1,k}$; see Lemma 3.8. This bound is a key ingredient of our main results—Theorem 3.11 (for projection operators) and Theorem 3.12 (for matrices).

We first note that, when $w_{h,\ell} \in \mathcal{V}^h_\ell$, $\Pi^h(\chi_\ell w_{h,\ell})$ is supported on Ω_ℓ and vanishes on $\partial\Omega_\ell$. Thus, by (2.20), for all $w_{h,\ell} \in \mathcal{V}^h_\ell$ and $v \in H^1(\Omega)$,

$$a_{\varepsilon,\ell}(Q^h_{\varepsilon,\ell} v, w_{h,\ell}) = a_{\varepsilon,\ell}(v, \Pi^h(\chi_\ell w_{h,\ell}))$$

and hence

$$a_{\varepsilon,\ell}(Q^h_{\varepsilon,\ell} v - \Pi^h(\chi_\ell v), w_{h,\ell}) = a_{\varepsilon,\ell}(v, \Pi^h(\chi_\ell w_{h,\ell})) - a_{\varepsilon,\ell}(\Pi^h(\chi_\ell v), w_{h,\ell}). \tag{3.20}$$

This shows that $Q^h_{\varepsilon,\ell} v - \Pi^h(\chi_\ell v)$ satisfies a local impedance problem with "data" given by the "commutator" (appearing on the right-hand side of (3.20)). To estimate this commutator we write

$$a_{\varepsilon,\ell}(v, \Pi^h(\chi_\ell w_{h,\ell})) - a_{\varepsilon,\ell}(\Pi^h(\chi_\ell v), w_{h,\ell}) = a_{\varepsilon,\ell}((I-\Pi^h)(\chi_\ell v), w_{h,\ell}) \tag{3.21}$$

$$- a_{\varepsilon,\ell}(v,(I-\Pi^h)(\chi_\ell w_{h,\ell}))$$
$$+ \; b_\ell(v,w_{h,\ell}),$$

where

$$b_\ell(v,w) \;:=\; a_{\varepsilon,\ell}(v,\chi_\ell w) - a_{\varepsilon,\ell}(\chi_\ell v,w) \;=\; (v,\chi_\ell w)_{1,k,\Omega_\ell} - (\chi_\ell v,w)_{1,k,\Omega_\ell}$$
$$= \int_{\Omega_\ell} \nabla\chi_\ell.(\overline{w}\nabla v - v\nabla\overline{w}) \; . \tag{3.22}$$

The following lemma provides estimates for each of the terms on the right-hand side of (3.2).

LEMMA 3.7. (i) *For all* $v,w\in H^1(\Omega_\ell)$,

$$|b_\ell(v,w)| \;\lesssim\; (k\delta_\ell)^{-1}\,\|v\|_{1,k,\Omega_\ell}\|w\|_{1,k,\Omega_\ell}.$$

(ii) *For all* $v_h,w_h\in\mathcal{V}_\ell^h$,

$$\max\Big\{|a_{\varepsilon,\ell}(v_h,(\mathrm{I}-\Pi^h)(\chi_\ell w_h))|,|a_{\varepsilon,\ell}((\mathrm{I}-\Pi^h)(\chi_\ell v_h),w_h)|\Big\}$$
$$\lesssim\; \left(1+\frac{1}{kH_\ell}\right)\frac{h_\ell}{\delta_\ell}\,\|v_h\|_{1,k,\Omega_\ell}\|w_h\|_{1,k,\Omega_\ell}.$$

Proof. Applying the Cauchy–Schwarz inequality to (3.22) and using (2.19), we obtain

$$|b_\ell(v,w)| \;\lesssim\; (k\delta_\ell)^{-1}\Big(k\|w\|_{L^2(\Omega_\ell)}|v|_{H^1(\Omega_\ell)} + k\|v\|_{L^2(\Omega_\ell)}|w|_{H^1(\Omega_\ell)}\Big),$$

and the result (i) follows after an application of the Cauchy–Schwarz inequality with respect to the Euclidean inner product in $\mathbb{R}^2$.

For (ii), recall Assumption 2.1, and use the continuity of $a_{\varepsilon,\ell}$ (from Lemma 2.4) and the fact that Ω_ℓ has characteristic length scale H_ℓ to obtain

$$|a_\varepsilon(v_h,(\mathrm{I}-\Pi^h)(\chi_\ell w_h))| \;\lesssim\; (1+(kH_\ell)^{-1})\|v_h\|_{1,k,\Omega_\ell}\,\|(\mathrm{I}-\Pi^h)(\chi_\ell w_h)\|_{1,k,\Omega_\ell}; \tag{3.23}$$

the result then follows on applying Lemma 3.3. ∎

Combining (3.20) with Lemma 3.7 and Theorem 2.8, we obtain the following estimate for the quantity $Q_{h,\ell}v_h - \Pi^h(\chi_\ell v_h)$. As we will see in (3.46), this quantity is related to the quality of the preconditioner on the subdomain Ω_ℓ.

LEMMA 3.8. *Under Assumption* 3.4 *and the assumptions of Theorem* 2.8*, for all* $v_h\in\mathcal{V}_\ell^h$*, and for all* ℓ*,*

$$\|Q^h_{\varepsilon,\ell}v_h - \Pi^h(\chi_\ell v_h)\|_{1,k,\Omega_\ell} \;\lesssim\; \frac{1}{k\delta_\ell}\,\Theta(\varepsilon,H_\ell,k)\,\|v_h\|_{1,k,\Omega_\ell}, \tag{3.24}$$

and

$$\|Q^h_{\varepsilon,\ell}v_h\|_{1,k,\Omega_\ell} \;\lesssim\; \left[1+\frac{1}{k\delta_\ell}\Theta(\varepsilon,H_\ell,k)\right]\,\|v_h\|_{1,k,\Omega_\ell}. \tag{3.25}$$

Proof. Let $v_h\in\mathcal{V}_\ell^h$. By (3.20) and (3.2), we have

$$a_{\varepsilon,\ell}(Q^h_{\varepsilon,\ell}v_h - \Pi^h\chi_\ell v_h, w_{h,\ell}) \;= F(w_{h,\ell}), \quad w_{h,\ell}\in\mathcal{V}_\ell^h, \tag{3.26}$$

where

$$F(w_{h,\ell}) \;:=\; a_{\varepsilon,\ell}((I-\Pi^h)(\chi_\ell v_h), w_{h,\ell}) - a_{\varepsilon,\ell}(v_h, (I-\Pi^h)(\chi_\ell w_{h,\ell})) + b_\ell(v_h, w_{h,\ell}).$$

Using Lemma 3.7 and (3.16), we have, for any $w_{h,\ell} \in \mathcal{V}_\ell^h$,

$$\begin{aligned} |F(w_{h,\ell})| \;&\lesssim\; \left(\left(1+\frac{1}{kH_\ell}\right)\frac{h_\ell}{\delta_\ell} + \frac{1}{k\delta_\ell}\right)\|v_h\|_{1,k,\Omega_\ell}\|w_{h,\ell}\|_{1,k,\Omega_\ell} \\ &=\; \frac{1}{k\delta_\ell}\left(kh_\ell + \frac{h_\ell}{H_\ell} + 1\right)\|v_h\|_{1,k,\Omega_\ell}\|w_{h,\ell}\|_{1,k,\Omega_\ell} \\ &\lesssim\; \frac{1}{k\delta_\ell}\,\|v_h\|_{1,k,\Omega_\ell}\|w_{h,\ell}\|_{1,k,\Omega_\ell}, \end{aligned} \tag{3.27}$$

where we have used (3.16) and the fact that $h_\ell \le H_\ell$. Then (3.24) follows from Theorem 2.8. To obtain (3.25), we write $\|Q_{\varepsilon,\ell}v_h\|_{1,k,\Omega_\ell} \le \|Q_{\varepsilon,\ell}v_h - \Pi^h(\chi_\ell v_h)\|_{1,k,\Omega_\ell} + \|\Pi^h(\chi_\ell v_h)\|_{1,k,\Omega_\ell}$, and then use (3.18) and (3.24). ∎

Combining Lemma 3.8 with the definition of Θ in (2.15), (2.16), we have the immediate corollary.

COROLLARY 3.9. *Under the assumptions of Theorem* 2.8*, we get the following:*
(i) *If* $|\varepsilon| > 0$*, then*

$$\|Q_{\varepsilon,\ell}^h v_h - \chi_\ell v_h\|_{1,k,\Omega_\ell} \;\lesssim\; \frac{k}{|\varepsilon|\delta_\ell}\|v_h\|_{1,k,\Omega_\ell}. \tag{3.28}$$

(ii) *If* $|\varepsilon| \ge 0$*,* $h \le \overline{h}(k,r)$*, and each* Ω_ℓ *is starshaped with respect to a ball uniformly in* ℓ*, then*

$$\|Q_{\varepsilon,\ell}^h v_h - \chi_\ell v_h\|_{1,k,\Omega_\ell} \;\lesssim\; \left(\frac{H_\ell}{\delta_\ell} + \frac{1}{k\delta_\ell}\right)\|v_h\|_{1,k,\Omega_\ell} \;\; \textit{uniformly in } \ell. \tag{3.29}$$

3.3. Bounds on the norm and field of values. To aid the reader, we recap all the assumptions made so far: Both the fine mesh $\mathcal{T}_h$ and the subdomains $\{\Omega_\ell\}$ are assumed shape-regular and have overlap described in (2.7) and (2.8), with $\delta > 0$. We make the finite-overlap assumption (2.9) and the partition of unity functions $\{\chi_\ell\}$ are assumed to be continuous, piecewise linear, and satisfy (2.19). We assume that k and ε satisfy Assumption 2.1, and either $\eta = \operatorname{sign}(\varepsilon)k$ or $\eta = \sqrt{k^2 + \mathrm{i}\varepsilon}$. All of these will be assumed without comment in what follows, but we will explicitly state when we need Assumption 3.4 and the following slightly stronger assumption.

ASSUMPTION 3.10. *We have*

$$k\delta \to \infty \quad \textit{as} \quad k \to \infty. \tag{3.30}$$

This assumption requires the overlap to contain an increasing number of oscillations as k increases (although the rate of increase can be arbitrarily slow).

THEOREM 3.11. *Let Assumption* 3.4 *hold and suppose that for each* $\ell = 1, \dots, N$ *there exists* $\sigma_\ell > 0$ *such that*

$$\left\|Q_{\varepsilon,\ell}^h v_h - \Pi^h(\chi_\ell v_h)\right\|_{1,k,\Omega_\ell} \le \sigma_\ell \|v_h\|_{1,k,\Omega_\ell} \quad \textit{for all} \quad v_h \in \mathcal{V}^h, \quad \ell = 1, \dots, N. \tag{3.31}$$

Set $\sigma = \max\{\sigma_\ell : \ell = 1, \dots, N\}$.

(i) *Then,*

$$\max_{v_h\in\mathcal{V}^h}\frac{\left\|Q_\varepsilon^h v_h\right\|_{1,k}}{\|v_h\|_{1,k}} \lesssim \Lambda\,(1+\sigma)\,. \tag{3.32}$$

(ii) *If, in addition, Assumption* 3.10 *holds, then, for k sufficiently large,*

$$\min_{v_h\in\mathcal{V}^h}\frac{\left|(v_h,Q_\varepsilon^h v_h)_{1,k}\right|}{\|v_h\|_{1,k}^2} \geq \left(\frac{1}{\Lambda}-\sqrt{2}\sigma\Lambda\right) + R, \tag{3.33}$$

where the remainder R satisfies the estimate

$$|R| \leq C\,\frac{\Lambda}{k\delta}\,(1+\sigma)\,, \tag{3.34}$$

where C is a constant independent of all parameters. Note that (3.33) *is a genuine lower bound, and the unspecified constant C appears only in R.*

Proof. Throughout the proof, we use the notation

$$z_l := Q^h_{\varepsilon,\ell}v_h - \Pi^h(\chi_\ell v_h), \quad \text{so that, by (3.31)}, \quad \|z_\ell\|_{1,k,\Omega_\ell}\leq\sigma_\ell\|v_h\|_{1,k,\Omega_\ell}. \tag{3.35}$$

To obtain (3.32), we use the triangle inequality, then (3.18) and (3.31), to obtain

$$\|Q^h_{\varepsilon,\ell}v_h\|_{1,k,\Omega_\ell} \leq \|\Pi^h(\chi_\ell v_h)\|_{1,k,\Omega_\ell} + \|z_l\|_{1,k,\Omega_\ell} \leq (1+\sigma_\ell)\,\|v_h\|_{1,k,\Omega_\ell}. \tag{3.36}$$

Then, using Lemma 3.6, (3.18), and (3.36),

$$\begin{aligned}\|Q_\varepsilon^h v_h\|_{1,k}^2 &= \left\|\sum_\ell \Pi^h\left(\chi_\ell Q^h_{\varepsilon,\ell}v_h\right)\right\|_{1,k}^2 \leq \Lambda\sum_\ell\left\|\Pi^h\left(\chi_\ell Q^h_{\varepsilon,\ell}v_h\right)\right\|^2_{1,k,\Omega_\ell}\\ &\lesssim \Lambda\sum_\ell\left\|Q^h_{\varepsilon,\ell}v_h\right\|^2_{1,k,\Omega_\ell} \lesssim \Lambda\,(1+\sigma)^2\sum_\ell\|v_h\|^2_{1,k,\Omega_\ell}\end{aligned}$$

and (3.32) then follows on using (2.10).

To obtain (3.33), we first use Lemma 3.3 and (3.16) to obtain

$$(v_h,\Pi^h(\chi_\ell Q^h_{\varepsilon,\ell}v_h))_{1,k,\Omega_\ell} = (v_h,\chi_\ell Q^h_{\varepsilon,\ell}v_h)_{1,k,\Omega_\ell} + \mathcal{O}\left(\frac{h}{\delta}\right)\|v_h\|_{1,k,\Omega_\ell}\,\|Q^h_{\varepsilon,\ell}v_h\|_{1,k,\Omega_\ell}. \tag{3.37}$$

Also, using (3.22) and Lemma 3.7, we have that

$$\begin{aligned}\left|(v_h,\chi_\ell Q^h_{\varepsilon,\ell}v_h)_{1,k,\Omega_\ell} - (\chi_\ell v_h,Q^h_{\varepsilon,\ell}v_h)_{1,k,\Omega_\ell}\right| &= \left|b_\ell(v_h,Q^h_{\varepsilon,\ell}v_h)\right|\\ &\lesssim \mathcal{O}\left(\frac{1}{k\delta}\right)\|v_h\|_{1,k,\Omega_\ell}\,\|Q^h_{\varepsilon,\ell}v_h\|_{1,k,\Omega_\ell}.\end{aligned} \tag{3.38}$$

Moreover, by the definition of z_l and Lemma 3.3,

$$\begin{aligned}(\chi_\ell v_h,Q^h_{\varepsilon,\ell}v_h)_{1,k,\Omega_\ell} &= \|\chi_\ell v_h\|^2_{1,k,\Omega_\ell} + (\chi_\ell v_h,z_\ell)_{1,k,\Omega_\ell} + \left(\chi_\ell v_h,\Pi^h(\chi_\ell v_h)-\chi_\ell v_h\right)_{1,k,\Omega_\ell}\\ &= \|\chi_\ell v_h\|^2_{1,k,\Omega_\ell} + (\chi_\ell v_h,z_\ell)_{1,k,\Omega_\ell} + \mathcal{O}\left(\frac{h}{\delta}\right)\|\chi_\ell v_h\|_{1,k,\Omega_\ell}\,\|v_h\|_{1,k,\Omega_\ell}\,.\end{aligned} \tag{3.39}$$

Combining (3.37), (3.38), and then using (3.39), we obtain

$$\begin{aligned}(v_h, Q^h_\varepsilon v_h)_{1,k} &= \sum_\ell \left(v_h, \Pi^h(\chi_\ell Q^h_{\varepsilon,\ell} v_h)\right)_{1,k,\Omega_\ell} \\ &= \sum_\ell \left[(\chi_\ell v_h, Q^h_{\varepsilon,\ell} v_h)_{1,k,\Omega_\ell} + \mathcal{O}\left(\frac{1}{k\delta} + \frac{h}{\delta}\right) \|v_h\|_{1,k,\Omega_\ell} \|Q^h_{\varepsilon,\ell} v_h\|_{1,k,\Omega_\ell}\right] \\ &= \sum_\ell \left[\|\chi_\ell v_h\|^2_{1,k,\Omega_\ell} + (\chi_\ell v_h, z_l)_{1,k,\Omega_\ell}\right] \\ &+ \sum_\ell \left[\mathcal{O}\left(\frac{1}{k\delta} + \frac{h}{\delta}\right) \|v_h\|_{1,k,\Omega_\ell} \|Q^h_{\varepsilon,\ell} v_h\|_{1,k,\Omega_\ell} + \mathcal{O}\left(\frac{h}{\delta}\right) \|v_h\|_{1,k,\Omega_\ell} \|\chi_\ell v_h\|_{1,k,\Omega_\ell}\right]. \end{aligned} \tag{3.40}$$

Using (3.16), (3.36), (3.1), and (2.10), the second sum in (3.40) can be estimated by

$$\frac{1}{k\delta} \sum_\ell (1+\sigma_\ell) \|v_h\|^2_{1,k,\Omega_\ell} \lesssim \frac{\Lambda(1+\sigma)}{k\delta} \|v_h\|^2_{1,k}.$$

Also, using the Cauchy–Schwarz inequality, and then (3.1) and (3.35), the modulus of the first sum in (3.40) can be estimated from below by

$$\sum_\ell \|\chi_\ell v_h\|^2_{1,k,\Omega_\ell} - \left|\sum_\ell (\chi_\ell v_h, z_\ell)_{1,k,\Omega_\ell}\right| \geq \sum_\ell \left(\|\chi_\ell v_h\|^2_{1,k,\Omega_\ell} - \|\chi_\ell v_h\|_{1,k,\Omega_\ell} \|z_\ell\|_{1,k,\Omega_\ell}\right) \tag{3.41}$$

$$\geq \sum_\ell \|\chi_\ell v_h\|^2_{1,k,\Omega_\ell} - \sqrt{2}\sigma \sum_\ell \|v_h\|^2_{1,k,\Omega_\ell} + \mathcal{O}\left(\frac{\sigma}{k\delta}\right) \sum_\ell \|v_h\|^2_{1,k,\Omega_\ell}. \tag{3.42}$$

The result (3.33) then follows from using (3.5) and (2.10). ∎

Using Theorem (2.10), we now convert this to a statement about matrices.

THEOREM 3.12. *Let Assumption* 3.4 *hold, and let* $\sigma > 0$ *be such that*

$$\|A^{-1}_{\varepsilon,\ell} R_\ell A_\varepsilon - R_\ell\|_{D_k} \leq \sigma, \quad \ell = 1, \ldots, N. \tag{3.43}$$

Then

$$\|B^{-1}_\varepsilon A_\varepsilon\|_{D_k} \lesssim \Lambda\,(1+\sigma)\,. \tag{3.44}$$

If, in addition, Assumption 3.10 *holds, then for* k *sufficiently large,*

$$\min_{\mathbf{V}\in\mathbb{C}^n} \frac{\left|\langle \mathbf{V}, B^{-1}_\varepsilon A_\varepsilon \mathbf{V}\rangle_{D_k}\right|}{\|\mathbf{V}\|^2_{D_k}} \geq \left(\frac{1}{\Lambda} - \sqrt{2}\sigma\Lambda\right) + R, \tag{3.45}$$

with R *satisfying* (3.34).

Proof. First note that, from (1.14) and (1.15), if $v_h \in \mathcal{V}^h$ is a finite-element function with nodal vector $\mathbf{V}$, then $\|v_h\|_{1,k} = \|\mathbf{V}\|_{D_k}$. By Theorem 2.10, the nodal vectors of $Q^h_{\varepsilon,\ell} v_h$ and $Q^h_\varepsilon v_h$ are $A^{-1}_{\varepsilon,\ell} R_\ell A_\varepsilon \mathbf{V}$ and $B^{-1}_\varepsilon A_\varepsilon \mathbf{V}$, respectively. By (1.12), the nodal vector of $\Pi^h(\chi_\ell v_h)$ is $R_\ell \mathbf{V}$. Thus

$$\|Q^h_\varepsilon\|_{1,k} = \|B^{-1}_\varepsilon A_\varepsilon\|_{D_k} \quad \text{and} \quad \|Q^h_{\varepsilon,\ell} v_h - \Pi^h(\chi_\ell v_h)\|_{1,k} = \|A^{-1}_{\varepsilon,\ell} R_\ell A_\varepsilon \mathbf{V} - R_\ell \mathbf{V}\|_{D_k}. \tag{3.46}$$

From these relations, and also (2.23), Theorem 3.11 implies Theorem 3.12. ∎

This theorem immediately yields the following corollary about the convergence of GMRES.

COROLLARY 3.13. *Suppose Assumptions* 3.4 *and* 3.10 *hold and that* (3.43) *holds, with*

$$\sigma \;<\; \frac{1}{\sqrt{2}\Lambda^2}. \tag{3.47}$$

If GMRES is applied to (1.5) *in the inner product induced by* D_k *with* B_ε^{-1} *as a left preconditioner, then the number of iterations needed to achieve a prescribed accuracy remains bounded as* $k \to \infty$.

Proof. This follows directly from Theorem 3.12 and the GMRES convergence theory in [15]. ∎

As explained above, Assumptions 3.4 and 3.10 are quite mild requirements. However (3.47) is a stronger constraint and may lead to restrictions on ε and H. Essentially, it says that for each ℓ, the "local impedance solve" $A_{\varepsilon,\ell}^{-1}$ should be a sufficiently good left inverse for A_ε when it is restricted to $\overline{\Omega_\ell}$. In the following corollary, whose proof follows from Corollary 3.9, part (i) gives conditions under which σ can be bounded (hence useful for the upper bound (3.44)), while parts (ii) and (iii) give conditions for σ to be small (and hence are relevant to ensuring (3.45)).

COROLLARY 3.14. *Let the assumptions of Theorem* 3.11 *hold.*

(i) *Assume that* $h \le \overline{h}(k,r)$, *and each* Ω_ℓ *is starshaped with respect to a ball uniformly in* ℓ. *Then, for all* ε *with* $0 \le |\varepsilon| \le k^2$, *we have* $\sigma \lesssim H/\delta$.

(ii) *If* $|\varepsilon| > 0$, $\varepsilon \sim k^{1+\beta}$ *for* $0 < \beta < 1$, $\delta \sim H \sim k^{-\alpha}$ *for* $0 < \alpha < 1$, *then* $\sigma \lesssim k^{\alpha-\beta}$.

(iii) *If* $|\varepsilon| > 0$ *and* δ *is fixed, then there exist constants* C *and* k_0 *so that when* $\varepsilon = Ck$ *and* $k \ge k_0$,

$$\sigma \le \frac{1}{2\sqrt{2}\Lambda^2}.$$

Using the bounds of Corollary 3.14 in Theorem 3.12, we obtain the following results about $B_\varepsilon^{-1}A_\varepsilon$.

COROLLARY 3.15 (upper bound on the norm of $B_\varepsilon^{-1}A_\varepsilon$). *Assume that* $h \le \overline{h}(k,r)$, *and each* Ω_ℓ *is starshaped with respect to a ball uniformly in* ℓ. *Assume that* $\delta \sim H$. *Then, for all* $0 \le |\varepsilon| \le k^2$,

$$\|B_\varepsilon^{-1}A_\varepsilon\|_{D_k} \;\lesssim\; 1.$$

COROLLARY 3.16 (lower bound on the distance of the field of values from the origin).

(i) *If* $|\varepsilon| \sim k^{1+\beta}$ *for* $0 < \beta < 1$, $\delta \sim H$, *and* $H \sim k^{-\alpha}$ *for* $0 < \alpha < 1$, *then*

$$\min_{\mathbf{V}\in\mathbb{C}^n} \frac{\left|\langle \mathbf{V}, B_\varepsilon^{-1}A_\varepsilon\rangle_{D_k}\right|}{\|\mathbf{V}\|_{D_k}^2} \;\ge\; 1 - \mathcal{O}(k^{\alpha-\beta}) \quad \text{as} \quad k \to \infty.$$

(ii) *If* δ *is fixed, then there exist constants* C *and* k_0 *so that when* $|\varepsilon| = Ck$ *and* $k \ge k_0$,

$$\min_{\mathbf{V}\in\mathbb{C}^n} \frac{\left|\langle \mathbf{V}, B_\varepsilon^{-1}A_\varepsilon\rangle_{D_k}\right|}{\|\mathbf{V}\|_{D_k}^2} \;\ge\; \frac{1}{2\Lambda}\,.$$

Remark 3.17 (right preconditioning). The results in [30, Theorem 5.8]—see also [31, section 3]—show how results about right preconditioning (working in the D_k^{-1} inner product) can be obtained from analogous results about left preconditioning of the adjoint problem (working in the D_k inner product). The results in section 2 and subsections 3.1 and 3.2 all hold when the problem (1.1), (1.2) is replaced by its adjoint (see, in particular, Remark 2.5); therefore, the results in this section about left preconditioning (in the D_k inner product) also hold for right preconditioning (in the D_k^{-1} inner product).

Remark 3.18 (Dirichlet boundary conditions). Some parts of the analysis presented in this paper hold in the case when the boundary conditions on the subdomains are changed from impedance to Dirichlet, i.e., when the integral over $\partial\Omega_\ell \setminus \Gamma_D$ is removed from (2.11). However, parts (ii) of Theorem 2.8 and Corollary 3.9 no longer hold in this case. Additionally, the upper bound on the norm for $\varepsilon = 0$ in Corollary 3.15 does not hold either. We see in Experiment 5 that when the impedance boundary conditions are replaced by Dirichlet, the preconditioner performs poorly for the pure Helmholtz equation.

4. Numerical experiments. In this section we give numerical experiments illustrating the performance of the preconditioners defined in section 1.2 and analyzed in section 3.3. We consider problem (1.1)–(1.2) with Ω being the unit square in 2-d. We first choose a uniform coarse mesh $\mathcal{T}^H$ of equal square elements of side length $H = 1/M$ on Ω. Let $\boldsymbol{x}_{\ell,m} = (\ell H, mH)$, $\ell, m = 0, \ldots, M$, denote the coarse mesh nodes. We introduce subdomains $\Omega_{\ell,m}$, defined to be interior of the union of all the coarse mesh elements that touch $\boldsymbol{x}_{\ell,m}$ for $\ell, m = 0, \ldots, M$. These subdomains have generous overlap in the sense of (2.7). Let $\chi_{\ell,m}$ denote the piecewise bilinear nodal basis functions with respect to the coarse mesh, i.e., $\chi_{\ell,m}$ is bilinear with respect to the coarse mesh and $\chi_{\ell,m}(\boldsymbol{x}_{\ell',m'}) = \delta_{\ell-\ell',m-m'}$. Then $\{\chi_{\ell,m} : \ell, m = 0, \ldots, M\}$ form a partition of unity and we use this to define the preconditioner (1.13).

Illustration of the notation used in the proof of (3.4). Recalling (1.11), we can see that for each $\ell, m \in \{0, \ldots, M\}$, $\operatorname{supp} \chi_{\ell,m} \subseteq \overline{\Omega_{\ell,m}}$. Moreover, for $\boldsymbol{x} \in \overline{\Omega}$ and $m(\boldsymbol{x})$ defined by (3.6), we have

$$
\begin{aligned}
&m(\boldsymbol{x}) = 1 \quad \text{when } \boldsymbol{x} \text{ is a node} \quad \boldsymbol{x} \in \{\boldsymbol{x}_{\ell,m} : \ \ell, m \in \{0, \ldots, M\}\},\\
&m(\boldsymbol{x}) = 2 \quad \text{when } \boldsymbol{x} \text{ is an interior point of any edge of the coarse mesh,}\\
&m(\boldsymbol{x}) = 4 \quad \text{when } \boldsymbol{x} \text{ is an interior point of any coarse mesh element.}
\end{aligned}
$$

Hence $\Lambda = 4$. Note that D_1 contains all the nodes of the coarse grid, D_2 contains all interior points of edges of the coarse grid, and D_4 contains all interior points of coarse grid elements. Note $\mu(D_1) = \mu(D_2) = 0$, with μ denoting Lebesgue measure, and $\mu(D_4) = \mu(\Omega)$. Moreover, the index sets $\mathcal{D}(1), \mathcal{D}(2)$, and $\mathcal{D}(4)$ actually contain all indices (ℓ, m) with $\ell, m \in \{0, \ldots, M\}$.

The coarse mesh is then refined uniformly to obtain a fine triangular mesh $\mathcal{T}^h$. The space $\mathcal{V}^h$ which is used to obtain the linear system (1.5) is the space of piecewise-linear finite-element functions on $\mathcal{T}^h$. The linear system (1.5) is therefore characterized by two parameters: the fine mesh diameter h and ε in (1.1) denoted by h_{prob} and $\varepsilon_{\text{prob}}$, respectively. In all of the experiments here we choose $h \sim k^{-3/2}$ (the level of refinement generally believed to keep the relative error of the finite-element solution bounded independently of k as $k \to \infty$; see Remark 2.9). Although these are 2-d problems, the dimension $n = (k^{3/2})^2 = k^3$ of the systems grows very quickly with k, and is well over 10^6 when $k = 140$ (considered below). The preconditioner is

characterized by the coarse grid diameter and the level of absorption used, denoted by H and $\varepsilon_{\rm prec}$, respectively.

In Experiments 1 and 2, we verify the theory by illustrating the performance of the preconditioner on some problems with $\varepsilon_{\rm prob} > 0$. In Experiments 3, 4, and 5, we solve the "pure Helmholtz" problem, i.e., $\varepsilon_{\rm prob} = 0$. Unless otherwise stated, the data f, g in (1.5) is chosen so that the exact solution of (1.3)–(1.4) is a plane wave $u(x) = \exp(\mathrm{i}kx.\widehat{d})$, where $\widehat{d} = (1/\sqrt{2}, 1/\sqrt{2})^\top$. Note that oscillations in the solution are resolved by the fine grid but are not resolved by the subdomains. We choose $\Gamma_D = \emptyset$, so that $\Gamma = \Gamma_I$. Except in Experiment 4, the initial guess for GMRES is chosen to be a random (uniformly distributed in $[0,1]^m$) vector in $\mathbb{R}^n$. In all cases the GMRES stopping criterion is based on requiring the initial residual to be reduced by 10^{-6}. Standard GMRES (with residual minimization in the Euclidean norm) is used, even though the estimates in Theorem 3.12 are with respect to the norm induced by D_k; the numerical experiments in [30, 6] (for a similar method) found the iteration counts to be essentially identical when minimization in the Euclidean norm is replaced by minimization in the norm induced by D_k.

Experiment 1. *We choose*

$$h_{\rm prob} \sim k^{-3/2}, \quad \varepsilon_{\rm prob} = \varepsilon_{\rm prec} = k^{1+\beta}, \ H_{\rm prec} = k^{-\alpha}\ , \quad \textit{where} \quad \beta = \alpha + 0.1. \tag{4.1}$$

Corollary 3.14 predicts a wavenumber-independent iteration count for GMRES, and this behavior is clearly visible in Table 1(a). Reading across this table, for fixed k, larger α corresponds to smaller subdomains (and thus the preconditioner becomes cheaper per iterate). The number of iterations increases (slightly) as α increases but remains bounded as k increases for fixed α. We also note that if we read diagonally across Table 1(a) (thus increasing the rate of decrease of H as k increases), we see roughly logarithmic growth in the number of iterations, although the analogous growth is somewhat faster in later tables.

Table 1

$k\backslash\alpha$	0.2	0.3	0.4	0.5
40	4	6	7	9
60	4	5	7	10
80	3	6	8	9
100	5	6	7	9
120	4	5	7	9
140	4	5	7	9

(a) GMRES iterations for case (4.1).

$k\backslash\alpha$	0.2	0.3	0.4	0.5
40	4	7	10	17
60	4	7	12	22
80	4	9	13	21
100	6	8	13	23
120	5	8	15	24
140	5	7	13	25

(b) GMRES iterations for case (4.2).

Based on Experiment 1, and recalling the discussion in the introduction (in particular, (1.8)), we now investigate how well the preconditioner performs when we reduce the absorption in the problem being solved to $\varepsilon_{\rm prob} = k$.

Experiment 2. *We choose*

$$h_{\rm prob} \sim k^{-3/2}, \quad \varepsilon_{\rm prob} = \varepsilon_{\rm prec} = k, \quad \textit{and} \quad H_{\rm prec} = k^{-\alpha}. \tag{4.2}$$

Comparing Tables 1 (left and right), we see an increase in the iteration numbers (especially for larger α), but growth with k appears to be avoided provided $\alpha \leq 0.4$. This shows that B_k^{-1} is a good preconditioner for A_k, and so by the heuristic argument

centered on (1.8), we expect B_k^{-1} to be a good preconditioner for A. Experiment 3 shows this to be true. Here $\varepsilon_{\text{prob}}$ is reduced from k to 0; we see a slight increase in iteration numbers compared to $\varepsilon_{\text{prob}} = k$, but still apparent robustness to increasing k, for fixed $\alpha \leq 0.4$.

EXPERIMENT 3. *We choose*

$$h_{\text{prob}} \sim k^{-3/2}, \ \varepsilon_{\text{prob}} = 0, \quad \textit{and} \quad H_{\text{prec}} = k^{-\alpha}. \tag{4.3}$$

TABLE 2
Number of GMRES iterations for the case (4.3).

	$\varepsilon_{\text{prec}} = k$				$\varepsilon_{\text{prec}} = 0$			
$k\backslash\alpha$	0.2	0.3	0.4	0.5	0.2	0.3	0.4	0.5
40	6	8	12	20	5	8	11	19
60	5	8	14	25	5	7	14	25
80	5	10	15	25	4	10	15	24
100	7	9	15	27	7	9	15	27
120	6	9	17	29	6	9	17	29
140	6	9	17	31	6	8	16	31

We make two observations from the results of Experiments 1–3.

1. The one-level Schwarz method provides an optimal preconditioner for the pure Helmholtz problem—the iteration numbers appear bounded independently of k (and hence n) as k increases—provided the subdomain diameter does not shrink too quickly. Robustness is maintained when the subdomain diameters shrink no faster than $\mathcal{O}(k^{-0.4})$.
2. The performance of the preconditioner is virtually the same whether it is built from the absorptive system $\varepsilon_{\text{prec}} = k$ or from the pure Helmholtz system $\varepsilon_{\text{prec}} = 0$. While the results of the present paper give theoretical support for the observed robustness when $\varepsilon_{\text{prec}} = k$ (see the discussion in section 1 and Appendix A), with existing theoretical tools it seems very difficult to prove results for the case $\varepsilon_{\text{prec}} = 0$.

EXPERIMENT 4. *As a more extreme case we consider subdomains which are fixed as $h \to 0$. While this is not a practical method (the subproblems have the same order of complexity as the global problem), it can provide a useful starting point for methods based on recursive application of the one-level method, as described in section* 1.4. *We therefore consider*

$$h \sim k^{-3/2}, \ \varepsilon_{\text{prob}} = 0, \ H = 1/M. \tag{4.4}$$

In the left-hand panel of Table 3, *$\varepsilon_{\text{prec}} = k$ and a random starting guess is chosen. In the middle panel, $\varepsilon_{\text{prec}} = 0$ and a random starting guess is chosen. In the right-hand panel, $\varepsilon_{\text{prec}} = k$ and a zero starting guess is chosen. Again, there is little effect from switching off the absorption in the preconditioner. Surprisingly, a random starting guess leads to consistently lower iteration counts than a zero starting guess; we have no explanation for this observation.*

Finally, we study the effect of changing the boundary condition on the subdomains from impedance to Dirichlet (recall Remark 3.18).

EXPERIMENT 5. *We choose Dirichlet conditions on subdomains with*

$$h_{\text{prob}} \sim k^{-3/2}, \ \varepsilon_{\text{prob}} = 0, \quad \textit{and} \quad H_{\text{prec}} = k^{-\alpha}. \tag{4.5}$$

TABLE 3
Number of GMRES iterations for the case (4.4).

	Random starting guess $\varepsilon_{\text{prec}} = k$			Random starting guess $\varepsilon_{\text{prec}} = 0$			Zero starting guess $\varepsilon_{\text{prec}} = k$		
$k\backslash M$	4	8	16	4	8	16	4	8	16
40	12	27	61	11	27	61	16	36	82
60	11	25	56	10	25	56	15	36	81
80	10	22	52	10	22	52	15	33	75
100	9	21	48	9	21	48	15	33	71
120	9	20	45	9	20	45	15	31	69
140	8	18	41	8	18	41	14	31	70

In Table 4 *we see that this yields a very poor preconditioner for the pure Helmholtz problem (compare Experiment* 5 *with Experiment* 3*). Similar observations are made in* [30]*, where coarse grids were also used to improve the robustness.*

TABLE 4
Number of GMRES iterations for the case (4.5) *with homogeneous Dirichlet condition on subdomain boundaries.*

	$\varepsilon_{\text{prec}} = k$				$\varepsilon_{\text{prec}} = 0$			
$k\backslash\alpha$	0.2	0.3	0.4	0.5	0.2	0.3	0.4	0.5
10	7	7	12	12	6	6	15	15
20	7	7	17	25	5	5	20	29
40	6	16	34	86	5	22	43	110
60	6	16	68	102	5	25	83	121
80	5	46	127	239	5	78	173	256
100	14	58	130	242	22	121	222	429

Appendix A. A rigorous basis for the discussion around (1.8).

LEMMA A.1. *Let* $(\cdot,\cdot)$ *be an inner product with associated norm* $\|\cdot\|$*. Assume that* (1.6) *holds with* $\|\cdot\|_2$ *replaced by* $\|\cdot\|$ *and with* $K > 0$ *independent of* ε *and* k*. Assume also that for all* ε *in some neighbourhood of the origin, there exist positive numbers* $C_1(\varepsilon)$ *and* $C_2(\varepsilon)$ *(which may depend on* ε *but are independent of all other parameters), such that*

$$\text{(A.1)}\qquad \|B_\varepsilon^{-1}A_\varepsilon\| \le C_1(\varepsilon) \quad \text{and} \quad \frac{|(\mathbf{V}, B_\varepsilon^{-1}A_\varepsilon\mathbf{V})|}{\|\mathbf{V}\|^2} \ge C_2(\varepsilon) \quad \text{for all} \quad \mathbf{V}\in\mathbb{C}^n.$$

Then

(A.2)

$$\|B_\varepsilon^{-1}A\| \le C_1(\varepsilon)\left(1+K\frac{|\varepsilon|}{k}\right) \quad \text{and} \quad \frac{|(\mathbf{V}, B_\varepsilon^{-1}A\mathbf{V})|}{\|\mathbf{V}\|^2} \ge C_2(\varepsilon) - K\,C_1(\varepsilon)\frac{|\varepsilon|}{k}$$

for all $\mathbf{V}\in\mathbb{C}^n$*.*

Remark A.2. Observe that for the norm in (A.2) to remain bounded we simply need $C_1(\varepsilon)$ to be bounded, while for the field of values to be bounded away from the origin we need the stronger condition

$$C_2(\varepsilon) > KC_1(\varepsilon)\frac{|\varepsilon|}{k}.$$

Proof of Lemma A.1. The first bound in (A.2) follows from (1.7), (1.6), and the first equation in (A.1). To obtain the second bound in (A.2), we use (1.7), the first bound in (A.2), and the inverse triangle inequality to obtain

$$\left|(\mathbf{V}, B_\varepsilon^{-1} A\mathbf{V})\right| \;\geq\; \left|(\mathbf{V}, B_\varepsilon^{-1} A_\varepsilon \mathbf{V})\right| - KC_1(\varepsilon)\frac{|\varepsilon|}{k}\|\mathbf{V}\|^2,$$

and we then use the second equation in (A.1). ∎

Acknowledgments. The first author thanks the Department of Mathematics at the Chinese University of Hong Kong for providing generous sup- port and a stimulating research environment during his visits; he also thanks Paul Childs for first motivating him to study this problem. The authors thank Eero Vainikko (University of Tartu) for generously computing the numerical experiments given in the paper, Eric Chung (Chinese University of Hong Kong), and Shihua Gong (University of Bath) for very useful discussions and the anonymous referees for their careful reading of the manuscript and perceptive comments which have improved this paper.

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