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DISSIPATIVITY AND CONTRACTIVITY ANALYSIS FOR FRACTIONAL FUNCTIONAL DIFFERENTIAL EQUATIONS AND THEIR NUMERICAL APPROXIMATIONS*

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Abstract. We first present a new delay-dependent fractional generalization of Halanay-like inequality to characterize the asymptotic behavior of fractional functional differential equations (F-FDEs). Then we study the dissipativity of F-FDEs with a bounded absorbing set and the asymptotic stability and the contractivity of F-FDEs with algebraically contractive rate. Two numerical schemes are further constructed for F-FDEs based on Grünwald–Letnikov formula and $L1$ method for Caputo fractional derivative, together with linear interpolation for the functional terms. These two schemes are proved to be dissipative and contractive and can preserve the exact decay rate as the continuous equations. These results can be directly applied to some special cases of F-FDEs, such as the fractional delay differential equations, fractional integro-differential equations, and fractional delay integro-differential equations. Finally, several numerical examples are given to illustrate the advantages of the structure-preserving numerical methods. In particular, we shall compare the numerical performance of our schemes and the popular predictor-corrector algorithms for F-FDEs and demonstrate that our schemes are more efficient and robust, especially for some stiff systems.

Key words. fractional functional differential equations, dissipativity, contractivity, Grünwald–Letnikov formula, $L1$ method

AMS subject classifications. 34A08, 34D05, 65L03

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1. Introduction.

Fractional differential equations have found recently many physics and engineering applications in modeling anomalous transport dynamics [27, 41]. Time fractional differential operators may arise naturally in many models whenever the recover time is power-law distribution [2]. Many interesting fractional models in various practical applications are established, such as fractional model of human immunodeficiency virus (HIV) infection of CD4+ T cells with time delay [54], fractional order financial delay system [51], and fractional order recovery SIR model [2]. More applications on fractional models may be found in [27, 41]. Practical application models are mostly nonlinear and involve some time delay or integral terms, and often have rich and complex dynamical structures. It is usually difficult to find the analytical solutions of those fractional differential equations. This motivates us to study the basic theory and efficient numerical methods of fractional functional differential equations (F-FDEs).

Let $\mathbb{R}^n$ be the n-dimensional Euclidian space with the standard inner product $\langle \cdot, \cdot \rangle$ and norm $||\cdot||$. Let $C(I)$ $(I = [0, T])$ be a Banach space consisting of all continuous mapping $y : I \to \mathbb{R}^n$. Consider the initial value problem of F-FDEs

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$$
\begin{cases} {}_0^C D_t^\alpha y(t) = f(t, y(t), y(\cdot)), & 0 < t \le T, \\ y(t) = \varphi(t), & -\sigma \le t \le 0, \end{cases} \tag{1}
$$

where $0 < \alpha < 1$, $T > 0$ (T may be $+\infty$) and $\sigma > 0$ is a given constant, $\varphi \in C[-\sigma, 0]$ is a given initial function. The Caputo fractional derivative is defined by

$$
{}_0^C D_t^\alpha y(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{y'(s)}{(t-s)^\alpha} ds, \qquad t > 0. \tag{2}
$$

Throughout this work, we assume that $f(t, y(t), y(\cdot))$ is independent of the values of the function $y(\xi)$ for $t < \xi \le T$, i.e., $f(t, y(t), y(\cdot))$ is a Volterra functional; see [30]. The function $y(\cdot)$ may take several different forms, such as $y(t - \tau(t))$ for a variable delay $\tau(t) \ge 0$, $y(qt)$ for a proportional delay or $\int_{t-\tau(t)}^t k(t, s, y(s))ds$ for a delay of integral form. And the combinations of these forms are also allowed. Many concrete examples can be found in [4, 6, 30].

Next, we discuss briefly about the regularity of the solution to the equation F-FDEs (1). There are two main factors that may affect the regularity, i.e., the Caputo fractional order operator ${}_0^C D_t^\alpha$ and the delay term $\tau(t)$. In general, solutions to time fractional-order evolution equations, including time fractional-order ODEs and PDEs, are usually only in $C^\alpha[0, T]$, namely, Hölder continuous, and cannot be in $C^1[0, T]$ due to the weak singular kernel of the fractional operator. More precisely, the first derivative of the solution is usually unbounded around the initial time, i.e., $y'(t) \sim C_\alpha t^{\alpha-1}$ as $t \to 0^+$, where C_α is a nonzero constant. For linear fractional ordinary differential equations (F-ODEs), a rigorous proof of this regularity of the solution can be found in [7]. For time fractional semilinear parabolic equations, the Hölder continuity in time were given in [25]. In particular, the Hölder continuity was proved in [42] for the solutions to a class of linear delay differential equations.

For functional differential equations with some delay term $\tau(t)$, the solution $y(t)$ may not be connected smoothly to the initial function $\varphi(t)$ at $t_0 = 0$, i.e., $\varphi'(0)^- \ne y'(0)^+ = f(0, y(0), y(\cdot))$. This nonsmoothness at t_0 may lead to the discontinuity of the higher derivatives at some later moments. But if we assume that initial function $\varphi(t)$, delay $\tau(t)$, and f are continuous, then $y'(t)$ will exists for $t > t_0$ for classical functional differential equations (FDEs); see [4]. The local existence for classical FDEs, i.e., $\alpha = 1$ in (1), can be proved under the assumption that $f(t, y(t), y(\cdot))$ is continuous and locally Lipschitz continuous with respect to the second and third variables. Furthermore, if the solution is bounded, the solution may exist globally [4].

In view of the above observations, if we consider the fractional order operator and the delay effect of F-FDEs, we may assume that the equation F-FDEs (1) has a unique solution $y \in C^\alpha[0, T]$, ${}_0^C D_t^\alpha y \in C[0, T]$, and being piecewise C^1, under the conditions that functions $\varphi(t)$ and $\tau(t)$ are continuous, and $f(t, y(t), y(\cdot))$ is continuous and locally Lipschitz continuous with respect to the second and third variables.

We shall consider two types of closely related problems. The first type of problems assumes the dissipative structural condition for the continuous mapping $f : [0, T) \times \mathbb{R}^n \times C[-\sigma, T) \to \mathbb{R}^n$:

$$
2 \langle u, f(t, u, \psi(\cdot)) \rangle \le \gamma + a\|u\|^2 + b \max_{t-\mu_2(t) \le \xi \le t-\mu_1(t)} \|\psi(\xi)\|^2, \tag{3}
$$

where the constants γ, a, and b satisfy that $\gamma, b \ge 0$, and $a + b < 0$, and the functions $\mu_1(t), \mu_2(t)$ stay in the following range:

$$
0 < \mu_0 \le \mu_1(t) \le \mu_2(t) \le t + \sigma \;\; \forall t > 0. \tag{4}
$$

Throughout this paper, we make the usual assumptions that the delay $\tau(t) = t - \mu_1(t) \geq \mu_0 > 0$ to avoid the possible clustering of discontinuous points due to vanishing delay [4]. Under the condition (3) there exists a bounded absorbing set for the F-FDE (1) as t goes to infinity; see Lemma 1.

The second type of problems assumes the one-sided Lipschitz condition

$$2\langle u_1 - u_2, f(t, u_1, \psi(\cdot)) - f(t, u_2, \psi(\cdot))\rangle \leq c\|u_1 - u_2\|^2, \tag{5}$$

and the Lipschitz condition

$$2\|f(t, u, \psi_1(\cdot)) - f(t, u, \psi_2(\cdot))\| \leq d \max_{t-\mu_2(t)\leq\xi\leq t-\mu_1(t)} \|\psi_1(\xi) - \psi_2(\xi)\| \tag{6}$$

for the continuous mapping f, where the constants c and d satisfy that $d \geq 0$ and $c + d < 0$, and the functions $\mu_1(t), \mu_2(t)$ also stay in the range as in (4). As we will see in Lemma 1, the F-FDE (1) is contractive and asymptotically stable under the conditions (5)–(6).

The Volterra F-FDE (1) provides a unified framework for the mathematical study of several special F-FDEs, including fractional delay differential equations (F-DDEs), fractional integro-differential equations, and fractional delay integro-differential equations, and also for the numerical analysis of F-FDEs. Specific examples of F-FDEs include fractional delay HIV model [54], fractional financial delay system [51], population dynamics model [7], and fractional parabolic PDEs with functional terms [43].

For $\alpha = 1$, the F-FDE (1) reduces to a classical integer-order FDE, for which the dissipativity was studied in [52] under the assumption (3). It was proved also that the classical FDE has a bounded absorbing set, which means all trajectories enter in a finite time and thereafter remain inside. One important special case, the DDE, was considered in [22], and some natural FDEs were analyzed in [50]. Dissipativity is a common important feature to many dynamical systems, including many PDEs [20, 44]. In addition, we remark that all these results on dissipative FDEs have a common characteristic, i.e., the solutions decay exponentially into a given ball.

For classical FDEs, the strict contractivity and asymptotic stability were systematically studied in [30] and [4] under conditions (5)–(6), with the help of some important stability inequalities established in terms of different initial functions. As pointed out in [31], an important property of conditions (5)–(6) is that they admit stiffness, i.e., the classical Lipschitz constant of $f(t, y(t), y(\cdot))$ with respect to $y(t)$ may be very large. This allows us to deal with some strong stiff FDEs coming from the space semidiscrete parabolic-like PDEs with functional terms. As in dissipative FDEs, we emphasize that the decay in contractive FDEs is also exponential, which often leads to exponential stability for the perturbation with respect to the initial functions.

In comparison with the rich and profound studies of FDEs, not much has been done yet in the research on qualitative theoretical and numerical analysis of F-FDEs. The existence, uniqueness, and stability for F-DDEs were established in [29, 39], and some asymptotic properties of linear F-DDEs were studied in [28]. We notice that the stability and asymptotic behavior of nonlinear F-ODEs and F-FDEs have attracted more and more attention recently. The fractional generalization of Lyapunov theorem and Mittag–Leffler stability was developed in [33], and the dissipativity and asymptotic stability of nonlinear F-ODEs and F-FDEs were established by Wang et.al [47, 48] under almost the same assumptions as that for classical integer equations ($\alpha = 1$). More precisely, the following results were shown in [48].

LEMMA 1. (i) *Let $y(t)$ be a solution of the F-FDE* (1)*; then under the dissipative condition* (3) *with* $\lim_{t\to+\infty}(t-\mu_2(t))=+\infty$*, the F-FDE* (1) *is dissipative, and for any given* $\varepsilon>0$*, the ball* $B(0,\sqrt{\frac{-\gamma}{a+b}}+\varepsilon)$ *is an absorbing set.*

(ii) *Let $z(t)$ be the solution to the F-FDE* (1) *with another initial function* $\chi(t)$*. Then it holds for* $t>0$ *under conditions* (5)–(6) *that*

$$\|y(t)-z(t)\|^2 \le \frac{c}{c+d}u_0 + M_0\,, \tag{7}$$

where $M_0=\max_{-\sigma\le\xi\le 0}\|\psi(\xi)-\chi(\xi)\|^2$, $u_0=\|\varphi(0)-\chi(0)\|^2$*. Moreover,* $\lim_{t\to+\infty}(t-\mu_2(t))=+\infty$*, then for any given* $\varepsilon>0$*, there exists* $t_*=t_*(M_0,\varepsilon)>0$ *such that*

$$\|y(t)-z(t)\|^2 \le \frac{c}{c+d}\varepsilon \quad \forall\ t>t_*,$$

and with the asymptotic stability $\lim_{t\to+\infty}\|y(t)-z(t)\|=0$.

The first part of Lemma 1 implies that the the F-FDE (1) possesses a bounded absorbing set, but the trajectories may not necessarily be asymptotic to a fixed point. The dynamics in the absorbing set can be rather complex, such as chaotic, or with bifurcation. The second part characterizes the stability and asymptotic stability of the solutions. Note that the dissipativity and contractivity of ODEs or FDEs are two classes of closely related but distinct properties. As one typical example, the famous Lorenz system is dissipative but does not meet the one-sided Lipschitz condition [23].

Lemma 1 provides only the qualitative asymptotic behavior of F-FDEs for $t\to+\infty$. In fact, the condition that $\lim_{t\to+\infty}(t-\mu_2(t))=+\infty$ plays a critical role in its proof. However, from the quantitative point of view, it does not give any hint on the rate of the evolution process. It is well known that the true solutions of classical ODEs or FDEs often decay exponentially while the F-ODEs and F-FDEs decay polynomially. This is a very significant difference between FDEs and F-FDEs. Hence, as the first objective of this work, we will present a new delay-dependent fractional Halanay-like inequality, which is used to essentially improve the results of Lemma 1. The new results not only provide the asymptotic behaviors but also give the polynomial decay rate of the true solutions of F-FDEs. Special attentions will be paid to F-DDEs. Different decay rates reveal some essential differences between FDEs and F-FDEs and reflect the nonlocal nature of fractional derivative in a certain sense.

Numerical methods have been one of the main tools in the study of fractional differential equations since it is very difficult or even impossible to obtain the true solutions to most fractional equations, especially when the fractional equations involve some nonlinear terms or functional parts. In addition to the accuracy and efficiency, another basic requirement of numerical methods is to preserve some important properties of the original equations, such as the energy, asymptotic stability, and bounded absorbing set. The second objective of this paper is to design some numerical schemes which can inherit the similar dissipativity, contractivity, and asymptotic stability as the continuous F-FDEs.

We now recall some existing numerical methods for FDEs and F-FDEs in the literature. In 1975, Dahlquist [14] first introduced the one-sided Lipschitz condition for classical ODEs to deal with stiff nonlinear systems and proposed the concept of G-stability for linear multistep methods (LMMs) and one-leg methods. Butcher [8] considered the contractivity for Runge–Kutta methods and introduced the concept of B-stability. We refer to the monograph [19] for more details. In 1994, Humphries and Stuart [23] first studied the numerical dissipativity of Runge–Kutta methods for ODEs satisfying dissipative conditions. Numerical dissipativity of LMMs or one-leg methods for ODEs was analyzed in [21].

The numerical contractivity and dissipativity for ODEs were soon extended to FDEs. The early study of FDEs mainly focused on delay differential equations (DDEs) with single constant delay and some low order methods; see [4, 6, 22, 30, 50]. Wen, Yu, and Li [53] established the dissipativity of Runge–Kutta methods together with piecewise interpolation operator for FDEs under the condition that $\lim_{t\to+\infty}(t-\mu_2(t))=+\infty$. In 2012, Li [31] made a breakthrough about numerical contractivity for FDEs, where a family of high order contractive Runge–Kutta methods were identified for stiff FDEs, which broke the conjecture of order barrier on contractive numerical methods for FDEs [4]. Further, Li [32] established the B-convergence of Runge–Kutta methods for stiff FDEs for long-time stiff computations.

Predictor-corrector methods are among the most popular numerical methods for F-DDEs [17, 5]. These methods are easy to implement and often have an acceptable accuracy and stability for nonstiff problems. A spectrally accurate Petrov–Galerkin method for F-DDEs was developed in [55] based on a new spectral theory for fractional Sturm–Liouville problems. This method has the advantage of high accuracy and low storage. Other related methods may be found in [13, 39]. We point out that all the methods above for F-FDEs do not consider the special structure of the equations.

The rest of the paper is arranged as follows. In section 2, a new delay-dependent fractional generalization of Halanay-like inequality is presented, which is used to essentially improve the existing results in Lemma 1 about the asymptotic behavior of F-FDEs. In section 3, the Grünwald–Letnikov (G-L) scheme and $L1$ method, along with linear interpolation operators, are employed to construct numerical methods for F-FDEs. Some important properties of the coefficients in the numerical methods are derived, and two schemes are shown to be dissipative and contractive, and can preserve the exact decay rate of the continuous equations. Some numerical examples are presented in section 4, and the numerical performance of the new schemes and some existed ones are compared. Some concluding remarks, including several possible extensions, are given in section 5.

2. Improved dissipativity and contractivity. In this section we establish some quantitative results about the dissipativity and contractivity of solutions to the F-FDE in (1). For this development, we shall often use the Mittag–Leffler and generalized Mittag–Leffler functions $E_\alpha(z)$ and $E_{\alpha,\beta}(z)$ defined for all $z\in\mathbb{C}$:

$$E_\alpha(z)=\sum_{k=0}^{\infty}\frac{z^k}{\Gamma(\alpha k+1)},\ \alpha>0;\quad E_{\alpha,\beta}(z)=\sum_{k=0}^{\infty}\frac{z^k}{\Gamma(\alpha k+\beta)},\ \alpha,\beta>0.$$

These functions are the fractional generalization of exponential functions and play a key role in fractional calculus and have some nice properties as stated below [27].

Lemma 2. *The following properties hold for $0<\alpha<1$ and $t\in\mathbb{R}$ that*

(1) $E_\alpha(t)>0,\ E_{\alpha,\alpha}(t)>0,\ \lim\limits_{t\to-\infty}E_\alpha(t)=0$ *and* $\lim\limits_{t\to-\infty}E_{\alpha,\alpha}(t)=0$;

(2) $\dfrac{d}{dt}E_\alpha(t)=\dfrac{1}{\alpha}E_{\alpha,\alpha}(t)$ *and* ${}_0^CD_t^\alpha E_\alpha(\lambda t^\alpha)=\lambda E_\alpha(\lambda t^\alpha)$ *for* $\lambda\in\mathbb{C}$;

(3) $E_{\alpha,\beta}(\lambda t)=-\displaystyle\sum_{k=1}^{N}\frac{1}{\Gamma(\beta-k\alpha)}\frac{1}{(\lambda t)^k}+O\left(\frac{1}{(\lambda t)^{N+1}}\right)$ *as* $t\to+\infty$

for $N\in\mathbb{N}^+$ *and* $\lambda<0$.

In [26], the authors proved a very useful fractional comparison principle under the regularity assumption $y \in C[0,T] \cap C^1(0,T]$ for some $T > 0$. We generalize this result to piecewise continuous and differentiable functions with monotonic properties at discontinuous points.

LEMMA 3. *Suppose the function h is piecewise continuous and differentiable on $[0,T]$, i.e., $h \in C[0,t_0] \cap C^1(0,t_0)$ and $h \in C[t_0,T] \cap C^1(t_0,T]$ for $t_0 \in (0,T]$. Furthermore, we assume the left and right derivatives of h exists at t_0, and it holds that $h(t_0^-) \geq h(t_0^+)$. Then if $h(t_1) = 0$ for $t_1 > t_0$ and $h(t) < 0$ for $t \in [0,t_1)$, we have ${}_0^C D_{t_1}^\alpha h(t_1) \geq 0$ for $\alpha \in (0,1)$.*

Proof. The proof is similar to Lemma 2.1 in [26], but we replace the first order derivative in the definition of Caputo operator at the discontinuous point t_0 by the left and right derivatives. Let $g(t) = h(t) - h(t_1)$, we define and compute by making the integral by parts to obtain

$$\begin{aligned}&\int_0^{t_1} (t_1 - s)^{-\alpha} g'(s) ds :\equiv \int_0^{t_0} (t_1 - s)^{-\alpha} g'(s) ds + \int_{t_0}^{t_1} (t_1 - s)^{-\alpha} g'(s) ds \\ &= -t_1^\alpha g(0) + (t_1 - t_0)^\alpha \left(g(t_0^-) - g(t_0^+) \right) - \alpha \int_0^{t_1} (t_1 - s)^{-\alpha - 1} g(s) ds.\end{aligned}$$

Since $g'(t) = h'(t)$ for all t except $t = t_0$, which implies that ${}_0^C D_{t_1}^\alpha h(t_1) \geq 0$ for $\alpha \in (0,1)$. ◻

LEMMA 4 (fractional Halanay inequality). *Assume that the non-negative continuous function $y(t)$ satisfies that*

$$\begin{cases} {}_0^C D_t^\alpha y(t) \leq \gamma + a y(t) + b \max\limits_{t-\tau(t) \leq \xi \leq t} y(\xi), & 0 < t \leq T, \\ \quad y(t) = |\varphi(t)|, & -\sigma \leq t \leq 0, \end{cases} \tag{8}$$

where the constants $\gamma, b \geq 0$, $a + b < 0$, $\sigma = -\inf\limits_{t \geq 0} (t - \tau(t)) > 0$, and the delay $\tau(t) \geq \mu_0 > 0$. Then the following estimate holds

$$y(t) \leq -\frac{\gamma}{a+b} + M E_\alpha(\lambda^* t^\alpha) \tag{9}$$

for all t such that $t \geq \tau(t)$, where $M = \|\varphi(t)\|_\infty := \max\limits_{t \in [-\delta,0]} |\varphi(t)|$, and the parameter λ^* is defined by

$$\lambda^* = \sup_{t-\tau(t) \geq 0} \{\lambda : \lambda - a - b \left(E_\alpha(\lambda(t - \tau(t))^\alpha) / E_\alpha(\lambda t^\alpha) \right) = 0\}, \tag{10}$$

and it holds that $\lambda^* \in [a+b, 0]$.

Further, if the delay is bounded, i.e., $\tau(t) \leq \tau_0$ for some constant $\tau_0 > 0$, then the parameter λ^* defined by

$$\lambda^* = \sup_{t-\tau(t) \geq 1} \{\lambda : \lambda - a - b \left(E_\alpha(\lambda(t - \tau(t))^\alpha) / E_\alpha(\lambda t^\alpha) \right) = 0\}$$

is strictly negative, namely, there exists some positive constant ϵ_0 satisfying $a+b < -\epsilon_0$ such that $\lambda^* \in [a+b, -\epsilon_0]$, and the estimate in (9) holds for all t such that $t \geq \tau(t)+1$.

Before proving this lemma, we give a heuristic comment on how to justify our important estimate (9). Let us consider the equation

$$ {}_0^C D_t^\alpha y(t) = \gamma + a y(t) + b \max_{t-\tau(t)\le \xi \le t} y(\xi), \ \ for \ \ t > 0. \tag{11} $$

The difficulty in finding a solution to this equation lies in the maximum term involved. But under the assumption $\gamma, b \geq 0$, and $a+b<0$, we know the solution decays in time. On the other hand, we notice the nice property of the fractional derivative of Mittag–Leffler function in part (ii) of Lemma 2, so we guess there should exist some negative constant $\lambda < 0$ such that the solution of (11) may involve a term like $E_\alpha(\lambda t^\alpha)$. If it is the case, then the maximum term involved in (11) can be simplified, as it holds that $\max_{t-\tau(t)\le\xi\le t} E_\alpha(\lambda\xi^\alpha) = E_\alpha(\lambda(t-\tau(t))^\alpha)$. This is because the function $E_\alpha(\lambda\xi^\alpha)$ is monotone decreasing with respect to $\xi \geq 0$ for $\alpha \in (0,1)$ and $\lambda < 0$. Furthermore, we know from our early result in [48] that the limit of the solution is $-\gamma/(a+b)$. These facts suggest us to consider a solution of the form $y(t) = -\gamma/(a+b) + ME_\alpha(\lambda t^\alpha)$. We substitute this solution into (11) to obtain

$$ M\lambda E_\alpha(\lambda t^\alpha) = \gamma + a\left(-\frac{\gamma}{a+b} + ME_\alpha(\lambda t^\alpha)\right) + b\left(-\frac{\gamma}{a+b} + ME_\alpha\left(\lambda(t-\tau(t))^\alpha\right)\right), $$

which shows the crucial parameter λ may be given by the solutions to the equation

$$ \lambda - a - b\left(E_\alpha(\lambda(t-\tau(t))^\alpha)/E_\alpha(\lambda t^\alpha)\right) = 0. $$

Proof of Lemma 4. We write $h(\lambda) = \lambda - a - b\left(E_\alpha(\lambda(t-\tau(t))^\alpha)/E_\alpha(\lambda t^\alpha)\right)$. For any fixed $t \geq 0$, we first show that there exists some root $\lambda \in [a+b, 0)$ such that $h(\lambda) = 0$.

In fact, we first see $h(0) = -a-b > 0$. Let $e_\alpha = E_\alpha(\lambda(t-\tau(t))^\alpha)/E_\alpha(\lambda t^\alpha)$. We know that for $\lambda < 0$, the function $E_\alpha(\lambda\xi^\alpha)$ is positive and monotone decreasing with respect to $\xi \geq 0$. This implies that $E_\alpha(\lambda(t-\tau(t))^\alpha) \geq E_\alpha(\lambda t^\alpha)$ by noting that $t \geq t-\tau(t) \geq 0$, namely, we have $e_\alpha \geq 1$ for $\lambda < 0$. Therefore $h(\lambda) = \lambda - a - be_\alpha \leq \lambda - a - b$ for $\lambda < 0$, which implies $h(a+b) \leq 0$.

Similarly, we can derive that $0 < e_\alpha \leq 1$ for $\lambda \geq 0$, which implies that $h(\lambda) = \lambda - a - be_\alpha \geq \lambda - a - b > 0$. Therefore, there exists no positive roots for the equation $h(\lambda) = 0$, so concluding that the roots of $h(\lambda) = 0$ satisfy that $\lambda \in [a+b, 0)$. Clearly, it then follows from the definition of λ^* in (10) that $\lambda^* \in [a+b, 0]$.

Now we show that the parameter λ^* defined in (10) is strictly less than zero for the case where the delay $\tau(t)$ is bounded, i.e., there exists positive constant τ_0 such that $\tau(t) \leq \tau_0$. Noting the fact that $h(0) = -a-b > 0$, it suffices to prove that $h'(\lambda)|_{\lambda=0}$ is bounded. By direct computing, we have

$$ \begin{aligned} h'(0) &= 1 - \\ &\quad b\, \frac{(t-\tau(t))^\alpha E_\alpha(\lambda t^\alpha) E_{\alpha,\alpha}(\lambda(t-\tau(t))^\alpha) - t^\alpha E_\alpha(\lambda(t-\tau(t))^\alpha) E_{\alpha,\alpha}(\lambda t^\alpha)}{\alpha E_\alpha^2(\lambda t^\alpha)}\Bigg|_{\lambda=0} \\ &= 1 + \frac{b}{\alpha\Gamma(\alpha)}\left(t^\alpha - (t-\tau(t))^\alpha\right) = 1 + \frac{b}{\Gamma(\alpha)}\frac{\tau(t)}{\xi_t^{1-\alpha}} \leq 1 + \frac{b\tau_0}{\Gamma(\alpha)} < +\infty, \end{aligned} $$

where we have used the assumption that $t - \tau(t) \geq 1$ and the fact that $1 \leq \xi_t \in (t-\tau(t), t)$. By the Taylor expansion, we can write

$$ h(-\epsilon) = h(0) - h'(0)\epsilon + O(\epsilon^2) $$

for $\epsilon > 0$. The facts that $h(0) = -a - b > 0$ and $h'(0)$ is bounded allow us to choose some small positive constant $\epsilon_0 > 0$ such that $h(-\epsilon_0) \geq 0$, which implies that $\lambda^* \in [a + b, -\epsilon_0]$.

Finally we come to prove our main estimate (9). This estimate follows from Lemma 2.3 of [48] for the case that $M = 0$. Now we consider $M > 0$. For any given $\varepsilon > 0$, we prove that

$$y(t) < -\frac{\gamma + \varepsilon}{a + b} + M E_\alpha(\lambda^* t^\alpha) \ \ \forall \ t \geq \tau(t). \tag{12}$$

In fact, if this is not true, then there exists some t such that $t \geq \tau(t)$ and

$$y(t) \geq z(t) :\equiv -\frac{\gamma + \varepsilon}{a + b} + M E_\alpha(\lambda^* t^\alpha). \tag{13}$$

Let t_* be the first time for $z(t) = y(t)$, namely,

$$t_* = \inf\{t \geq \tau(t) : y(t) \geq z(t)\}. \tag{14}$$

To continue our proof, we set $\delta(t) = y(t) - z(t)$. Then we know from the definition that $\delta(t_*) = 0$, and $\delta(t) < 0$ for $t_* - \tau(t_*) \leq t < t_*$. On the interval $[0, t_* - \tau(t_*)]$, we may define the function $z(t)$ appropriately, e.g., $z(t) = y(t) + \varepsilon_*$ for a sufficiently small $\varepsilon_* > 0$, so that $z(t) > y(t)$ for all $t \in [0, t_*)$. Now we can apply the fractional comparison principle in Lemma 3 and conclude that ${}_0^C D_{t_*}^\alpha \delta(t_*) \geq 0$.

On the other hand, we can estimate

$$\begin{aligned}
{}_0^C D_{t_*}^\alpha \delta(t_*) = &\ {}_0^C D_{t_*}^\alpha y(t_*) - {}_0^C D_{t_*}^\alpha z(t_*) \\
\leq &\left(\gamma + a y(t_*) + b \max_{t_* - \tau(t_*) \leq \xi \leq t_*} y(\xi)\right) - M\lambda^* E_\alpha(\lambda^* t_*^\alpha) \\
\leq &\left(\gamma + a\left(-\frac{\gamma + \varepsilon}{a + b} + M E_\alpha(\lambda^* t_*^\alpha)\right) + b \max_{t_* - \tau(t_*) \leq \xi \leq t_*} y(\xi)\right) - M\lambda^* E_\alpha(\lambda^* t_*^\alpha).
\end{aligned} \tag{15}$$

In view of $t_* - \tau(t_*) \geq 0$, we have that

$$\max_{t_* - \tau(t_*) \leq \xi \leq t_*} y(\xi) \leq \max_{t_* - \tau(t_*) \leq \xi \leq t_*} z(\xi) = -\frac{\gamma + \varepsilon}{a + b} + M E_\alpha(\lambda^* (t_* - \tau(t_*))^\alpha).$$

Now it follows from (15) that

$$\begin{aligned}
{}_0^C D_{t_*}^\alpha \delta(t_*) \leq &\ \gamma - \frac{a}{a + b}(\gamma + \varepsilon) + a M E_\alpha(\lambda^* t_*^\alpha) \\
&+ b\left(-\frac{\gamma + \varepsilon}{a + b} + M E_\alpha(\lambda^* (t_* - \tau(t_*))^\alpha)\right) - M\lambda^* E_\alpha(\lambda^* t_*^\alpha) \\
= &- M E_\alpha(\lambda^* t_*^\alpha)\left(\lambda^* - a - b\frac{E_\alpha(\lambda^* (t_* - \tau(t_*))^\alpha)}{E_\alpha(\lambda^* t_*^\alpha)}\right) - \varepsilon \\
\leq &- \varepsilon < 0,
\end{aligned}$$

which contradicts with the fact that ${}_0^C D_{t_*}^\alpha \delta(t_*) \geq 0$. This completes the proof of Lemma 4. ∎

Earlier in [48], we proved a fractional Halanaly inequality under the condition that $\lim_{t \to +\infty}(t - \tau(t)) = +\infty$, and only asymptotic limit was achieved there, without any hints on the decay rate of the solution. However, the polynomial decay rate of the solutions is an important characteristic for fractional differential equations and is also the main difference between fractional and integer differential equations. The case that $\tau(t) \leq \tau_0$ for some constant $\tau_0 > 0$ in Lemma 4 is quite appropriate for most F-FDEs, which excludes a class of unbounded delay differential equations, namely, the so-called proportion DDEs, for which it holds that $\tau(t) = qt$ for $q \in (0, 1)$. But we will see in our numerical Example 4.2 that the decay rate estimation (as stated in (21)) still holds for the proportion DDEs. One of the direct applications of our new fractional Halanaly inequality is to help us establish the fractional dissipativity and contractivity below for F-FDEs, which also improve essentially our early results [48]. For our later analysis, we now introduce an important fractional version of the Leibniz formula [1, 15].

LEMMA 5. *The following equality holds for $0 < \alpha < 1$ and any two absolutely continuous functions $x(t)$ and $y(t) \in \mathbb{R}^n$ for $n \geq 1$ on $[0, T]$:*

$$x^T(t) \cdot {}_0^C D_t^\alpha y(t) + y^T(t) \cdot {}_0^C D_t^\alpha x(t) =$$
$${}_0^C D_t^\alpha \left(x^T(t) \cdot y(t)\right) + \frac{\alpha}{\Gamma(1-\alpha)} \int_0^t \frac{1}{(t-\xi)^{1-\alpha}} \left(\int_0^\xi \frac{x'(\eta) d\eta}{(t-\eta)^\alpha} \cdot \int_0^\xi \frac{y'(s) ds}{(t-s)^\alpha} \right) d\xi,$$

where the sign " $\cdot$ " is the usual vector product between vectors in $\mathbb{R}^n$ for $n \geq 1$.

We readily see from the identity in Lemma 5 by taking $x(t) = y(t)$ that

$$(16) \qquad {}_0^C D_t^\alpha \left(y^T(t) \cdot y(t)\right) \leq 2y^T(t) \cdot {}_0^C D_t^\alpha y(t) \quad for \quad 0 < \alpha < 1\,.$$

This inequality and its discrete version given in Lemma 9 later are crucial for our subsequent analysis. This inequality can be viewed as the fractional extension of the Leibniz formula in classical calculus for the product of two functions and enable us to multiply both sides of the F-FDEs (1) and its discrete scheme by appropriate functions and derive some desired a priori bounds of the solution and the approximate solution.

THEOREM 6. *Let $y(t)$ is a solution to the F-FDE* (1) *and $\mu_2(t)$ be the function in* (3) *or* (6) *satisfying that $\mu_2(t) \leq \mu_*$ for a constant $\mu_* > 0$.*

(i) *Under the dissipative condition* (3), *it holds that*

$$(17) \qquad \|y(t)\|^2 \leq -\frac{\gamma}{a+b} + M E_\alpha(\lambda^* t^\alpha)$$

for all t such that $t \geq \mu_2(t)+1$, where $M = \|\varphi(t)\|_\infty$ and the parameter $\lambda^* \in [a+b, -\epsilon_1]$ is given by

$$(18) \qquad \lambda^* = \sup_{t-\mu_2(t)\geq 1} \{\lambda : \lambda - a - b\left(E_\alpha(\lambda(t-\mu_2(t))^\alpha)/E_\alpha(\lambda t^\alpha)\right) = 0\},$$

where ϵ_1 is some positive constant such that $a + b < -\epsilon_1 < 0$. Moreover, the F-FDE (1) *is dissipative, i.e., the ball $B(0, \sqrt{-\gamma/(a+b)} + \varepsilon)$ is an absorbing set as $t \to +\infty$ for any given $\varepsilon > 0$, and follows the dissipative rate*

$$\|y(t)\|^2 \leq -\frac{\gamma}{a+b} + M \frac{C_\alpha}{t^\alpha} \quad as\ t \to +\infty,$$

where $C_\alpha > 0$ is a constant independent of t.

(ii) *Under the conditions* (5)–(6), *the stability inequality holds*

$$(19) \qquad \|y(t) - z(t)\|^2 \leq M_0 E_\alpha(\lambda^{**} t^\alpha)$$

for all t such that $t \geq \mu_2(t)+1$, where $M_0 = \max_{-\sigma \leq \xi \leq 0} \|\psi(\xi) - \chi(\xi)\|^2$ and $z(t)$ is the solution to the F-FDE (1) *with initial function $\chi(t)$, and the parameter $\lambda^{**} \in [c+d, -\epsilon_2]$ is given by*

$$\lambda^{**} = \sup_{t-\mu_2(t)\geq 1} \{\lambda : \lambda - c - d\,(E_\alpha(\lambda(t-\mu_2(t))^\alpha)/E_\alpha(\lambda t^\alpha)) = 0\}, \tag{20}$$

where ϵ_2 is some positive constant such that $c+d < -\epsilon_2 < 0$. And as $t \to +\infty$, we have the following asymptotic contractive rate

$$\|y(t) - z(t)\|^2 \leq M_0 \frac{C_\alpha}{t^\alpha}, \tag{21}$$

where $C_\alpha > 0$ is a constant independent of t.

Proof. We first show (i). It follows from the condition (3) that

$$2y^T(t) \cdot {}_0^C D_t^\alpha y(t) \leq \gamma + a\|y(t)\|^2 + b \max_{t-\mu_2(t)\leq \xi \leq t-\mu_1(t)} \|y(\xi)\|^2,$$

using this we readily derive from Lemma 5 that

$${}_0^C D_t^\alpha \left(\|y(t)\|^2\right) \leq \gamma + a\|y(t)\|^2 + b \max_{t-\mu_2(t)\leq \xi \leq t} \|y(\xi)\|^2.$$

Now the desired results follow directly from the fractional Halanay inequality and the asymptotic expansion of Mittag–Leffler function given in Lemma 2.

Next we prove (ii). By the definition of $y(t)$ and $z(t)$ from the F-FDEs (1), we can write

$$\begin{aligned}
&\left\langle {}_0^C D_t^\alpha (y(t) - z(t)),\ y(t) - z(t)\right\rangle \\
&\quad = \langle f(t, y(t), y(\cdot)) - f(t, z(t), z(\cdot)),\ y(t) - z(t)\rangle \\
&\quad = \langle f(t, y(t), y(\cdot)) - f(t, y(t), z(\cdot)),\ y(t) - z(t)\rangle \\
&\qquad + \langle f(t, y(t), z(\cdot)) - f(t, z(t), z(\cdot)),\ y(t) - z(t)\rangle.
\end{aligned}$$

Using the Cauchy–Schwarz inequality and conditions (5)–(6), we derive

$$\begin{aligned}
&2\left\langle {}_0^C D_t^\alpha (y(t) - z(t)),\ y(t) - z(t)\right\rangle \\
&\leq d \max_{t-\mu_2(t)\leq \xi \leq t-\mu_1(t)} \|y(\xi) - z(\xi)\| \cdot \|y(t) - z(t)\| + c\|y(t) - z(t)\|^2.
\end{aligned}$$

This implies that $u(t) = y(t) - z(t)$ satisfies

$$2u^T(t) \cdot {}_0^C D_t^\alpha u(t) \leq d \max_{t-\mu_2(t)\leq \xi \leq t-\mu_1(t)} \|u(\xi)\| \cdot \|u(t)\| + c\|u(t)\|^2.$$

But using Lemma 5, we can further deduce

$$\begin{aligned}
{}_0^C D_t^\alpha \left(\|u(t)\|^2\right) &\leq 2u^T(t) \cdot {}_0^C D_t^\alpha u(t) \\
&\leq d \max_{t-\mu_2(t)\leq \xi \leq t-\mu_1(t)} \|u(\xi)\| \cdot \|u(t)\| + c\|u(t)\|^2 \\
&\leq d \max_{t-\mu_2(t)\leq \xi \leq t} \|u(\xi)\| \cdot \|u(t)\| + c\|u(t)\|^2 \\
&\leq d \max_{t-\mu_2(t)\leq \xi \leq t} \|u(\xi)\|^2 + c\|u(t)\|^2.
\end{aligned}$$

Now the result (19) follows directly from Lemma 4, while the estimate (21) of the decay rate is a consequence of the asymptotic expansion of Mittag–Leffler function given in Lemma 2. ∎

3. Numerical approximations for F-FDEs. In this section we propose two numeric schemes for the F-FDE (1) and investigate their numerical dissipativity and contractivity. To do so, we consider a partition of the interval $[0,T]$, $t_n = nh$, $n = 0,1,2,3,\ldots$, with $h > 0$ being the step-size. We shall write y_n for the approximation of $y(t_n)$, and employ the canonical interpolation operator Π^h introduced by Li [31] to approximate the true solution $y(t)$, which satisfies the canonical condition

$$\max_{\bar{t}\le t\le t_n} \left\|\Pi^h(t,\varphi,y_0,y_1,\ldots,y_n) - \Pi^h(t,\chi,z_0,z_1,\ldots,z_n)\right\|$$
$$\le \begin{cases} c_\pi \max\limits_{\eta(\bar{t})\le j\le n} \|y_j - z_j\| & for\ -\tau\le\bar{t}\le t_n, \eta\left(\bar{t}\right)\ge 0,\\ c_\pi \max\left\{\max\limits_{1\le j\le n}\|y_j - z_j\|, \max\limits_{-\tau\le t\le 0}\|\phi(t)-\chi(t)\|\right\} & for\ -\tau\le\bar{t}\le t_n, \eta\left(\bar{t}\right) < 0,\end{cases}$$

where the function $\eta(t)$ is defined by

$$\eta(t) = \min\{m : \text{the integer } m \ge 0, t_m \ge t\} - p,$$

with p being a non-negative integer depending only on the procedure of interpolation, and the canonical constant $c_\pi \ge 1$ is independent of $\bar{t}, n, y_j, z_j, \phi$ and χ. The general method of construction a high order interpolation operator Π^h based on Lagrangian polynomial can be found in [31]. We will consider in this work only the piecewise linear interpolation of first order defined by

$$\Pi^h(t,\varphi,y_0,y_1,...y_n) = \begin{cases}\phi(t) & for\ t\in[-\tau,0],\\ \dfrac{(t_j - t)y_{j-1} + (t - t_{j-1})y_j}{h} & for\ t\in(t_{j-1},t_j]\end{cases} \tag{22}$$

for $j = 1,2,\ldots,n$. It is easy to check that the canonical condition is satisfied with $c_\pi = 1$ and $p = 1$.

In the numerical treatment of F-ODEs, methods based on the discretization of the fractional derivative are referred as backward differentiation formulas (BDFs). BDFs are often implicit and have good stability. Combining BDFs for Caputo derivative and canonical interpolation operator for functional term lead to the following numerical method for the F-FDE (1) in the full-term recursion:

$$\begin{cases} y^h(t) = \Pi^h(t,\varphi,y_0,y_1,\ldots,y_n),\\ \sum\limits_{j=0}^{n}\omega_{n-j}y_j = h^\alpha f(t_n,y_n,y^h(\cdot)), \quad n = 1,2,3,\ldots.\end{cases} \tag{23}$$

There are several different approaches to construct the weights $\{\omega_n\}_{n=0}^{\infty}$ in (23) in literature, resulting in a wide variety of schemes with different accuracy and stability. We will consider only two popular schemes: the weights are given by G-L formula [27] and $L1$ method [34, 45], while the piecewise linear interpolation is given in (22).

3.1. Grünwald–Letnikov formula. For $0 < \alpha < 1$, the G-L fractional derivative ${}_0^{GL}D_t^\alpha y(t)$ is defined by

$${}_0^{GL}D_t^\alpha y(t) = \lim_{h\to 0^+}\frac{\left(\Delta_h^\alpha\right)y(t)}{h^\alpha} = \lim_{h\to 0^+}\frac{1}{h^\alpha}\sum_{k=0}^{m=[t/h]}(-1)^k\begin{pmatrix}\alpha\\ k\end{pmatrix}y(t-kh). \tag{24}$$

If we do not perform the limit operation $h \to 0^+$ but let $h > 0$ be the step-size, then we get the discretized version of the operator ${}_0^{GL}D_t^{\alpha} y(t)$:

$$ {}_0^{GL}D_t^{\alpha} y(t_n) = \frac{1}{h^{\alpha}} \sum_{k=0}^{n} (-1)^k \begin{pmatrix} \alpha \\ k \end{pmatrix} y_{n-k} + O(h) = \frac{1}{h^{\alpha}} \sum_{k=0}^{n} \omega_k y_{n-k} + O(h), $$

where $\omega_k = (-1)^k \begin{pmatrix} \alpha \\ k \end{pmatrix}$. For well-behaved functions, the G-L derivative is equivalent to the Riemann–Liouville (R-L) fractional derivative defined by ${}_0^{RL}D_t^{\alpha} y(t) := \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_0^t \frac{y(s)}{(t-s)^{\alpha}} ds$ for $t > 0$. Hence, the G-L scheme is also a good first order approximation of the R-L derivative, i.e., ${}_0^{RL}D_t^{\alpha} y(t_n) = \frac{1}{h^{\alpha}} \sum_{k=0}^{n} \omega_k y_{n-k} + O(h)$. By making use of the relationship that ${}_0^{C}D_t^{\alpha} y(t) = {}_0^{RL}D_t^{\alpha} (y(t) - y(0))$ between the Caputo and R-L fractional derivatives, we can see that it is natural to present the first order numerical approximation ${}_0^{C}D_{t_n}^{\alpha} y(t_n) = {}_0^{RL}D_{t_n}^{\alpha} (y(t_n) - y(0)) = \frac{1}{h^{\alpha}} \sum_{k=0}^{n} \omega_k (y_{n-k} - y_0) + O(h)$, which can be rewritten as

$$ {}_0^{C}D_{t_n}^{\alpha} y(t_n) = \frac{1}{h^{\alpha}} \left(\sum_{k=1}^{n} \omega_{n-k} y_k + \delta_n y_0 \right) + O(h). \tag{25} $$

This choice also yields good numerical stability [18]. For any fixed $n \geq 1$, the weights ω_k can be recursively evaluated as

$$ \begin{cases} \omega_0 = 1, \quad \omega_k = \left(1 - \dfrac{\alpha+1}{k}\right) \omega_{k-1}, \quad k = 1, 2, \ldots, n-1, \\ \delta_n = -\displaystyle\sum_{j=0}^{n-1} \omega_j. \end{cases} \tag{26} $$

The G-L formula is a simple and effective numerical scheme with first order accuracy. It has a wide range of applications and can be used to construct a variety of high order and shifted schemes. The weights ω_k are the coefficients of the generating function $\omega(\xi) = (1-\xi)^{\alpha}$ and have the following properties [18].

LEMMA 7. *For $0 < \alpha < 1$, it holds for the coefficients $\omega_k = (-1)^k \begin{pmatrix} \alpha \\ k \end{pmatrix}$ that*

(i) $\omega_0 = 1,\ \omega_n < 0, |\omega_{n+1}| < |\omega_n|, \quad n = 1, 2, \ldots;$

(ii) $\omega_0 = -\sum_{j=1}^{\infty} \omega_j > -\sum_{j=1}^{n} \omega_j, \quad n \geq 1;$

(iii) $\omega_n = O(n^{-1-\alpha}),\ \delta_n = O(n^{-\alpha}) \quad$ *as* $n \to \infty$.

3.2. *L*1 method. The $L1$ method is a very popular algorithm for the numerical treatment of Caputo derivatives and can be written as the discrete convolution quadrature of the form

$$ {}_0^{C}D_t^{\alpha} y(t)|_{t=t_n} = \frac{1}{h^{\alpha}} \sum_{k=0}^{n} \omega_{n-k} y_k + O(h^q), \tag{27} $$

where the parameter q is the order of the local truncation error and determined by the regularity of the solution, and the coefficients are given by

$$ \omega_0 = \frac{1}{\Gamma(2-\alpha)}, \quad \omega_n = \frac{1}{\Gamma(2-\alpha)} \left((n-1)^{1-\alpha} - n^{1-\alpha} \right), $$

$$\omega_k = \frac{1}{\Gamma(2-\alpha)} \left((k+1)^{1-\alpha} - 2k^{1-\alpha} + (k-1)^{1-\alpha} \right) , \quad k = 1, \ldots, n-1 \, .$$

The $L1$ method often leads to unconditionally stable algorithms and has the accuracy $O(h^{2-\alpha})$ for smooth data and solutions (see, e.g., [34, 45]) but only a first order convergence for nonsmooth initial data [24].

By direct computing, we can derive the following properties about the coefficients in (27).

LEMMA 8. *The coefficients of the $L1$ method satisfy the following relations:*

(i) $\omega_0 > 0, \omega_1 < \omega_2 < \cdots < \omega_{n-1} < 0 \; \omega_n < 0 for\ any \quad n \geq 1;$

(ii) $k^{1+\alpha}\omega_k \to -\dfrac{\alpha}{\Gamma(1-\alpha)}$ *as* $k \to \infty$ *for* $k \neq n,$

and $n^{\alpha}\omega_n \to -\dfrac{1}{\Gamma(1-\alpha)}$ *as* $n \to \infty.$

Other important numerical approximations to Caputo or R-L fractional derivatives in the literature include fractional LMMs [18, 35, 36], shifted Grünwald difference operators [38, 46], fourth order weighted and shifted difference operators [11], Diethelm's method [16], high-order schemes according to polynomial interpolation [9], and numerical methods based on the rational approximation of the generating functions of F-BDFs [40], etc.

In general, we can combine some proper numerical methods for F-ODEs and interpolation operators for functional terms to get the corresponding numerical methods for F-FDEs. But in this work, we consider only the two numerical schemes we described earlier, due to two main reasons. First, higher order schemes often have poor stability. Generally speaking, the order of an A-stable F-LMMs cannot exceed two [36]. Second, the analysis for the global F-LMMs will be much more technical and complicated than that for classical LMMs in the framework of G-norm.

3.3. Numerical dissipativity and contractivity analysis. We know that the Dahlquist's G-stability plays a core role in the dissipativity and contractivity of LMMs and one-leg methods for classical ODEs and FDEs [14, 19, 21]. For a fixed k, let $X_m = (x_{m+k-1}, x_{m+k-2}, \ldots, x_m)^T$, and consider the inner product norm in $\mathbb{R}^{n \cdot k}$:

$$\|X_m\|_G^2 = \sum_{i=1}^{k} \sum_{j=1}^{k} g_{ij} \left\langle x_{m+i-1}, x_{m+j-1} \right\rangle , \tag{28}$$

where the matrix $G = (g_{ij})_{i,j=1,2,\ldots,k}$ is assumed to be real, symmetric, and positive definite. Then all the analysis for classical LMMs will be performed under the G-norm. In essence, G-norm concept can change the LMMs into the corresponding one-leg methods so that one may make use of the dissipativity and contractivity conditions. A key feature of the analysis for classical k-step LMMs under the G-norm is that the dimension of the matrix G is fixed all the time, namely, it is equal to k, i.e., the number of steps of the corresponding LMMs. However, due to the nonlocal nature of fractional operators and the growth of the matrix size along with time, the G-norm can not be applied to F-LMMs. It is still a fundamental issue how to develop an effective novel analysis framework for global F-LMMs for nonlinear F-ODEs and F-FDEs.

We intend to make some initial efforts in this direction. For this purpose, we first introduce several auxiliary results. The first one is the discrete version of the inequality (16) we established earlier in [49].

LEMMA 9. *If the weights $\{\omega_n\}_{n=0}^{\infty}$ from some numerical approximation to the Caputo fractional derivative in* (2) *have the following properties*

$$\begin{cases} \text{(i) } \omega_0 > 0, \\ \text{(ii) } \omega_j < 0 \ \forall \ j \geq 1, \\ \text{(iii) } \sum\limits_{j=0}^{n} \omega_j \geq 0 \text{ for any given } n \geq 1, \end{cases}$$

then the following inequality holds:

$$\sum_{j=0}^{n} \omega_{n-j} \|x_j\|^2 \leq \left\langle 2x_n, \sum_{j=0}^{n} \omega_{n-j} x_j \right\rangle, \qquad n \geq 1. \tag{29}$$

It is easy to check that both G-L formula and $L1$ method satisfy the conditions in Lemma 9. For the proof of the numerical dissipativity of the F-FDEs, we shall first establish the boundedness of the solutions to a Volterra difference equation, by applying the discrete variant of a Paley–Wiener theorem, which was first introduced by Lubich [35].

LEMMA 10. *Consider the discrete Volterra equation*

$$x_n = p_n + \sum_{j=0}^{n} q_{n-j} x_j, \quad n \geq 0, \tag{30}$$

where the kernel $\{q_n\}_{n=0}^{\infty}$ belongs to l^1, i.e., $\sum_{j=0}^{\infty} |q_j| < \infty$. Then $x_n \to 0$ (resp., bounded) whenever $p_n \to 0$ (resp., bounded) as $n \to \infty$ if and only if the Paley–Wiener condition is satisfied, i.e.,

$$\sum_{j=0}^{\infty} q_j \zeta^j \neq 1 \ for \ |\zeta| \leq 1. \tag{31}$$

Moreover, let $\{r_n\}_{n=0}^{\infty}$ be the coefficients defined by the relation

$$\sum_{j=0}^{\infty} r_j \zeta^j = \left(1 - \sum_{j=0}^{\infty} q_j \zeta^j\right)^{-1}. \tag{32}$$

Then if $\{q_n\}_{n=0}^{\infty}$ belongs to l^1 and the condition (31) *holds, $\{r_n\}_{n=0}^{\infty}$ is in l^1 and has the estimate $\|x\|_{l^\infty} \leq \|r\|_{l^1} \|p\|_{l^\infty}$.*

Generally speaking, it is much more difficult to derive the exact decay rates for a numerical method than to achieve only the qualitative properties such as the stability and asymptotic stability of some equilibrium solutions. Wang, Xiao, and Zou [49] provided an exact convergence rate for the solutions to a Volterra difference equation based on some recent qualitative analysis of the discrete Volterra equation [3].

LEMMA 11. *Consider the Volterra difference equation*

$$x_{n+1} = f_n + \sum_{j=0}^{n} F_{n-j} x_j, \quad n \geq 1,$$

satisfying the spectral condition $\rho = \sum_{j=0}^{\infty} |F_j| < 1$.

(i) *If $\lim\limits_{n\to\infty} f_n = f_\infty$, then $\lim\limits_{n\to\infty} x_n = (1-\rho)^{-1} f_\infty$* [3].

(ii) *If $f_n \to \frac{c_1}{n^\alpha}$ as $n \to \infty$ for some $c_1 > 0$ and $0 < \alpha < 1$, then $x_n \to c_1 (1-\rho)^{-1} / n^\alpha$ as $n \to \infty$* [49].

We are now in the position to prove the main results of this section, which can be seen as the discrete version of Theorem 6.

THEOREM 12. *Consider the numerical method* (23) *for the F-FDE* (1)*, where the weights* $\{\omega_k\}_{n=0}^{\infty}$ *are given by either the G-L formula or L1 method, and the interpolation operator* $y^h = \Pi^h$ *is given by the piecewise linear interpolation* (22)*. Assume that* $\eta(\hat{t}_n) \geq 1$ *with* $\hat{t}_n = t_n - \mu_2(t_n)$*.*

(i) *The numerical solutions are dissipative under the dissipative condition* (3)*, i.e., for any given initial value* y_0 *and* $\varepsilon > 0$*, there is a bounded set* $B\,(0, r)$ *and* $n_0 \in N^+$ *such that* $\|y_n\| \in B\,(0, r)$ *for all* $n > n_0$*, with* $r = \sqrt{-\gamma/(a+b)} + \varepsilon$*.*

(ii) *Under the one-sided Lipschitz and Lipschitz condition* (5)–(6)*, the numerical solutions are contractive and can preserve the exact contractivity rate towards zero like* $n^{-\alpha}$ *as* $n \to \infty$*, i.e., the estimate* $\|y_n - z_n\|^2 \leq c_\alpha/n^\alpha$ *holds for some constant* $c_\alpha > 0$ *independent of* n*, where* y_n *and* z_n *are two numerical solutions given by* (23) *with initial functions* φ *and* χ*, respectively.*

Proof. We first prove part (i). Using the dissipativity condition (3), we easily see

$$\left\langle 2y_n, \sum_{j=0}^{n} \omega_{n-j} y_j \right\rangle = 2h^\alpha \left\langle y_n, f(t_n, y_n, y^h(\cdot)) \right\rangle$$
$$\leq h^\alpha \left(\gamma + a\|y_n\|^2 + b \max_{t_n-\mu_2(t_n) \leq t \leq t_n - \mu_1(t_n)} \|y^h(t)\|^2 \right).$$

But applying Lemma 9 and the canonical condition satisfied by the linear interpolation defined in (22) with canonical constant $c_\pi = 1$, we derive

$$\sum_{j=0}^{n} \omega_{n-j} \|y_j\|^2 \leq h^\alpha \left(\gamma + a\|y_n\|^2 + b \max_{\eta(\hat{t}_n) \leq j \leq n} \|y_j(t)\|^2 \right),$$

which can be written in the following equivalent convolution Volterra inequality:

$$\|y_n\|^2 \leq \frac{h^\alpha \gamma}{\omega_0 - h^\alpha a} + \sum_{j=0}^{n-1} \frac{|\omega_{n-j}|}{\omega_0 - h^\alpha a} \|y_j\|^2 + \frac{h^\alpha b}{\omega_0 - h^\alpha a} \max_{\eta(\hat{t}_n) \leq j \leq n} \|y_j\|^2. \tag{33}$$

For the sake of simplicity, we introduce the coefficients

$$A = \frac{h^\alpha \gamma}{\omega_0 - h^\alpha a}, \; C_{n-j} = \frac{|\omega_{n-j}|}{\omega_0 - h^\alpha a}, \; B = \frac{h^\alpha b}{\omega_0 - h^\alpha a}.$$

We now consider (33) in two cases:

$$(a) \max_{\eta(\hat{t}_n) \leq j \leq n} \|y_j\|^2 = \|y_n\|^2; \qquad (b) \max_{\eta(\hat{t}_n) \leq j \leq n} \|y_j\|^2 = \max_{\eta(\hat{t}_n) \leq j \leq n-1} \|y_j\|^2.$$

For case (a), the inequality (33) is equivalent to

$$\|y_n\|^2 \leq A + \sum_{j=0}^{n-1} C_{n-j} \left\|y_j\right\|^2 + B \left\|y_n\right\|^2. \tag{34}$$

By setting $z_0 := \|y_0\|^2$, $q_0 := B \in (0, 1)$, and $z_n := \|y_n\|^2$, $p_n := A$, $q_n := C_n$ for $n \geq 1$, we can rewrite (34) to

$$z_n \leq p_n + \sum_{j=0}^{n} q_{n-j} z_j, \quad n \geq 1. \tag{35}$$

To derive a desired bound of $\|y_n\|$, we further define a sequence $\{x_n\}$ by

$$x_n = p_n + \sum_{j=0}^{n} q_{n-j} x_j, \quad n \geq 0\,. \tag{36}$$

For convenience, we have included the case $n = 0$ in (36), for which we need to choose x_0 such that $0 \leq z_0 \leq x_0$ and then set $p_0 = (1 - q_0)x_0$. This is feasible as $z_0 = \|y_0\|^2$ is given. Now using the facts that $0 \leq z_0 \leq x_0$ and $p_0 = (1 - q_0)x_0$, we can see that (35) holds also for $n = 0$.

Next we claim that $z_n \leq x_n$ for $n \geq 1$. In fact, we readily see from (35)–(36) with $n = 1$ that $z_1 - x_1 \leq q_1(z_0 - x_0) + q_0(z_1 - x_1)$. Therefore, $(1 - q_0)(z_1 - x_1) \leq q_1(z_0 - x_0) \leq 0$, which implies that $z_1 \leq x_1$. Similarly we can derive from (35)–(36) for $n > 1$ that $(1-q_0)(z_n - x_n) \leq \sum_{j=0}^{n-1} q_{n-j}(z_j - x_j) \leq 0$, which concludes our claim.

Now we apply Lemma 10, i.e., the discrete Paley–Wiener theorem, to the sequence $\{x_n\}$ in (36), to get the bound of $\{x_n\}$, which then implies our desired bound of $z_n = \|y_n\|^2$. To do so, we need to verify two conditions in Lemma 10, i.e., the Paley–Wiener condition (31) and the sequence $\{r_n\}$ defined by (32) lying in l^1. For the condition (31), it suffices to show that $\rho = \sum_{j=0}^{\infty} q_j < 1$, which comes from the direct computing:

$$\rho = \sum_{j=0}^{\infty} q_j = \sum_{j=1}^{\infty} C_j + B = \begin{cases} \dfrac{1 + h^\alpha b}{1 - h^\alpha a} < 1 & \textit{for GL formula}, \\ \dfrac{1 + \Gamma(2-\alpha)h^\alpha b}{1 - \Gamma(2-\alpha)h^\alpha a} < 1 & \textit{for L1 method.} \end{cases} \tag{37}$$

To estimate the sequence $\{r_n\}$, we first notice the convergence of the sequence (32) for $|\zeta| \leq 1$ by using the Paley–Wiener condition (37). Now taking $\zeta = 1$ in (32) and using our above established fact that $\rho = \sum_{j=0}^{\infty} q_j \in (0, 1)$, we know $r_j \geq 0$ for $j \geq 0$. This implies that $\|r\|_{l^1} = \sum_{j=0}^{\infty} r_j = (1 - \sum_{j=0}^{\infty} q_j)^{-1} = (1 - \rho)^{-1}$; then a direct application of Lemma 10 yields

$$z_n = \|y_n\|^2 \leq x_n \leq \frac{A}{1-\rho} = \begin{cases} -\dfrac{\gamma}{a+b} & \textit{for GL formula}, \\ -\dfrac{\gamma}{a+b} & \textit{for L1 method} \end{cases} \tag{38}$$

as $n \to \infty$, which gives our desired bound.

For case (b), we can easily see that (33) is equivalent to

$$\begin{aligned} \|y_n\|^2 &\leq A + \sum_{j=0}^{n-1} C_{n-j} \|y_j\|^2 + B \max_{\eta(\hat{t}_n) \leq j \leq n-1} \|y_j\|^2 \\ &\leq A + \sum_{j=0}^{n-1} C_{n-j} \|y_j\|^2 + B \max_{0 \leq j \leq n-1} \|y_j\|^2. \end{aligned} \tag{39}$$

Define the characteristic function $\chi_j = 1$ if $\max_{\eta(\hat{t}_n) \leq j \leq n-1} \|y_j\|^2 = \|y_j\|^2$ and $\chi_j = 0$ otherwise. Then the inequality (39) is equivalent to

$$\|y_n\|^2 \leq A + \sum_{j=0}^{n-1} (C_{n-j} + B\chi_j) \|y_j\|^2\,. \tag{40}$$

Now we can derive the desired estimate that $\|y_n\|^2 \le -\frac{\gamma}{a+b}$ by following the same argument as we bounded the sequence $\{y_n\}$ satisfying (34) in case (a) above, but using Lemma 11 (i) now instead of Lemma 10. We omit the detailed argument here.

It remains to prove part (ii). Let $v_j = y_j - z_j$; then it follows from (5)–(6) that

$$\begin{aligned}\left\langle 2v_n, \sum_{j=0}^{n} \omega_{n-j} v_j \right\rangle &= 2h^\alpha \left\langle v_n, f(t_n, y_n, y^h(\cdot)) - f(t_n, z_n, z^h(\cdot))\right\rangle \\ &= 2h^\alpha \left\langle v_n, f(t_n, y_n, y^h(\cdot)) - f(t_n, z_n, y^h(\cdot))\right\rangle \\ &\quad + 2h^\alpha \left\langle v_n, f(t_n, z_n, y^h(\cdot)) - f(t_n, z_n, z^h(\cdot))\right\rangle \\ &\le h^\alpha \left(c\|v_n\|^2 + d \max_{t_n-\mu_2(t_n)\le t\le t_n-\mu_1(t_n)} \|v^h(t)\|^2 \right).\end{aligned}$$

But applying Lemma 9 and the canonical condition for linear interpolation given in (22) with canonical constant $c_\pi = 1$, we deduce that

$$\sum_{j=0}^{n} \omega_{n-j}\|v_j\|^2 \le h^\alpha \left(c\|v_n\|^2 + d \max_{\eta(\hat{t}_n)\le j\le n} \|v_j\|^2 \right),$$

which can be written in the following equivalent convolution Volterra inequality:

$$\begin{aligned}\|v_n\|^2 &\le \sum_{j=0}^{n-1} \frac{|\omega_{n-j}|}{\omega_0 - h^\alpha c} \|v_j\|^2 + \frac{h^\alpha d}{\omega_0 - h^\alpha c} \max_{\eta(\hat{t}_n)\le j\le n} \|v_j\|^2 \\ &\le \sum_{j=0}^{n-1} H_{n-j} \|v_j\|^2 + G \max_{1\le j\le n} \|v_j\|^2,\end{aligned} \tag{41}$$

where we have made use of the assumption $\eta(\hat{t}_n) \ge 1$, and $H_{n-j} = |\omega_{n-j}|/(\omega_0 - h^\alpha c)$, $G = h^\alpha d/(\omega_0 - h^\alpha c)$. Note that coefficients H_{n-j} and G are non-negative. Then as we carried out in part (i), we can also continue our argument with (41) in two cases:

$$(a) \ \max_{1\le j\le n} \|v_j\|^2 = \|v_n\|^2; \qquad (b) \ \max_{1\le j\le n} \|v_j\|^2 = \max_{1\le j\le n-1} \|v_j\|^2.$$

For case (a), we can easily see that inequality (41) is equivalent to

$$\|v_n\|^2 \le \frac{H_n}{1-G} \|v_0\|^2 + \sum_{j=1}^{n-1} \frac{H_{n-j}}{1-G} \|v_j\|^2. \tag{42}$$

We can check that $\rho_1 = \frac{1}{1-G}\sum_{j=1}^{\infty} H_j = \frac{\omega_0}{\omega_0 - h^\alpha(c+d)} < 1$ by noting that $c + d < 0$.

For case (b), the same as we derived (39) earlier, we can obtain

$$\|v_n\|^2 \le H_n \|v_0\|^2 + \sum_{j=1}^{n-1} (H_{n-j} + G\chi_j) \|v_j\|^2. \tag{43}$$

It is easy to check that

$$\rho_2 = \sum_{j=1}^{+\infty} H_j + G = \begin{cases} \dfrac{1 + h^\alpha d}{1 - h^\alpha c} < 1 & \mathit{for\ GL\ formula}, \\ \dfrac{1 + \Gamma(2-\alpha)h^\alpha d}{1 - \Gamma(2-\alpha)h^\alpha c} < 1 & \mathit{for\ L1\ method}. \end{cases} \tag{44}$$

On the other hand, we know from Lemmas 7 and 8 that there exists $c_\alpha > 0$ such that $H_n \le c_\alpha/n^\alpha$. Using this, we can derive the desired estimate that $\|v_n\|^2 \le c_\alpha/n^\alpha$

for both cases (a) and (b), by following the same argument as we bounded the sequence $\{y_n\}$ satisfying (34) or (40) in part (i), but using Lemma 11 (ii) now instead of Lemma 10 or Lemma 11 (i). We omit the detailed argument here. $\square$

4. Numerical experiments. In this section, we present some numerical experiments to validate our analytical results in previous sections. We will verify the numerical dissipativity and contractivity of F-FDEs, also reveal different decay rates between F-FDEs and integer FDEs. More importantly, we will compare the performance of our methods and the popular predictor-corrector type method proposed in [17].

4.1. Delay fractional financial system. Consider the system

$$
\begin{cases} {}_0^C D_t^\alpha x(t) = z(t) + (y(t-\tau) - 3)x(t), \\ {}_0^C D_t^\alpha y(t) = 1 - 0.1y(t) - x^2(t-\tau), \\ {}_0^C D_t^\alpha z(t) = -x(t-\tau) - z(t), \end{cases} \tag{45}
$$

where $\tau \geq 0$ is the delay parameter. The nondelay fractional financial system (i.e., $\tau = 0$) was proposed, and its complex dynamical behaviors were studied in [12]. Note that the system is dissipative for $\tau = 0$, and there exist a global absorbing set. For $\tau > 0$, we will observe numerically that there exist also bounded chaotic absorbing sets. Time delay in a financial system means that one policy form being made to take effect has to go through certain time, and its influence can not be negligible in many cases.

The system (45) was studied in [51] by using the fractional Adams–Bashforth–Moulton method. We shall mainly compare the numerical performance of our method and the popular predictor-corrector type method for this example. The initial values are taken as $x(t) = 0.1, y(t) = 4, z(t) = 0.5$ for $t \leq 0$.

Figure 1 plots the numerical solutions for different delay τ and Figure 2 presents numerical solutions for different fractional order α. They are computed by G-L formula with the step size $h = 0.05$ and $T = 100$. We can see that both the delay τ and fractional order α can significantly affect the shape and size of the solution phase. Some appropriate modifications in delay or fractional order can enhance or suppress the emergence of chaotic or periodic motions.

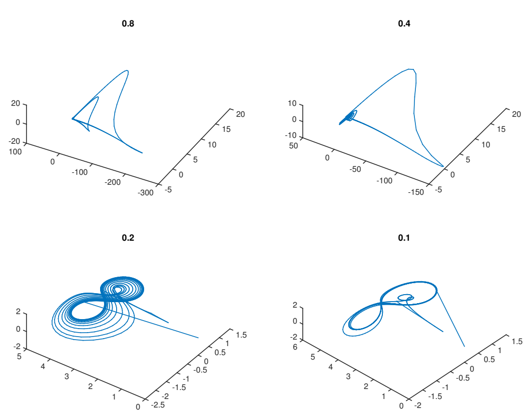

FIG. 1. *Numerical solutions for* $\alpha = 0.7, h = 0.05$ *with delay* $\tau = 0.8, 0.4, 0.2,$ *and* 0.1.

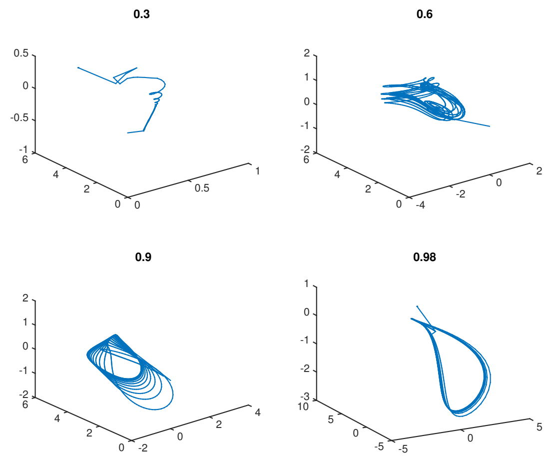

FIG. 2. *Numerical solutions for* $\tau = 0.2, h = 0.05$ *with fractional order* $\alpha = 0.3, 0.6, 0.9,$ *and* 0.98.

Our numerical method can correctly simulate various different qualitative characteristics of this system for all the delay $\tau \geq 0$ and fractional order $\alpha \in (0, 1)$ with accepted step size. If the fractional Adams–Bashforth–Moulton method is used to simulate this system, the step size must be restricted, i.e., $h < h_0(\alpha)$ for some $h_0(\alpha) > 0$ to ensure numerical stability. Moreover, when the order α is small, this limitation becomes very demanding and heavily restricts long time computation; see Table 1.

TABLE 1
Numerical performances of the fractional Adams–Bashforth–Moulton method for $\tau = 20h$.

	$\alpha = 0.98$	$\alpha = 0.8$	$\alpha = 0.6$	$\alpha = 0.3$	$\alpha = 0.1$
Blowup for $h =$	0.02	0.02	0.005	0.0002	0.0000000002
Stable for $h =$	0.01	0.01	0.002	0.0001	0.0000000001

As for integer DDEs, a good numerical scheme for F-DDEs should be mostly $\tau(0)$-stable [22]. We guess that the two difference schemes constructed in this work are fractional $\tau(0)$-stable while the fraction predictor-corrector type method proposed in [17] is not. In the next article we will further study the fractional $\tau(0)$-stability of fractional linear multistep methods for F-DDEs.

4.2. Fractional stiff F-DDE I. We consider the F-DDE

$$
\begin{cases}
{}_0^C D_t^\alpha y_1(t) = \left(8\cdot 10^7 + 2\right) y_1(t) - \left(4\cdot 10^7 - 4\right) y_2(t) \\
\qquad\qquad + \dfrac{2}{5}\sin(t/2)\left(-y_1(t/2) + 2y_2(t/2)\right) + 10\sin(t) + 1, \\
{}_0^C D_t^\alpha y_2(t) = \left(4\cdot 10^7 - 4\right) y_1(t) - \left(2\cdot 10^7 + 8\right) y_2(t) \\
\qquad\qquad + \dfrac{2}{5}\sin(t/2)\left(2y_1(t/2) - 4y_2(t/2)\right) - 20\sin(t) - 2, \\
\quad y_1(0) = 2, \quad y_2(0) = 1.
\end{cases} \tag{46}
$$

It is easy to check that this problem satisfies the condition (5)–(6) with $c = -10$, $d = 2$, and $\mu_1(t) = \mu_2(t) = \frac{t}{2}$, so it is strictly contractive. When $\alpha = 1$, this system admits a unique true solution [31]:

$$y_1(t) = \sin(t) + 2\exp(-10^8 t), \quad y_2(t) = -2\sin(t) + 2\exp(-10^8 t).$$

We note that this DDE for $\alpha = 1$ is not stiff in the transient phase, namely, $0 \leq t \leq 10^{-7}$, but becomes stiff beyond this phase. The true solution is unknown for $0 < \alpha < 1$. However, the transient phase interval depends strongly on the order α. We have tried to solve this problem in the interval $0 \leq t \leq t_0(\alpha)$ with extremely small step size to show this characteristics; see Figure 3.

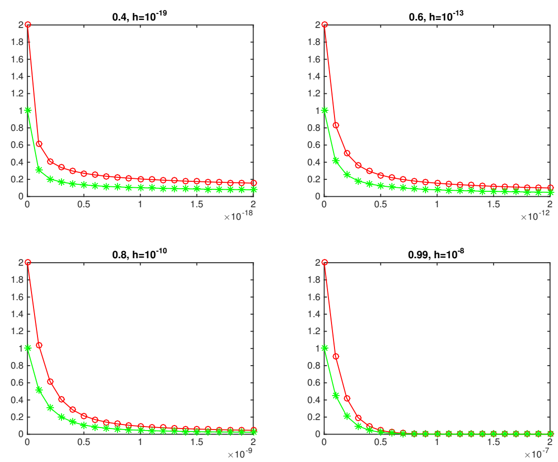

FIG. 3. *Numerical solutions for* $\alpha = 0.4, 0.6, 0.8$*, and* 0.99 *in the transient phase* $0 \leq t \leq t_0(\alpha)$.

Figure 4 plots the numerical solutions obtained by G-L formula for $\alpha = 0.2$ and 0.8 with $h = 0.1$. It shows that the new method is efficient and robust for this strong stiff F-DDEs uniformly for all α in $(0, 1)$. The results for the stability of the corrector-predictor method in [17] versus mesh size h and order α for various cases are listed in Table 2, which shows the fractional explicit corrector-predictor method does not work for stiff F-DDEs.

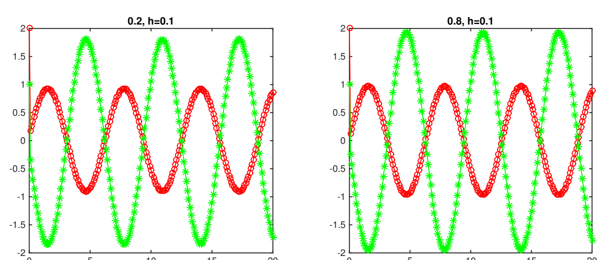

FIG. 4. *Numerical solutions for* $\alpha = 0.2$ *and* 0.8 *with* $h = 0.1$.

Figure 5 presents the contractivity $e(t) = \|y(t) - z(t)\|$ of the difference between two solutions $y(t)$ and $z(t)$ with two initial functionals ψ and ϕ. We take $\phi = (5, -3)^T$. It clearly shows that the contractivity rate depends on the order parameter α. The

TABLE 2
Numerical performances of the fractional Adams–Bashforth–Moulton method for strongly stiff F-DDEs.

	$\alpha = 0.99$	$\alpha = 0.8$	$\alpha = 0.6$	$\alpha = 0.3$
Blowup for $h =$	2e-8	2e-10	1e-13	1e-26
Stable for $h =$	1e-8	1e-10	1e-14	1e-27

greater the order α is, the faster the difference function $e(t)$ becomes contractive. But they all keep the contractivity of polynomial type, rather than of exponential type as for the integer ODEs ($\alpha = 1$).

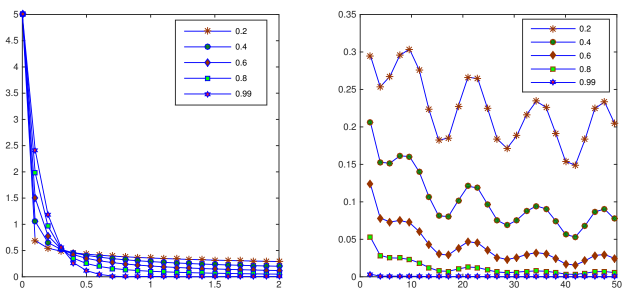

FIG. 5. *Errors for* $\alpha = 0.2, 0.4, 0.6, 0.8$, *and* 0.99 *in interval* $[0, 2]$ *(left) and* $[2, 50]$ *(right).*

As in [49], we introduce an index function p_α to measure the quantitative behavior of the contractivity rate corresponding to two different initial functionals ψ and ϕ:

$$p_\alpha = \frac{\ln\left(\|\psi - \phi\|^2 C_\alpha\right) - \ln\left(\|y(t) - z(t)\|^2\right)}{\ln(t)}, \quad t > 1. \tag{47}$$

The index p_α is used to characterize the polynomial decay rate of $e(t)$, i.e., $e(t) = O(1/t^{p_\alpha})$, which is derived from the estimation given in (21). Obviously, the index $p_\alpha \to -\ln(\|y(t) - z(t)\|^2)/\ln(t)$ as $t \to +\infty$ and is independent of the initial value $\|\psi - \phi\|^2 C_\alpha$. In the numerical simulations, we take $\|y(1) - z(1)\|^2 = \|\psi - \phi\|^2 C_\alpha$.

TABLE 3
The observed index function p_α *for Example* 4.2 *with* $h = 0.1$.

t	$\alpha = 0.2$	$\alpha = 0.4$	$\alpha = 0.6$	$\alpha = 0.8$	$\alpha = 0.99$
10	0.1758	0.5733	0.9992	1.4426	1.7276
20	0.2611	0.6660	1.0939	1.5325	1.8125
30	0.4185	0.8427	1.2939	1.7558	2.0698
40	0.4729	0.9153	1.3902	1.8895	2.2637
50	0.3256	0.7342	1.1640	1.6043	1.9154

The observed index function p_α is presented in Table 3. The results show that the contractivity rate is about $\|e(t)\|^2 = O(t^{-2\alpha})$, which is about two times our analytically predicted rate. This should be mainly due to the linear equation. Indeed, for scalar F-ODE or essentially decoupled linear systems, we showed in [49] that the contractivity rate can be optimal as $\|e(t)\|^2 = O(t^{-2\alpha})$ both theoretically and numerically, as demonstrated in the current example. But for more general nonlinear vector-valued F-ODEs, the theoretical contractivity rate can only be $\|e(t)\|^2 = O(t^{-\alpha})$; see [49] for more details and a simple example. The main reason for this phenomenon is that we have used the energy-like estimate to bound $\|y(t_n)\|^2$ or $\|y_n\|^2$ by $E_\alpha(2\lambda t_n^\alpha)$. But we do not have the identity that $\sqrt{E_\alpha(2\lambda t^\alpha)} = E_\alpha(\lambda t^\alpha)$ for the Mittag–Leffler function, unlike the identity $\sqrt{e^{2\lambda t}} = e^{\lambda t}$ for the classical exponential function. However, we can easily find that the general contractivity rate of $\|e(t)\|$ remains to be polynomial, instead of the exponential decay for classical integer FDEs.

4.3. Fractional stiff F-FDE II. We consider a time fractional parabolic PDE with functional term of the form

$$\begin{cases} {}_0^C D_t^\alpha u(x,t) = 4u_{xx}(x,t) - u(x,t-\pi/2) + 32\sin(t) \\ \qquad\qquad + \displaystyle\int_{t-\pi/2}^{t} \cos(\theta)\cos(2\theta)u(x,\theta)d\theta, \quad 0 < x < 1, 0 < t < +\infty, \\ u(0,t) = u(1,t) = 0, \quad -\pi/2 \le t < +\infty, \\ u(x,t) = 4x(1-x)\sin(t), \quad 0 < x < 1, \quad -\pi/2 \le t \le 0. \end{cases} \tag{48}$$

We discretize the second spatial derivative by the standard central difference scheme on a grid of points $x_i = i/M, i = 1, \ldots, M-1$ to get the spatial semidiscrete F-FDE:

$$\begin{cases} {}_0^C D_t^\alpha u_i(t) = \dfrac{4}{\Delta x^2}\left(u_{i+1}(t) - 2u_i(t) + u_{i-1}(t)\right) - u_i(t-\pi/2) + 32\sin(t) \\ \qquad\qquad + \displaystyle\int_{t-\pi/2}^{t} \cos(\theta)\cos(2\theta)u_i(\theta)d\theta, \quad 0 < t < +\infty, \\ u_0(t) = u_M(t) = 0, \quad -\pi/2 \le t < +\infty, \\ u_i(t) = 4i\Delta x(1-i\Delta x)\sin(t), \quad -\pi/2 \le t \le 0, \end{cases} \tag{49}$$

where $u_i(t) = u(x_i,t)$ and $\Delta x = 1/M$. We take $M = 100$ in our simulations. It is easy to check the semidiscrete F-FDEs meet the condition (5)–(6).

In this example, we take the $L1$ method for the simulations. Figure 6 reports the numerical solutions for $\alpha = 0.6, h = \pi/20$. Figure 7 presents the contractivity of the error $\mathbf{e}(t) = \|\mathbf{u}(t) - \mathbf{v}(t)\|$ for $\alpha = 0.2, 0.4, 0.6, 0.8$, and 0.99 in the interval $[2\pi, 15\pi]$ with $h = \pi/32$, where $v(t)$ is the solution of (49) with the perturbed initial functional given by $\phi = 4x(1-x)\sin(4t)$ for $-\pi/2 \le t \le 0$.

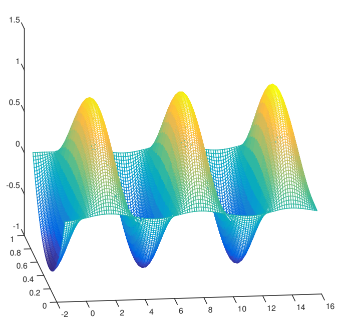

FIG. 6. *Numerical solutions.*

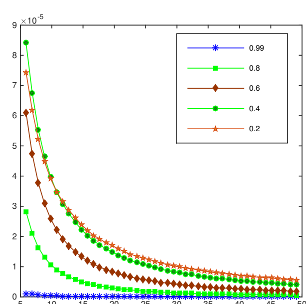

FIG. 7. *The errors* $\mathbf{e}(t)$.

It is also observed that the contractivity rate depends on the order parameter α polynomially. The greater the order α, the faster the difference function $e(t)$ becomes contractive. Similarly to the index function p_α we introduced for the previous Example

4.2, we list its values also for the current example (see Table 4), from which we see the contractivity rate is much better than our theoretically predicted one. We guess this may be due to the homogeneous initial condition for the linear F-FDEs, i.e., $\phi(0) = \psi(0) = 0$, and $e(0) = \phi(0) - \psi(0) = 0$. This can be roughly seen form the facts below. The equation of the error function $e(t)$ of the solution to the numerical scheme (49) can be written in the vectorial system ${}_0^C D_t^\alpha \mathbf{e}(t) = A\mathbf{e}(t) + \mathcal{E}(\cdot) + \mathbf{g}(t)$, where A is a constant coefficient matrix, and $\mathcal{E}(\cdot) = \int_{t-\pi/2}^{t} \cos(\theta)\cos(2\theta)U(\theta)d\theta - U(t-\pi/2)$ is a functional term. We can represent the solution $\mathbf{e}(t)$ by [27]

$$\mathbf{e}(t) = \mathbf{e}(0)E_\alpha(At^\alpha) + \int_0^t (t-s)^{\alpha-1}E_{\alpha,\alpha}(A(t-s)^\alpha)\,(\mathcal{E}(\cdot) + \mathbf{g}(s))\,ds. \tag{50}$$

The first term on the right-hand side of (50) has a polynomial decay rate by using the facts that the eigenvalues of matrix A are negative and that $\|\mathbf{e}(0)E_\alpha(At^\alpha)\| \le C\|\mathbf{e}(0)\|t^{-\alpha}$ as $t \to +\infty$. Hence, when the initial condition is homogeneous as in the current example the term $\mathbf{e}(0)E_\alpha(At^\alpha)$ drops and it may lead to higher contractivity rate of $e(t)$.

TABLE 4
The observed index function p_α for Example 4.3 with $h = \pi/32$.

t	$\alpha = 0.2$	$\alpha = 0.4$	$\alpha = 0.6$	$\alpha = 0.8$	$\alpha = 0.99$
10	6.6611	6.6047	6.9245	7.6262	10.411
20	5.6924	5.7409	6.0797	6.7116	8.9393
30	5.2708	5.3612	5.7075	6.3117	8.3185
40	5.0613	5.1765	5.5278	6.1166	7.9968
50	4.9041	5.0355	5.3896	5.9676	7.7273

We emphasize that the new method works well for the semi-discrete F-FDEs with relatively good step size of $h = 0.1$ and is much better than the explicit predictor-corrector method [17].

5. Concluding remarks. We have presented a new delay-dependent fractional generalization of the Halanay-like inequality to characterize the asymptotic behavior of F-FDEs, with which we have established the dissipativity and contractivity of the F-FDEs under the assumptions that are quite similar to the ones for classical integer FDEs. Moreover, we have worked out an estimate of the polynomial decay rate for the solutions to the F-FDEs, very different from the exponential decay rate for the classical integer FDEs. We have constructed two numerical schemes for the F-FDEs and demonstrated that they are both dissipative and contractive and can preserve the exact contractivity rate as that for the continuous F-FDEs. To the best of our knowledge, this is the first work about the analytical and numerical dissipativity and asymptotic behavior of nonlinear F-FDEs. Several numerical examples have shown that the new schemes are much more efficient and robust than the popular predictor-corrector algorithms for F-FDEs, especially for some strongly stiff systems.

There are a few possible and interesting extensions. For the R-L F-FDEs, the dissipativity and contractivity of our presented two difference schemes can be established similarly as we have done here. The contractivity rate for the R-L F-FDEs should be changed to $\|U(t) - V(t)\|^2 = O(t^{-1-\alpha})$ as $t \to +\infty$, and the numerical contractivity rate for the R-L F-FDEs to $\|U^n - V^n\|^2 = O(n^{-1-\alpha})$ as $n \to \infty$ accordingly. For the multi-order fractional systems with $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)^T$, $\alpha_i \in (0,1), i = 1, 2, \ldots, n$, the corresponding dissipativity and contractivity of those difference schemes can be studied in a similar manner.

We have studied in this work only two low order numerical schemes for F-FDEs and proved the numerical contractivity (or stability). Although high order schemes for F-FDEs may suffer from the low regularity of the true solutions of F-FDEs, we guess some second order schemes, such as second order fractional BDFs and fractional trapezoidal methods, may be derived. Several new techniques about the construction and analysis of high order schemes for F-ODEs proposed in [10, 11, 24, 37, 46, 56] could be helpful to our further study. It would be very interesting to study F-FDEs with nonsmooth solutions, propose some second order schemes for F-FDEs, and analyze their contractivity (or stability).

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