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Finite element methods and their convergence for elliptic and parabolic interface problems

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Summary. In this paper, we consider the finite element methods for solving second order elliptic and parabolic interface problems in two-dimensional convex polygonal domains. Nearly the same optimal L^2-norm and energy-norm error estimates as for regular problems are obtained when the interfaces are of arbitrary shape but are smooth, though the regularities of the solutions are low on the whole domain. The assumptions on the finite element triangulation are reasonable and practical.

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1. Introduction

Numerical solutions of second order elliptic and parabolic problems with discontinuous coefficients are often encountered in material sciences and fluid dynamics. It is the case when two distinct materials or fluids with different conductivities or densities or diffusions are involved. When the interface is smooth enough, the solution of the interface problem is also very smooth in individual regions occupied by materials or fluids, but the global regularity is usually very low, see Littman et al. [22], Kellogg [15,16], Ladyzenskaja et al. [17]. Because of the low global regularity and the irregular

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geometry of the interface, achieving the high order of accuracy seems difficult with finite element methods (cf. Babuska [1]), whose elements could not fit with the interface of general shape.

Babuska [1] studied the elliptic interface problem defined on a smooth domain with a smooth interface. The interface problem was formulated as an equivalent minimization problem with all the boundary and jump conditions incorporated in the cost functions. The finite element methods were then used to solve the minimization problems. Under some approximation assumptions on finite element spaces, the energy-norm error estimates were obtained. Xu [25] considered solving the elliptic interface problem assuming its solution and the normal derivatives of the solution continuous across the interface, by the standard finite element method. The algorithms in [1] and [25] require the exact calculation of line integrals on the boundary of the domain and on the interface, and exact integrals on interface finite elements are also needed. Han [12] proposed an infinite element method, which may be considered as a certain scheme of mesh refinement, for elliptic interface problems with interfaces consisting of straight lines, not suitable for curved interfaces. The energy-norm error estimates were achieved both in [12] and [25].

LeVeque-Li [19] proposed an immersed interface method for elliptic interface problems defined on a regular domain for which a uniform rectangular grid can be used. Then finite difference methods were constructed based on the uniform grid and the jump conditions on the interface. The authors applied their methods also for other interface problems, e.g. the Stokes flow problem [18], the one-dimensional moving interface problem [20] and Hele-Shaw flow [21]. The resultant linear systems from these methods are non-symmetric and indefinite even the original problems are self-adjoint and uniformly elliptic. The convergence proofs of these methods are still open.

In this paper, we propose the finite element method for solving second order both elliptic and parabolic interface problems and prove that the method converges nearly in the same optimal way as the usual non-interface elliptic and parabolic problems, both for the energy-norm and L^2-norm. In fact, the energy-norm error estimate can be shown to be optimal. The interface is allowed to be of arbitrary shape but is smooth. The resultant linear systems are always symmetric and positive definite when the original PDEs are self-adjoint and uniformly elliptic. And in particular, the domain decomposition methods, which have been investigated widely in recent years (cf. Chan-Zou [5] and Xu-Zou [26]), can be applied here to construct efficient preconditioned iterative methods for solving these large scale and sparse linear systems of equations. And different from the previous finite element methods, the calculations of the stiffness matrix and the interface integral related to the jumps of normal derivatives are much simpler and more practical here.

Considering we approximate the smooth interface by a polygon and the interface function by its interpolant, the approximation problem here seems similar to the classical finite element methods using straight triangles for solving elliptic problems with Neumann boundary conditions on smooth domains (see, for example, [3, 2, 8, 9] and the references therein). But there are some differences. In the classical case, one can assume full regularities (or even more) about solutions, coefficients, boundary value function etc. But for the current interface problems, we usually have very low global regularities about the solutions, coefficients and interface functions. So the classical analysis is difficult to apply for the convergence analysis in the interface problem. Among crucial technical tools used here are some Sobolev embedding inequality, Sobolev extension theorem, parabolic dual arguments in both L^2-norm and energy-norm error estimates, interface energy-norm projection and discrete L^2 projection etc.

Let us now end this section with some notation to be used in the paper. For each integer $m \geq 0$ and real p with $1 \leq p \leq \infty$, $W^{m,p}(\Omega)$ denotes the standard Sobolev space of real functions with their weak derivatives of order up to m in the Lebesgue space $L^p(\Omega)$. When $p = 2$, we use $H^m(\Omega)$ to stand for $W^{m,2}(\Omega)$. For a given Banach space B, we define

$$W^{m,p}(0,T;B) = \Big\{ u(t) \in B \text{ for a.e. } t \in (0,T) \text{ and } \sum_{k=0}^{m} \int_0^T \|u^{(k)}(t)\|_B^p \, dt < \infty \Big\}$$

equipped with the norm

$$\|u\|_{W^{m,p}(0,T;B)} = \Big[\sum_{k=0}^{m} \int_0^T \|u^{(k)}(t)\|_B^p \, dt \Big]^{1/p}.$$

As usual, we let

$$L^p(0,T;B) = W^{0,p}(0,T;B) \quad \text{and} \quad H^1(0,T;B) = W^{1,2}(0,T;B).$$

Throughout the paper, the generic constant C is always independent of the finite element mesh parameter h and the time step size τ.

2. Elliptic interface problems

Let Ω be a convex polygon in $\mathbb{R}^2$ and $\Omega_1 \subset \Omega$ be an open domain with C^2 boundary $\Gamma = \partial\Omega_1 \subset \Omega$. Let $\Omega_2 = \Omega \setminus \Omega_1$ (see Fig. 1). We consider the following elliptic interface problem

$$-\nabla \cdot (\beta \nabla u) = f \quad \text{in} \quad \Omega \tag{2.1}$$

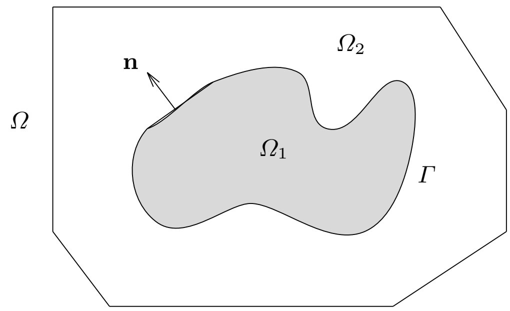

Fig. 1. Domain Ω, its subdomains Ω_1, Ω_2 and interface Γ

with Dirichlet boundary condition

$$u = 0 \quad \text{on} \quad \partial\Omega \tag{2.2}$$

and jump conditions on the interface

$$[u] = 0, \quad \left[\beta \frac{\partial u}{\partial \mathbf{n}}\right] = g \quad \text{across} \quad \Gamma, \tag{2.3}$$

where $[v]$ is the jump of a quantity v across the interface Γ and $\mathbf{n}$ the unit outward normal to the boundary $\partial\Omega_1$. For definiteness, we let $[v](x) = v_1(x) - v_2(x)$, $x \in \Gamma$, with v_1 and v_2 the restrictions of v on Ω_1 and Ω_2, respectively. For the ease of exposition, we assume that the coefficient function β is positive and piecewise constant, i.e.

$$\beta(x) = \beta_1 \quad \text{for} \quad x \in \Omega_1; \quad \beta(x) = \beta_2 \quad \text{for} \quad x \in \Omega_2.$$

But the results of this section can be easily extended to more general elliptic interface problems (see Sect. 4).

For the later analysis, we need the following space

$$X = H^1(\Omega) \cap H^2(\Omega_1) \cap H^2(\Omega_2)$$

equipped with the norm

$$\|v\|_X = \|v\|_{H^1(\Omega)} + \|v\|_{H^2(\Omega_1)} + \|v\|_{H^2(\Omega_2)}, \quad \forall\, v \in X.$$

By the Sobolev embedding theorem, for any $v \in X$, we have $v \in W^{1,p}(\Omega), \forall\, p > 2$.

Regarding the regularity for the solution of the interface problem (2.1)-(2.3), we have the following results:

Theorem 2.1 *Assume that $f \in L^2(\Omega)$ and $g \in H^{1/2}(\Gamma)$. Then the problem (2.1)-(2.3) has a unique solution $u \in X$ and u satisfies the a priori estimate*

$$\|u\|_X \le C\,(\|f\|_{L^2(\Omega)} + \|g\|_{H^{1/2}(\Gamma)}). \tag{2.4}$$

Proof. In the case of $g = 0$, the results are well-known (cf. Babuska [1] and Kellogg [14]). Let us consider the general case $g \in H^{1/2}(\Gamma)$. As Γ is of class C^2, we can find a function $\tilde{u} \in X \cap H_0^1(\Omega)$ such that

$$[\tilde{u}] = 0, \quad \left[\beta \frac{\partial \tilde{u}}{\partial \mathbf{n}}\right] = g \text{ on } \Gamma \quad \text{and} \quad \|\tilde{u}\|_X \le C\,\|g\|_{H^{1/2}(\Gamma)}.$$

Then Theorem 2.1 follows by observing that $v = u - \tilde{u}$ solves the problem (2.1)-(2.3) with f replaced by $f + \nabla \cdot (\beta \nabla \tilde{u})$ and $g = 0$. $\square$

Remark 2.1 One way to construct the function $\tilde{u} \in X \cap H_0^1(\Omega)$ used in the proof of Theorem 2.1 is as follows. First, let $\tilde{u}_1 \in H^1(\Omega_1)$ solve

$$-\Delta \tilde{u}_1 + \tilde{u}_1 = 0 \quad \text{in } \Omega_1; \quad \beta_1 \frac{\partial \tilde{u}_1}{\partial \mathbf{n}} = g \quad \text{on } \Gamma.$$

We know $\tilde{u}_1$ exists uniquely and $\tilde{u}_1 \in H^2(\Omega_1)$ satsifying (cf. Grisvard [11])

$$\|\tilde{u}_1\|_{H^2(\Omega_1)} \le C\,\|g\|_{H^{1/2}(\Gamma)}.$$

Then we solve the following biharmonic problem of the first kind in Ω_2 to get $\tilde{u}_2$:

$$\begin{aligned} &-\Delta^2 \tilde{u}_2 = 0 \quad \text{in } \Omega_2, \\ &\tilde{u}_2 = \tilde{u}_1, \quad \frac{\partial \tilde{u}_2}{\partial \mathbf{n}} = 0 \quad \text{on } \Gamma, \\ &\tilde{u}_2 = 0, \quad \frac{\partial \tilde{u}_2}{\partial \nu} = 0 \quad \text{on } \partial\Omega \end{aligned}$$

where ν is the unit outward normal to the boundary $\partial\Omega$.

It is well-known that there exists a unique $\tilde{u}_2 \in H^2(\Omega_2)$ to the above biharmonic problem satisfying the estimate (cf. Girault-Raviart [10], pp. 15-17)

$$\|\tilde{u}_2\|_{H^2(\Omega_2)} \le C\,\|\tilde{u}_1\|_{H^{3/2}(\Gamma)} \le C\,\|\tilde{u}_1\|_{H^2(\Omega_1)} \le C\|g\|_{H^{1/2}(\Gamma)}.$$

Thus the desired function $\tilde{u}$ can be taken to be $\tilde{u} = \tilde{u}_i$ in Ω_i for $i = 1, 2$.

To present the finite element method for the interface problem (2.1)-(2.3), we now introduce its weak formulation. We define a bilinear form $a(\cdot,\cdot) : H^1(\Omega) \times H^1(\Omega) \mapsto \mathbb{R}$ by

$$a(u, v) = \int_\Omega \beta(x) \nabla u \cdot \nabla v \, dx, \quad \forall\, u, v \in H^1(\Omega).$$

Then it is immediate to derive the weak formulation of the interface problem (2.1)-(2.3):

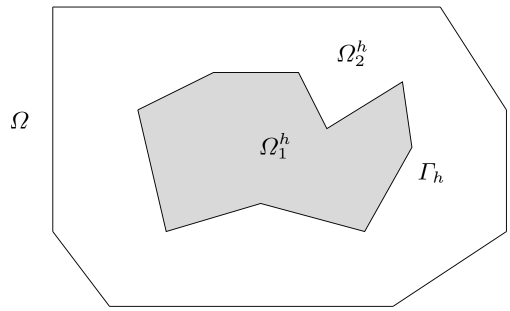

Fig. 2. The domain Ω, its approximate subdomains Ω_1^h, Ω_2^h and interface Γ_h

Problem (P). Find $u \in H_0^1(\Omega)$ such that

$$a(u,v) = (f,v) + \langle g, v\rangle\,, \quad \forall\, v \in H_0^1(\Omega). \tag{2.5}$$

Here and later, the notation $(\cdot,\cdot)$ and $\langle\cdot,\cdot\rangle$ are used to denote the scalar products of the $L^2(\Omega)$ space and the interface space $L^2(\Gamma)$, respectively.

We now describe the triangulation $\mathcal{T}_h$ of the domain Ω. We first approximate the domain Ω_1 by a domain Ω_1^h with a polygonal boundary Γ_h whose vertices all lie on the interface Γ. Let Ω_2^h stand for the domain with $\partial\Omega$ and Γ_h as its exterior and interior boundaries, respectively (see Fig. 2).

Now we triangulate Ω by a finite set of closed triangles $\mathcal{T}_h = \{K\}$ which satisfies the following conditions:

(A1) $\bar{\Omega} = \cup_{K\in\mathcal{T}_h} K$,
(A2) if $K_1, K_2 \in \mathcal{T}_h$ and $K_1 \neq K_2$, then either $K_1 \cap K_2 = \emptyset$ or $K_1 \cap K_2$ is a common vertex or edge of both triangles,
(A3) each $K \in \mathcal{T}_h$ is either in Ω_1^h or Ω_2^h, and has at most two vertices lying on Γ_h.

The triangles with one or two vertices on Γ_h are called interface triangles, the set of all interface triangles is denoted by $\mathcal{T}_h^*$ and we let $\Omega^* = \cup_{K\in\mathcal{T}_h^*} K$.

For each triangle $K \in \mathcal{T}_h$, we use h_K for its diameter, ρ_K and $\bar{\rho}_K$ for the diameters of its inscribed and circumscribed circles, respectively. Let $h = \max_{K\in\mathcal{T}_h} h_K$. We assume that the family of triangulations $\{\mathcal{T}_h\}_{h\in(0,h_0)}$, for some fixed $h_0 > 0$, is quasi-uniform, i.e. there are two positive constants C_0 and C_1 independent of h such that

$$C_0\,\rho_K \le h \le C_1\,\bar{\rho}_K, \quad \forall\, K \in \mathcal{T}_h, \quad \forall\, h \in (0, h_0). \tag{2.6}$$

We introduce some more notations. We first rewrite the set $\mathcal{T}_h^*$ of interface elements as

$$\mathcal{T}_h^* = \{K \in \mathcal{T}_h;\ K \cap \Gamma \neq \emptyset\}.$$

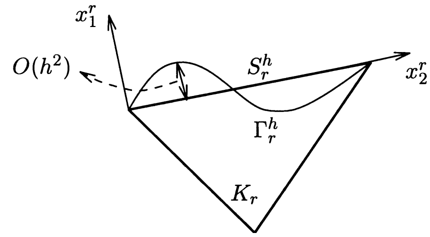

Fig. 3. The arc Γ_r^h and the local coordinates x_1^r, x_2^r

Let

$$\Gamma_h = \cup_{r=1}^{m_h} S_r^h \quad \text{and} \quad \Gamma = \cup_{r=1}^{m_h} \Gamma_r^h,$$

where $S_r^h \subset \Gamma_h$ is the edge of some triangle in $\mathcal{T}_h$, denoted by K_r, and Γ_r^h is the part of Γ corresponding to S_r^h (see Fig. 3). As the interface Γ is of class C^2, there exists a constant $h_0 > 0$ such that for $h \in (0, h_0)$, one can introduce a local coordinate x_1^r, x_2^r for each Γ_r^h ($r = 1, 2, \cdots, m_h$). We take the x_2^r-axis along the edge S_r^h and x_1^r-axis in the normal to S_r^h (see Fig. 3). Then the arc Γ_r^h can be expressed in the parametric form

$$\Gamma_r^h = \{(x_1^r, x_2^r);\ x_1^r = \phi_r^h(x_2^r),\ x_2^r \in [0, s_r^h]\}$$

where s_r^h is the length of S_r^h. As the interface Γ is of class C^2, we have $\phi_r^h \in C^2([0, s_r^h])$ for $r = 1, \cdots, m_h$. It is easy to prove that (cf. Feistauer-Ženišek [9])

$$|\phi_r^h(x_2^r)| \le C\,(s_r^h)^2 \le C\,h^2, \quad \forall\, x_2^r \in [0, s_r^h], \tag{2.7}$$

$$\left|\frac{d}{dx_2^r}\phi_r^h(x_2^r)\right| \le C\,h, \quad \forall\, x_2^r \in [0, s_r^h]. \tag{2.8}$$

For any interface element $K \in \mathcal{T}_h^*$, let $\mathcal{K}_1 = K \cap \Omega_1$ and $\mathcal{K}_2 = K \cap \Omega_2$, then using (2.7), we know

$$\text{either} \quad \text{meas}(\mathcal{K}_1) \le C\,h_K^3 \quad \text{or} \quad \text{meas}(\mathcal{K}_2) \le C\,h_K^3.$$

Later on, we will always use $\tilde{K}$ to denote one of two subregions $\mathcal{K}_1$ and $\mathcal{K}_2$ which satisfies meas$(\mathcal{K}_i) \le C\,h_K^3$, $i = 1$ or 2.

Now we define V_h to be the standard linear finite element space defined on the triangulation $\mathcal{T}_h$ and V_h^0 the subspace of V_h with its functions vanishing on the boundary $\partial\Omega$. For the coefficient function $\beta(x)$, we define its approximation $\beta_h(x)$ as follows: for each triangle $K \in \mathcal{T}_h$, let $\beta_K(x) = \beta_i$ if $K \subset \Omega_h^i$, $i = 1$ or 2. Then β_h is defined by

$$\beta_h(x) = \beta_K(x), \quad \forall\, K \in \mathcal{T}_h.$$

It is easy to verify that

$$\operatorname{supp}(\beta - \beta_h) \cap K = \tilde{K}, \quad \forall\, K \in \mathcal{T}_h^*.$$

Corresponding to the bilinear form $a(\cdot,\cdot) : H^1(\Omega) \times H^1(\Omega) \mapsto \mathbb{R}^1$ defined previously, we introduce its discrete form $a_h(\cdot,\cdot) : H^1(\Omega)\times H^1(\Omega) \mapsto \mathbb{R}^1$ by

$$a_h(u,v) = \sum_{K\in\mathcal{T}_h} \int_K \beta_K(x)\nabla u \cdot \nabla v\, dx, \quad \forall\, u, v \in H^1(\Omega).$$

Furthermore, we need an approximation g_h to the interface function g on Γ. Let $\{P_j\}_{j=1}^{m_h}$ be the set of all nodes of the triangulation $\mathcal{T}_h$ lying on the interface Γ, and $\{\phi_j^h\}_{j=1}^{m_h}$ the set of standard nodal basis functions corresponding to $\{P_j\}_{j=1}^{m_h}$ in the space V_h. Assume that $g \in C(\Gamma)$. Then we define $g_h \in V_h$ by

$$g_h = \sum_{j=1}^{m_h} g(P_j)\phi_j.$$

Now we are in a position to define the finite element approximation to Problem (P):

Problem (P$_h$). Find $u_h \in V_h^0$ such that

$$\text{(2.9)} \qquad a_h(u_h, v_h) = (f, v_h) + \langle g_h, v_h\rangle_h\,, \quad \forall\, v_h \in V_h^0.$$

Here $\langle\cdot,\cdot\rangle_h$ denotes the scalar product in the space $L^2(\Gamma_h)$. It is easy to see that the finite element problem (2.9) has a unique solution u_h.

The main results of this section are stated in the following theorem:

Theorem 2.2 *Let u and u_h be the solutions to Problem (P) and Problem (P$_h$), respectively. Then, for $0 < h < h_0$, we have*

$$\text{(2.10)} \quad \|\nabla(u-u_h)\|_{L^2(\Omega)} \le C\,h\,|\log h|^{1/2}\left(\|f\|_{L^2(\Omega)} + \|g\|_{H^2(\Gamma)}\right),$$

$$\text{(2.11)} \qquad \|u-u_h\|_{L^2(\Omega)} \le C\,h^2\,|\log h|\;\left(\|f\|_{L^2(\Omega)} + \|g\|_{H^2(\Gamma)}\right).$$

Remark 2.2 The definition of Sobolev spaces on manifolds, e.g. $\|\cdot\|_{H^2(\Gamma)}$, can be found in Grisvard [11]. The regularity of g in Theorem 2.2 can be weakened, see Remark 2.3.

Before proving Theorem 2.2, we first show two lemmas needed. Let $\Pi_h : C(\bar{\Omega}) \mapsto V_h$ be the standard linear interpolation operator corresponding to the space V_h (cf. Ciarlet [7]). As the solutions concerned are only in $H^1(\Omega)$ globally (cf. Theorem 2.1), one can not apply the standard interpolation theory directly. Instead we are able to show the following properties of the interpolant Π_h:

Lemma 2.1 *For the linear interpolation operator $\Pi_h : C(\bar{\Omega}) \mapsto V_h$, we have*

$$\text{(2.12)}\quad \|v - \Pi_h v\|_{L^2(\Omega)} + h\,\|\nabla(v - \Pi_h v)\|_{L^2(\Omega)} \le C\,h^2\,|\log h|^{1/2}\|v\|_X, \quad \forall\, v \in X\,.$$

Proof. For any $v \in X$, let v_i be the restriction of v on Ω_i for $i = 1, 2$. As the interface Γ is of class C^2, we can extend the function $v_i \in H^2(\Omega_i)$ onto the whole domain Ω and obtain the function $\tilde{v}_i \in H^2(\Omega)$ such that $\tilde{v}_i = v_i$ on Ω_i and

$$\|\tilde{v}_i\|_{H^2(\Omega)} \le C\,\|v\|_{H^2(\Omega_i)} \quad \text{for } \ i = 1, 2,$$

See Stein [24] for the existence of such extensions.

Now for any element K in $\mathcal{T}_h \setminus \mathcal{T}_h^*$, the standard finite element interpolation theory (cf. Ciarlet [7]) implies that

$$\text{(2.13)}\qquad \|v - \Pi_h v\|_{H^m(K)} \le C\,h^{2-m}\|v\|_{H^2(K)}, \quad m = 0, 1.$$

Next, we consider any element K in $\mathcal{T}_h^*$. Recall that $\mathcal{K}_i = K \cap \Omega_i$ for $i = 1, 2$. Without loss of generality, we can assume that meas$(\mathcal{K}_2) \le C h_K^3$. Thus, using Hölder's inequality we derive that for any $p > 2$ and $m = 0, 1$,

$$\begin{aligned}
\|v - \Pi_h v\|_{H^m(\mathcal{K}_2)} &\le C\,h_K^{\frac{3(p-2)}{2p}}\,\|v - \Pi_h v\|_{W^{m,p}(\mathcal{K}_2)} \\
&\le C\,h_K^{\frac{3(p-2)}{2p}}\,\|v - \Pi_h v\|_{W^{m,p}(K)} \\
&\le C\,h_K^{\frac{3(p-2)}{2p}+1-m}\|v\|_{W^{1,p}(K)} \qquad \text{(2.14)}
\end{aligned}$$

where we have used the standard finite element interpolation result in the last inequality. On the other hand, it follows by means of the previously defined extention $\tilde{v}_i$ of v_i that for $m = 0, 1$,

$$\begin{aligned}
\|v - \Pi_h v\|_{H^m(\mathcal{K}_1)} &= \|\tilde{v}_1 - \Pi_h \tilde{v}_1\|_{H^m(\mathcal{K}_1)} \\
&\le \|\tilde{v}_1 - \Pi_h \tilde{v}_1\|_{H^m(K)} \\
&\le C\,h_K^{2-m}\|\tilde{v}_1\|_{H^2(K)}. \qquad \text{(2.15)}
\end{aligned}$$

Now by (2.14)–(2.15) we obtain for $m = 0, 1$ that

$$\begin{aligned}
\sum_{K \in \mathcal{T}_h^*} \|v - \Pi_h v\|^2_{H^m(K)} &\le C\,h^{4-2m}\{\|\tilde{v}_1\|^2_{H^2(\Omega)} + \|\tilde{v}_2\|^2_{H^2(\Omega)}\} \\
&\quad + C \sum_{K \in \mathcal{T}_h^*} h^{2-2m+\frac{3(p-2)}{p}}\,\|v\|^2_{W^{1,p}(K)} \\
&\le C\,h^{4-2m}\|v\|^2_X
\end{aligned}$$

$$+C\,h^{2-2m+\frac{2(p-2)}{p}}\{\sum_{K\in\mathcal{T}_h^*}\|v\|^p_{W^{1,p}(K)}\}^{\frac{2}{p}}$$
$$\le C\,h^{4-2m}\|v\|^2_X + C\,h^{4-2m-\frac{4}{p}}\|v\|^2_{W^{1,p}(\Omega)}$$

where in the second inequality we have used the discrete Hölder's inequality and the fact that $\sum_{K\in\mathcal{T}_h^*}1 \le C\,h^{-1}$ due to the quasi-uniformity of the triangulations $\mathcal{T}_h$. Now using the Sobolev embedding inequality for two dimensions (cf. Ren-Wei [23]):

$$\|\phi\|_{L^p(\Omega_i)} \le C\,p^{1/2}\|\phi\|_{H^1(\Omega_i)}, \quad \forall\,p>2, \qquad \phi\in H^1(\Omega_i),\ i=1,2, \tag{2.16}$$

we conclude that for $m=0,1$ and any $p>2$,

$$\left\{\sum_{K\in\mathcal{T}_h^*}\|v-\Pi_h v\|^2_{H^m(K)}\right\}^{1/2} \le C\,h^{2-m}\|v\|_X + Ch^{2-m-\frac{2}{p}}p^{\frac{1}{2}}\|v\|_X, \tag{2.17}$$

where the constant C is independent of h and $p>2$. Taking $p=|\log h|$ in (2.17) and combining (2.13) with (2.17), we finally obtain for $m=0,1$ that

$$\|v-\Pi_h v\|_{H^m(\Omega)} \le C\,h^{2-m}|\log h|^{1/2}\,\|v\|_X.$$

This completes the proof of Lemma 2.1. □

The second lemma is on the approximation property of g_h to the interface function g:

Lemma 2.2 *Assume that $g\in H^2(\Gamma)$. Then we have*

$$\left|\int_\Gamma g\,v_h\,ds - \int_{\Gamma_h} g_h\,v_h\,ds\right| \le C\,h^{3/2}\|g\|_{H^2(\Gamma)}\|v_h\|_{H^1(\Omega^*)}, \qquad \forall\,v_h\in V_h, \tag{2.18}$$

where $\Omega^* = \cup_{K\in\mathcal{T}_h^*}K$.

Proof. To simplify the notation, we will make no difference between a function defined in the original x_1x_2-coordinate system and its transformed version in the local $x_1^r x_2^r$-coordinate system introduced before. For example, the interface function $g(x_1,x_2)$ in the x_1x_2-coordinate system will be denoted by $g(\phi_r^h(x_2^r),x_2^r)$ in the $x_1^r x_2^r$-coordinate system. This abuse of notation will not affect any results involved in the following proof but it simplifies the notation greatly. Thus, under the local coordinate (x_1^r, x_2^r), the restriction of g_h on S_r^h, denoted by $\tilde{g}_h$, can be written as

$$\tilde{g}_h(0, x_2^r) = \frac{x_2^r}{s_r^h} g(0, s_r^h) + \frac{s_r^h - x_2^r}{s_r^h} g(0,0), \quad \forall\, x_2^r \in [0, s_r^h]. \tag{2.19}$$

Then for $r = 1, \cdots, m_h$, we have

$$\begin{aligned}
&\int_{\Gamma_r^h} g\, v_h\, ds - \int_{S_r^h} g_h\, v_h\, ds \\
&= \int_0^{s_r^h} g(\phi_r^h(x_2^r), x_2^r)\, v_h(\phi_r^h(x_2^r), x_2^r) \sqrt{1 + \Big|\frac{d}{dx_2^r}\phi_r^h(x_2^r)\Big|^2}\, dx_2^r \\
&\quad - \int_0^{s_r^h} \tilde{g}_h(0, x_2^r)\, v_h(0, x_2^r) dx_2^r \\
&= \int_0^{s_r^h} g(\phi_r^h(x_2^r), x_2^r) \Big[v_h(\phi_r^h(x_2^r), x_2^r) - v_h(0, x_2^r)\Big] \\
&\quad \times \sqrt{1 + \Big|\frac{d}{dx_2^r}\phi_r^h(x_2^r)\Big|^2}\, dx_2^r \\
&\quad + \int_0^{s_r^h} \Big[g(\phi_r^h(x_2^r), x_2^r) - \tilde{g}_h(0, x_2^r)\Big]\, v_h(0, x_2^r) \sqrt{1 + \Big|\frac{d}{dx_2^r}\phi_r^h(x_2^r)\Big|^2}\, dx_2^r \\
&\quad + \int_0^{s_r^h} \tilde{g}_h(0, x_2^r)\, v_h(0, x_2^r) \left[\sqrt{1 + \Big|\frac{d}{dx_2^r}\phi_r^h(x_2^r)\Big|^2} - 1\right] dx_2^r \\
&\equiv: (\mathrm{I})_1 + (\mathrm{I})_2 + (\mathrm{I})_3\,.
\end{aligned} \tag{2.20}$$

We now estimate $(\mathrm{I})_1$, $(\mathrm{I})_2$ and $(\mathrm{I})_3$ one by one. First from (2.7) we obtain

$$|(\mathrm{I})_1| \le C\, h^3\, \|g\|_{L^\infty(\Gamma_r^h)} \sum_{K \in \mathcal{T}_h^r} \|\nabla v_h\|_{L^\infty(K)},$$

where $\mathcal{T}_h^r = \{K \in \mathcal{T}_h;\ K \cap \Gamma_r^h \ne \emptyset\}$. Obviously, the number of elements in $\mathcal{T}_h^r$ is bounded by some constant independent of h. Using the inverse inequality, we get

$$|(\mathrm{I})_1| \le C\, h^2\, \|g\|_{L^\infty(\Gamma)} \sum_{K \in \mathcal{T}_h^r} \|\nabla v_h\|_{L^2(K)}. \tag{2.21}$$

We know from (2.19) that $\tilde{g}_h(0, x_2^r)$ is the linear interpolant of $g(\phi_r^h(x_2^r), x_2^r)$ on $[0, s_r^h]$. Thus, the standard finite element interpolation theory and inverse inequality imply

$$
\begin{aligned}
|(\mathrm{I})_2| &\le C\,h^{1/2}\,\|g(\phi_r^h(\cdot),\cdot) - \tilde{g}_h(0,\cdot)\|_{L^2(S_r^h)}\,\|v_h\|_{L^\infty(S_r^h)} \\
&\le Ch^{5/2}\,\|g\|_{H^2(\Gamma_r^h)}\,\|v_h\|_{L^\infty(K_r)} \\
&\le C\,h^{3/2}\|g\|_{H^2(\Gamma_r^h)}\,\|v_h\|_{L^2(K_r)}. \qquad (2.22)
\end{aligned}
$$

We now turn to estimate $(\mathrm{I})_3$. We know from (2.7) that

$$
\Big|\sqrt{1+\Big|\frac{d}{dx_2^r}\phi_r^h(x_2^r)\Big|^2}-1\Big| \le \frac{1}{2}\Big|\frac{d}{dx_2^r}\phi_r^h(x_2^r)\Big|^2 \le C\,h^2, \quad \forall\, x_2^r \in [0, s_r^h].
$$

Using this and inverse inequality we derive

$$
|(\mathrm{I})_3| \le C\,h^3\|g\|_{L^\infty(\Gamma)}\|v_h\|_{L^\infty(S_r^h)} \le Ch^2\,\|g\|_{L^\infty(\Gamma)}\|v_h\|_{L^2(K_r)}. \qquad (2.23)
$$

Thus we conclude from (2.20)–(2.23) that

$$
\begin{aligned}
&\Big|\int_\Gamma g\,v_h\,ds - \int_{\Gamma_h} g_h\,v_h\,ds\Big| \\
&\le Ch^2\sum_{r=1}^{m_h}\Big[\|g\|_{L^\infty(\Gamma)}\sum_{K\in\mathcal{T}_h^r}\|v_h\|_{H^1(K)}\Big] \\
&\quad +Ch^{3/2}\sum_{r=1}^{m_h}\|g\|_{H^2(\Gamma_r^h)}\|v_h\|_{L^2(K_r)} \\
&\le Ch^2\,\|g\|_{L^\infty(\Gamma)}\sum_{K\in\mathcal{T}_h^*}\|v_h\|_{H^1(K)} + Ch^{3/2}\|g\|_{H^2(\Gamma)}\|v_h\|_{L^2(\Omega^*)} \\
&\le Ch^{3/2}\|g\|_{H^2(\Gamma)}\|v_h\|_{H^1(\Omega^*)},
\end{aligned}
$$

where in the second inequality we have used the fact the number of elements in $\mathcal{T}_h^r$ is bounded by some constant independent of h. This completes the proof of Lemma 2.2. □

Remark 2.3 The regularity requirement on the interface function g in Lemma 2.2 can be much weaker. In fact, g needs only to be piecewise in H^2, i.e. $g \in H^2(\Gamma_r^h)$ for $r = 1, 2, \cdots, m_h$. In this case, all the discontinuous points on Γ must be taken as the finite element nodes. On the other hand, if we don't use the approximation g_h for g in Problem (P_h), i.e. replace the term $\langle g_h, v_h\rangle_h$ by $\langle g, v_h\rangle$, then $g \in H^{1/2}(\Gamma)$ is enough for Lemma 2.2 and thus also enough for the main results of this section, i.e. Theorem 2.2.

Proof of Theorem 2.2. We know from (2.5) and (2.9) that

$$\begin{aligned} & a(u_h - \Pi_h u, v_h) \\ = \; & a(u - \Pi_h u, v_h) + \{a(u_h, v_h) - a_h(u_h, v_h)\} \\ & + \{\langle g_h, v_h\rangle_h - \langle g, v_h\rangle\} \\ \equiv\!: \; & \text{(II)}_1 + \text{(II)}_2 + \text{(II)}_3, \quad \forall\, v_h \in V_h^0. \end{aligned} \tag{2.24}$$

By Lemma 2.1, we can bound the term (II)_1 by

$$\begin{aligned} |\text{(II)}_1| &\le C\, \|\nabla(u - \Pi_h u)\|_{L^2(\Omega)} \|\nabla v_h\|_{L^2(\Omega)} \\ &\le C\, h\, |\log h|^{1/2} \|u\|_X\, \|\nabla v_h\|_{L^2(\Omega)}\,. \end{aligned} \tag{2.25}$$

The term (II)_3 can be bounded immediately by using Lemma 2.2 and Poincare's inequality:

$$\begin{aligned} |\text{(II)}_3| &\le C h^{3/2}\, \|g\|_{H^2(\Gamma)} \|v_h\|_{H^1(\Omega^*)} \\ &\le C h^{3/2}\, \|g\|_{H^2(\Gamma)} \|\nabla v_h\|_{L^2(\Omega)}. \end{aligned} \tag{2.26}$$

For the term (II)_2, we recall that for any $K \in \mathcal{T}_h^*$, $\tilde{K} = \text{supp}\,(\beta - \beta_h) \cap K$ and meas$(\tilde{K}) \le C\, h_K^3$, then we have

$$\begin{aligned} |\text{(II)}_2| &= \Big| \sum_{K \in \mathcal{T}_h^*} \int_K (\beta - \beta_K) \nabla u_h \cdot \nabla v_h \, dx \Big| \\ &= \Big| \sum_{K \in \mathcal{T}_h^*} \int_{\tilde{K}} (\beta - \beta_K) \nabla u_h \cdot \nabla v_h \, dx \Big| \\ &\le C \sum_{K \in \mathcal{T}_h^*} \|\nabla u_h\|_{L^2(\tilde{K})} \|\nabla v_h\|_{L^2(\tilde{K})} \\ &\le C\, h \sum_{K \in \mathcal{T}_h^*} \|\nabla u_h\|_{L^2(K)} \|\nabla v_h\|_{L^2(K)} \\ &\le C\, h\, \|\nabla u_h\|_{L^2(\Omega)} \|\nabla v_h\|_{L^2(\Omega)} \\ &\le C\, h\, (\|f\|_{L^2(\Omega)} + \|g\|_{L^\infty(\Gamma)})\, \|\nabla v_h\|_{L^2(\Omega)}, \end{aligned} \tag{2.27}$$

where we have used the fact that ∇u_h and ∇v_h are constant in $K, \forall\, K \in \mathcal{T}_h$, in the second inequality and the inequality

$$\|\nabla u_h\|_{L^2(\Omega)} \le C\, (\|f\|_{L^2(\Omega)} + \|g\|_{L^\infty(\Gamma)}),$$

which follows directly from (2.9) by using $\|g_h\|_{L^2(\Gamma_h)} \le C\, \|g_h\|_{L^\infty(\Gamma_h)} \le C\, \|g\|_{L^\infty(\Gamma)}$.

From the estimates (2.25) and (2.27), we conclude by taking $v_h = u_h - \Pi_h u$ in (2.24) that

$$\|\nabla(u_h - \Pi_h u)\|_{L^2(\Omega)} \le C\, h\, |\log h|^{1/2}\, (\|f\|_{L^2(\Omega)} + \|g\|_{H^2(\Gamma)}). \tag{2.28}$$

Now the desired estimate (2.10) follows from (2.28), Lemma 2.1 and the triangle inequality.

To show the L^2-estimate in (2.11), we use the Nitsche's trick. Let $w \in H_0^1(\Omega)$ be the solution of the following auxiliary problem: find $w \in H_0^1(\Omega)$ satisfying

$$a(w,v) = (u - u_h, v), \quad \forall\, v \in H_0^1(\Omega) \tag{2.29}$$

and its finite element approximation: find $w_h \in V_h^0$ satisfying

$$a_h(w_h,v) = (u - u_h, v), \quad \forall\, v \in V_h^0. \tag{2.30}$$

By Theorem 2.1, we know that $\|w\|_X \le C\,\|u-u_h\|_{L^2(\Omega)}$. Now noting the fact the jump $[\beta(\partial w/\partial \mathbf{n})] = 0$ across the interface Γ, and applying the previously proved result (2.10) for problems (2.29)-(2.30), we have

$$\|\nabla(w-w_h)\|_{L^2(\Omega)} \le C\,h\,|\log h|^{1/2}\,\|u-u_h\|_{L^2(\Omega)}. \tag{2.31}$$

Taking $v = u - u_h \in H_0^1(\Omega)$ in (2.29) and using (2.5) and (2.9), we get

$$\begin{aligned} \|u-u_h\|^2_{L^2(\Omega)} &= a(w, u-u_h) \\ &= a(w-w_h, u-u_h) + a(w_h, u-u_h) \\ &= a(w-w_h, u-u_h) + [a_h(u_h,w_h) - a(u_h,w_h)] \\ &\quad + [\,\langle g, w_h\rangle - \langle g_h, w_h\rangle_h\,] \\ &\equiv: \text{(III)}_1 + \text{(III)}_2 + \text{(III)}_3. \end{aligned} \tag{2.32}$$

By (2.10) and (2.31) we immediately have

$$|\text{(III)}_1| \le C\,h^2\,|\log h|\,\|u-u_h\|_{L^2(\Omega)}\Big(\|f\|_{L^2(\Omega)} + \|g\|_{H^2(\Gamma)}\Big). \tag{2.33}$$

We next estimate the term (III)_2 in (2.32).

Arguing as deriving (2.27), we can deduce

$$\begin{aligned} |\text{(III)}_2| &\le C\,h \sum_{K\in\mathcal{T}_h^*} \|\nabla u_h\|_{L^2(K)}\|\nabla w_h\|_{L^2(K)} \\ &\le C\,h\,\|\nabla u_h\|_{L^2(\Omega^*)}\|\nabla w_h\|_{L^2(\Omega^*)} \\ &\le C\,h\,\|\nabla (u-u_h)\|_{L^2(\Omega^*)}\|\nabla w_h\|_{L^2(\Omega^*)} \\ &\quad + C\,h\,\|\nabla u\|_{L^2(\Omega^*)}\|\nabla (w-w_h)\|_{L^2(\Omega^*)} \\ &\quad + C\,h\,\|\nabla u\|_{L^2(\Omega^*)}\|\nabla w\|_{L^2(\Omega^*)} \\ &\le C\,h^2|\log h|^{1/2}\Big(\|f\|_{L^2(\Omega)} + \|g\|_{H^2(\Gamma)}\Big)\,\|u-u_h\|_{L^2(\Omega)} \\ &\quad + C\,h\,\|\nabla u\|_{L^2(\Omega^*)}\|\nabla w\|_{L^2(\Omega^*)}, \end{aligned} \tag{2.34}$$

where we have used (2.10), (2.31), (2.4) and the following inequality

$$\|\nabla w_h\|_{L^2(\Omega)} \le C\,\|u-u_h\|_{L^2(\Omega)}.$$

The last term in (2.34) can be further estimated as follows:

$$\|\nabla u\|_{L^2(\Omega^*)} \le C\, h^{\frac{p-2}{2p}} \|\nabla u\|_{L^p(\Omega^*)} \le C\, p^{\frac{1}{2}}\, h^{\frac{p-2}{2p}} \|u\|_X$$

by Hölder's inequality, meas$(\Omega^*) \le C\,h$ and Sobolev inequality (2.16). Taking $p = |\log h|$ and using Theorem 2.1 yield

$$\|\nabla u\|_{L^2(\Omega^*)} \le C\, h^{1/2}\, |\log h|^{1/2} \left(\|f\|_{L^2(\Omega)} + \|g\|_{H^2(\Gamma)} \right). \tag{2.35}$$

Similarly, we have

$$\begin{aligned} \|\nabla w\|_{L^2(\Omega^*)} &\le C\, h^{1/2}\, |\log h|^{1/2} \|w\|_X \\ &\le C\, h^{1/2}\, |\log h|^{1/2} \|u - u_h\|_{L^2(\Omega)}\,. \end{aligned} \tag{2.36}$$

Thus, we have derived from (2.34)-(2.36) that

$$|(\mathrm{III})_2| \le Ch^2\, |\log h| \left(\|f\|_{L^2(\Omega)} + \|g\|_{H^2(\Gamma)} \right) \|u - u_h\|_{L^2(\Omega)}. \tag{2.37}$$

To estimate the last term $(\mathrm{III})_3$ in (2.32), using Lemma 2.2 yields

$$\begin{aligned} |(\mathrm{III})_3| &\le Ch^{3/2} \|g\|_{H^2(\Gamma)} \|w_h\|_{H^1(\Omega^*)} \\ &\le Ch^{3/2} \|g\|_{H^2(\Gamma)} \|w - w_h\|_{H^1(\Omega^*)} + Ch^{3/2} \|g\|_{H^2(\Gamma)} \|w\|_{H^1(\Omega^*)} \\ &\le Ch^{5/2} |\log h|^{1/2}\, \|g\|_{H^2(\Gamma)} \|u - u_h\|_{L^2(\Omega)} \\ &\quad + Ch^{3/2} \|g\|_{H^2(\Gamma)} \|w\|_{H^1(\Omega^*)}\,. \end{aligned}$$

Now using (2.36) we obtain

$$|(\mathrm{III})_3| \le Ch^2 |\log h|^{1/2} \left(\|f\|_{L^2(\Omega)} + \|g\|_{H^2(\Gamma)} \right) \|u - u_h\|_{L^2(\Omega)}. \tag{2.38}$$

Now the L^2-error estimate (2.11) follows from (2.32)–(2.38). This completes the proof of Theorem 2.2. □

Remark 2.4 The error estimates (2.10)-(2.11) are optimal up to the factor $|\log h|$. However, the H^1-error estimate can be improved by using a little bit more detailed argument. It was proved in Ladyzhenskaya et al. [17] that $u \in W^{1,\infty}(\Omega_0 \cap \Omega_i)$ for $i = 1, 2$, where Ω_0 is some neighborhood of the interface Γ. By checking the proof of Lemma 2.1, it is easy to see that for $m = 0, 1$ and any $v \in X \cap W^{1,\infty}(\Omega_1 \cap \Omega_0) \cap W^{1,\infty}(\Omega_2 \cap \Omega_0)$ we have

$$\begin{aligned} &\|v - \Pi_h v\|_{H^m(\Omega)} \\ &\le C\, h^{2-m} \left(\|v\|_X + \|v\|_{W^{1,\infty}(\Omega_1 \cap \Omega_0)} + \|v\|_{W^{1,\infty}(\Omega_2 \cap \Omega_0)} \right). \end{aligned}$$

Here we have to choose h_0 so small such that $\Omega^* \subset \Omega_0$. This estimate enables us to achieve the improved error estimate following the same proof of Theorem 2.2:

$$\|u - u_h\|_{H^1(\Omega)} \\ \le C\,h\left(\|f\|_{L^2(\Omega)} + \|g\|_{H^2(\Gamma)} + \|u\|_{W^{1,\infty}(\Omega_1\cap\Omega_0)} + \|u\|_{W^{1,\infty}(\Omega_2\cap\Omega_0)}\right).$$

This is the optimal energy-norm error estimate.

3. Parabolic interface problems

Let Ω be a convex polygon in $\mathbb{R}^2$ and $\Omega_1 \subset \Omega$ be an open domain with C^2 boundary $\Gamma = \partial\Omega_1 \subset \Omega$, $\Omega_2 = \Omega \setminus \Omega_1$ (see Fig. 1) and $Q_T = \Omega \times (0,T)$. In this section, we consider the following parabolic interface problem

$$\frac{\partial u}{\partial t} - \nabla\cdot(\beta\nabla u) = f(x,t) \quad \text{in} \quad Q_T \tag{3.1}$$

with the initial and boundary conditions

$$u(x,0) = u_0(x) \quad \text{in } \Omega; \quad u = 0 \quad \text{on} \quad \partial\Omega \times (0,T) \tag{3.2}$$

and jump conditions on the interface

$$[u] = 0, \quad \left[\beta\frac{\partial u}{\partial \mathbf{n}}\right] = g(x,t) \quad \text{across} \quad \Gamma \times (0,T), \tag{3.3}$$

where $[v]$ and $\mathbf{n}$ are specified as in Sect. 2. For the ease of exposition, we assume that the coefficient β is positive and piecewise constant, i.e.

$$\beta(x) = \beta_1 \quad \text{for} \quad x \in \Omega_1; \quad \beta(x) = \beta_2 \quad \text{for} \quad x \in \Omega_2.$$

As stated in Sect. 2, the results of this section can be easily extended to more general parabolic interface problems (see Sect. 4).

For convenience, we introduce a space Y defined by

$$Y = L^2(\Omega) \cap H^1(\Omega_1) \cap H^1(\Omega_2)$$

equipped with the norm

$$\|u\|_Y = \|u\|_{L^2(\Omega)} + \|u\|_{H^1(\Omega_1)} + \|u\|_{H^1(\Omega_2)}.$$

Regarding the regularity for the solutions of the interface problem (3.1)-(3.3), we have the following results (cf. Ladyzhenskaya et al. [17]):

Theorem 3.1 *Assume that $f \in H^1(0,T;L^2(\Omega))$, $u_0 \in H^1(\Omega)$ and $g \in L^2(0,T;H^{1/2}(\Gamma))$. Then the problem (3.1)–(3.3) has a unique solution $u \in L^2(0,T;X) \cap H^1(0,T;Y)$.*

We are now going to formulate the fully-discrete approximation to the problem (3.1)-(3.3). We shall make use of the backward difference scheme to discretize the problem in time and the piecewise linear finite element method in space.

We first divide the time interval $(0, T)$ into M equally-spaced subintervals by the following points:

$$0 = t^0 < t^1 < \cdots < t^M = T$$

with $t^n = n\tau$, $\tau = T/M$ the time step size. Let $I^n = (t^{n-1}, t^n]$ be the n-th subinterval. For a given sequence $\{w^n\}_{n=0}^M \subset L^2(\Omega)$, we introduce the backward difference quotient:

$$\partial_\tau w^n = \frac{w^n - w^{n-1}}{\tau}.$$

For a continuous mapping $w : [0, T] \to L^2(\Omega)$, we define $w^n = w(\cdot, n\tau)$, $0 \le n \le M$.

The piecewise linear finite element spaces V_h and V_h^0, and all other notation used in this section relevant to finite element discretizations are the same as in Sect. 2. To approximate the interface function $g(x,t)$, for $n = 1, 2, \cdots, M$, we define $g_h^n \in V_h$ as

$$\text{(3.4)} \qquad \bar{g}_h^n = \sum_{j=1}^{m_h} \bar{g}^n(P_j)\phi_j, \qquad \bar{g}^n(\cdot) = \tau^{-1} \int_{I^n} g(\cdot, t)\, dt.$$

With the above notation, we now introduce the fully-discrete finite element approximation to the problem (3.1)–(3.3):

Problem (P$_{h,\tau}$). Let $u_h^0 = \Pi_h u_0$. For $n = 1, 2, \cdots, M$, find $u_h^n \in V_h^0$ such that

$$\text{(3.5)} \quad (\partial_\tau u_h^n, v_h) + a_h(u_h^n, v_h) = (f^n, v_h) + \langle \bar{g}_h^n, v_h \rangle_h\,, \quad \forall\, v_h \in V_h^0.$$

Evidently, for each $n = 1, 2, \cdots, M$, Problem (P$_{h,\tau}$) has a unique solution $u_h^n \in V_h^0$ by Lax-Milgram theorem.

For convenience, let us define a piecewise constant function $u_{h,\tau}$ in time by

$$\text{(3.6)} \quad u_{h,\tau}(x,t) = u_h^n(x), \quad \forall\, t \in (t^{n-1}, t^n], \quad n = 1, 2, \cdots, M.$$

The main results of this section are the following Theorem 3.2 for the L^2-norm error estimate:

Theorem 3.2 *Let u and $u_{h,\tau}$ be the solutions of the problem (3.1)–(3.3) and Problem ($P_{h,\tau}$), respectively. Assume that $u_0 \in X$, $f \in H^1(0,T; L^2(\Omega))$ and $g \in L^2(0,T; H^2(\Gamma))$. Then, for $0 < h < h_0$, we have*

$$\|u - u_{h,\tau}\|_{L^2(Q_T)} \leq B_1(u,g,f)(\tau + h^2\,|\log h|) \tag{3.7}$$

where $B_1(u,g,f)$ is a constant depending on u, g and f specified by

$$B_1(u,g,f) = C\|u_0\|_X + C\Big\{\int_0^T \Big(\|u\|_X^2 + \|u_t\|_{L^2(\Omega)}^2 + \|f_t\|_{L^2(\Omega)}^2 + \|g\|_{H^2(\Gamma)}^2\Big)\,dt\Big\}^{\frac{1}{2}},$$

and the following Theorem 3.3 for the energy-norm error estimate:

Theorem 3.3 *Let u and $u_{h,\tau}$ be the solutions of the problem (3.1)–(3.3) and Problem ($P_{h,\tau}$), respectively. Assume that $u_0 \in H^1(\Omega)$, $f \in H^1(0, T; L^2(\Omega))$ and $g \in L^2(0,T; H^2(\Gamma))$. Then, for $0 < h < h_0$, we have*

$$\|u - u_{h,\tau}\|_{L^2(0,T;H^1(\Omega))} \leq B_2(u,g,f)(\tau + h\,|\log h|^{1/2}) \tag{3.8}$$

where $B_2(u,g,f)$ is a constant depending on u, g and f specified by

$$B_2(u,g,f) = C\|u_0\|_{H^1(\Omega)} + C\Big\{\int_0^T \Big(\|u\|_X^2 + \|u_t\|_Y^2 + \|f_t\|_{L^2(\Omega)}^2 + \|g\|_{H^2(\Gamma)}^2\Big)\,dt\Big\}^{\frac{1}{2}}.$$

We shall prove the above two theorems by the parabolic duality method. As in (3.4), for any function $\xi \in L^2(0,T;B)$ with some Banach space B, we denote

$$\bar{\xi}^n(\cdot) = \tau^{-1}\int_{I^n} \xi(\cdot,t)\,dt, \quad n = 1,2,\cdots,M.$$

We first prove Theorem 3.2. To do so, we introduce the following auxiliary discrete dual problem:

Problem (A$_{h,\tau}$). Let $w_h^M = 0$. For $n = M, M-1, \cdots, 1$, find $w_h^{n-1} \in V_h^0$ such that

$$(-\partial_\tau w_h^n, v_h) + a_h(w_h^{n-1}, v_h) = (\bar{u}^n - u_h^n, v_h), \quad \forall\, v_h \in V_h^0. \tag{3.9}$$

Clearly, Problem (A$_{h,\tau}$) has a unique solution $w_h^{n-1} \in V_h^0$ for each $n = M, M-1, \cdots, 1$.

We shall need the following stability results for the solution w_h^n to Problem (A$_{h,\tau}$).

Lemma 3.1 *We have*

$$\max_{1\le n\le M} \|w_h^{n-1}\|^2_{H^1(\Omega)} + \sum_{n=1}^{M} \tau \, \|\partial_\tau w_h^n\|^2_{L^2(\Omega)} \le C \sum_{n=1}^{M} \tau \, \|\bar{u}^n - u_h^n\|^2_{L^2(\Omega)}. \tag{3.10}$$

Proof. The lemma can be easily proved by taking $v_h = -\tau(\partial_\tau w_h^n)$ in (3.9) and applying the standard arguments (cf. Chen-Hoffmann [6] and Hoffmann-Zou [13]). We omit the details. □

The following lemma is crucial to the proof of Theorem 3.2.

Lemma 3.2 *We have*

$$\sum_{n=1}^{M} \tau \, \|w_h^{n-1}\|^2_{H^1(\Omega^*)} \le C\, h \, |\log h| \left(\sum_{n=1}^{M} \tau \|\bar{u}^n - u_h^n\|^2_{L^2(\Omega)} \right). \tag{3.11}$$

Proof. Recall that $\Omega^* = \cup_{K\in\mathcal{T}_h^*} K$ and meas$(\Omega^*) \le C\, h$. For $n = M, M-1, \cdots, 1$, we define $w^{n-1} \in H_0^1(\Omega)$ to be the solution of the following elliptic interface problem:

$$a(w^{n-1}, v) = (\bar{u}^n - u_h^n, v) + (\partial_\tau w_h^n, v), \quad \forall\, v \in H_0^1(\Omega), \tag{3.12}$$

which corresponds to the following jump conditions

$$[w^{n-1}] = 0, \quad \left[\beta \frac{\partial w^{n-1}}{\partial \mathbf{n}}\right] = 0 \quad \text{across the interface} \quad \Gamma.$$

Then, by Theorem 2.1, we know that $w^{n-1} \in X$ and

$$\|w^{n-1}\|_X \le C \Big(\|\bar{u}^n - u_h^n\|_{L^2(\Omega)} + \|\partial_\tau w_h^n\|_{L^2(\Omega)} \Big). \tag{3.13}$$

We know from (3.9) that $w_h^{n-1} \in V_h^0$ is the finite element approximation of w^{n-1} defined in (3.12). Thus, by Theorem 2.2, we have

$$\|w_h^{n-1} - w^{n-1}\|_{H^1(\Omega)} \le C\, h \, |\log h|^{1/2} \Big(\|\bar{u}^n - u_h^n\|_{L^2(\Omega)} + \|\partial_\tau w_h^n\|_{L^2(\Omega)} \Big),$$

which with Lemma 3.1 implies

$$\sum_{n=1}^{M} \tau \|w^{n-1} - w_h^{n-1}\|^2_{H^1(\Omega)} \le C\, h^2 \, |\log h| \sum_{n=1}^{M} \tau \|\bar{u}^n - u_h^n\|^2_{L^2(\Omega)}. \tag{3.14}$$

Now using the argument leading to (2.35), the estimate (3.13) and Lemma 3.1, we obtain

$$\begin{aligned}\sum_{n=1}^{M}\tau\|w^{n-1}\|^2_{H^1(\Omega^*)} &\le C\,h\,|\log h|\sum_{n=1}^{M}\tau\|w^{n-1}\|^2_X\\ &\le C\,h\,|\log h|\sum_{n=1}^{M}\tau\|\bar u^n-u_h^n\|^2_{L^2(\Omega)}.\end{aligned}\tag{3.15}$$

Then (3.11) follows from (3.14)-(3.15) and the following triangle inequality

$$\sum_{n=1}^{M}\tau\|w_h^{n-1}\|^2_{H^1(\Omega^*)}\le\sum_{n=1}^{M}\tau\|w^{n-1}-w_h^{n-1}\|^2_{H^1(\Omega)}+\sum_{n=1}^{M}\tau\|w^{n-1}\|^2_{H^1(\Omega^*)}.$$

This completes the proof of Lemma 3.2. □

Proof of Theorem 3.2. For any $v\in X$, let $f^*=\beta_i\Delta v_i$ in Ω_i, $i=1,2$. Clearly, $f^*\in L^2(\Omega)$. With this f^* we define an operator $P_h: X\cap H_0^1(\Omega)\mapsto V_h^0$ by

$$a_h(P_hv,\,v_h)=(f^*,v_h),\quad\forall\,v\in X\cap H_0^1(\Omega),\ v_h\in V_h^0.$$

It is easy to verify that

$$\begin{aligned}&a_h(P_hv,\phi_h)=(f^*,\phi_h)=a(v,\phi_h),\\ &v\in X\cap H_0^1(\Omega),\quad\phi_h\in V_h^0.\end{aligned}\tag{3.16}$$

Thus by Theorem 2.2 and the definition of f^*, we have for any $v\in X\cap H_0^1(\Omega)$,

$$\begin{aligned}\|v-P_hv\|_{L^2(\Omega)}&\le C\,h^2\,|\log h|\,\|f^*\|_{L^2(\Omega)}\\ &\le C\,h^2\,|\log h|\,\|v\|_X,\end{aligned}\tag{3.17}$$

$$\begin{aligned}\|\nabla(v-P_hv)\|_{L^2(\Omega)}&\le C\,h\,|\log h|^{1/2}\,\|f^*\|_{L^2(\Omega)}\\ &\le C\,h\,|\log h|^{1/2}\,\|v\|_X.\end{aligned}\tag{3.18}$$

Now taking $v_h=\tau(P_h\bar u^n-u_h^n)\in V_h^0$ in (3.9) and then summing the resultant equations over n, we get

$$\begin{aligned}&\sum_{n=1}^{M}\tau\|\bar u^n-u_h^n\|^2_{L^2(\Omega)}\\ &=\sum_{n=1}^{M}\tau(\bar u^n-u_h^n,\bar u^n-P_h\bar u^n)+\sum_{n=1}^{M}\tau(\bar u^n-u_h^n,P_h\bar u^n-u_h^n)\end{aligned}$$

$$= \sum_{n=1}^{M} \tau(\bar{u}^n - u_h^n, \bar{u}^n - P_h \bar{u}^n) + \sum_{n=1}^{M} \tau(-\partial_\tau w_h^n, P_h \bar{u}^n - u_h^n)$$
$$+ \sum_{n=1}^{M} \tau a_h(w_h^{n-1}, P_h \bar{u}^n - u_h^n)$$
$$= \sum_{n=1}^{M} \tau(\bar{u}^n - u_h^n, \bar{u}^n - P_h \bar{u}^n) + \sum_{n=1}^{M} \tau(- \partial_\tau w_h^n, P_h \bar{u}^n - u^n)$$
$$+ \sum_{n=1}^{M} \tau(- \partial_\tau w_h^n,\, u^n - u_h^n)$$
$$+ \sum_{n=1}^{M} \tau \Big[a(w_h^{n-1}, \bar{u}^n) - a_h(w_h^{n-1}, u_h^n) \Big]$$
$$\equiv: (\mathrm{IV})_1 + (\mathrm{IV})_2 + (\mathrm{IV})_3 + (\mathrm{IV})_4. \tag{3.19}$$

where in the last equality we have used (3.16).

Before estimating the four terms in (3.19), we first rewrite $(\mathrm{IV})_3 + (\mathrm{IV})_4$ a little bit. To do so, we multiply both sides of (3.1) by any $v \in H_0^1(\Omega)$, then integrate the equation over $\Omega \times I^n$ and make use of the conditions (3.2)-(3.3) to obtain for $n = 1, 2, \cdots, M$,

$$(\partial_\tau u^n, v) + a\,(\bar{u}^n, v) = (\bar{f}^n, v) + \langle \bar{g}^n, v \rangle\,, \quad \forall\, v \in H_0^1(\Omega). \tag{3.20}$$

Now taking $v_h = v = w_h^{n-1}$ both in (3.5) and (3.20), then subtracting the two equations, and summing the resulting equations over n, we come to

$$\sum_{n=1}^{M} \tau \left(\partial_\tau(u^n - u_h^n), w_h^{n-1} \right)$$
$$+ \sum_{n=1}^{M} \tau \left[a(\bar{u}^n, w_h^{n-1}) - a_h(u_h^n, w_h^{n-1}) \right]$$
$$= \sum_{n=1}^{M} \tau(\bar{f}^n - f^n, w_h^{n-1}) + \sum_{n=1}^{M} \tau \left[\langle \bar{g}^n, w_h^{n-1} \rangle - \langle \bar{g}_h^n, w_h^{n-1} \rangle_h \right]. \tag{3.21}$$

Applying the identity

$$\sum_{n=1}^{M} (a_n - a_{n-1}) b_n = a_M b_M - a_0 b_0 - \sum_{n=1}^{M} a_{n-1}(b_n - b_{n-1}),$$

to $(\mathrm{IV})_3$ with $a_n = w_h^n$ and $b_n = u^n - u_h^n$, we obtain from (3.21) that

$$(\mathrm{IV})_3 + (\mathrm{IV})_4 \;=\; (u_0 - \Pi_h u_0, w_h^0) + \sum_{n=1}^{M} \tau(\bar{f}^n - f^n, w_h^{n-1})$$

$$+\sum_{n=1}^{M}\tau\left[\left\langle \bar g^n, w_h^{n-1}\right\rangle - \left\langle \bar g_h^n, w_h^{n-1}\right\rangle_h\right]$$

$$\equiv: (\mathrm{V})_1 + (\mathrm{V})_2 + (\mathrm{V})_3, \tag{3.22}$$

where we have used $w_h^M = 0$ and $u_h^0 = \Pi_h u_0$.

Now we estimate the terms $(\mathrm{IV})_1$, $(\mathrm{IV})_2$ and $(\mathrm{V})_1$, $(\mathrm{V})_2$ and $(\mathrm{V})_3$ one by one. First we have from (3.17) that

$$\begin{aligned}\Big|(\mathrm{IV})_1\Big| &\le \sum_{n=1}^{M}\tau\|\bar u^n - u_h^n\|_{L^2(\Omega)}\|\bar u^n - P_h\bar u^n\|_{L^2(\Omega)}\\ &\le C\,h^2\,|\log h|\sum_{n=1}^{M}\tau\|\bar u^n - u_h^n\|_{L^2(\Omega)}\|\bar u^n\|_X\\ &\le C\,h^2\,|\log h|\Big[\sum_{n=1}^{M}\tau\|\bar u^n - u_h^n\|^2_{L^2(\Omega)}\Big]^{1/2}\Big[\int_0^T\|u\|_X^2\,dt\Big]^{1/2}.\end{aligned}$$

Similarly, using (3.17) and Lemma 3.1 we have

$$\begin{aligned}\Big|(\mathrm{IV})_2\Big| &\le \Big|\sum_{n=1}^{M}\tau(\partial_\tau w_h^n, P_h\bar u^n - \bar u^n)\Big| + \Big|\sum_{n=1}^{M}\tau(\partial_\tau w_h^n, \bar u^n - u^n)\Big|\\ &\le C\Big(\tau + h^2|\log h|\Big)\Big[\sum_{n=1}^{M}\tau\|\partial_\tau w_h^n\|^2_{L^2(\Omega)}\Big]^{1/2}\\ &\quad\times\Big[\int_0^T(\|u\|_X^2 + \|u_t\|^2_{L^2(\Omega)})dt\Big]^{1/2}\\ &\le C\Big(\tau + h^2|\log h|\Big)\Big[\sum_{n=1}^{M}\tau\|\bar u^n - u_h^n\|^2_{L^2(\Omega)}\Big]^{1/2}\\ &\quad\times\Big[\int_0^T(\|u\|_X^2 + \|u_t\|^2_{L^2(\Omega)})dt\Big]^{1/2}.\end{aligned}$$

By Lemmas 2.1 and 3.1, we can easily show that

$$\begin{aligned}&\Big|(\mathrm{V})_1\Big| + \Big|(\mathrm{V})_2\Big| \le C\left(\tau + h^2\,|\log h|\right)\\ &\quad\times\Big[\sum_{n=1}^{M}\tau\|\bar u^n - u_h^n\|^2_{L^2(\Omega)}\Big]^{1/2}\Big[\|u_0\|_X^2 + \int_0^T\|f_t\|^2_{L^2(\Omega)}dt\Big]^{1/2}.\end{aligned}$$

The term $(\mathrm{V})_3$ can be bounded by using Lemmas 2.2 & 3.2,

$$\Big|(\mathrm{V})_3\Big|$$

$$\le Ch^{3/2} \sum_{n=1}^{M} \tau \|\bar{g}^n\|_{H^2(\Gamma)} \|w_h^{n-1}\|_{H^1(\Omega^*)}$$
$$\le Ch^{3/2} \Big[\sum_{n=1}^{M} \tau \|w_h^{n-1}\|_{H^1(\Omega^*)} \Big]^{1/2} \Big[\int_0^T \|g\|_{H^2(\Gamma)}^2 dt \Big]^{1/2}$$
$$\le C\, h^2 \,|\log h|^{1/2} \Big[\sum_{n=1}^{M} \tau \|\bar{u}^n - u_h^n\|_{L^2(\Omega)}^2 \Big]^{1/2} \Big[\int_0^T \|g\|_{H^2(\Gamma)}^2 dt \Big]^{1/2}.$$

Finally, by simple calculations we have

$$\|u - u_{h,\tau}\|_{L^2(Q_T)} \le C\tau \, \|u_t\|_{L^2(Q_T)} + \Big[\sum_{n=1}^{M} \tau \|\bar{u}^n - u_h^n\|_{L^2(\Omega)}^2 \Big]^{1/2},$$

then Theorem 3.2 follows immediately from this inequality, (3.19), (3.22) and the previous estimates for $(\mathrm{IV})_1$, $(\mathrm{IV})_2$ and $(\mathrm{V})_i$, $i = 1, 2, 3$. □

We now prove Theorem 3.3. As in the proof of Theorem 3.2, we introduce the following auxiliary discrete dual problem:

Problem ($\mathrm{A}_{h,\tau}^*$). Let $z_h^M = 0$. For $n = M, M-1, \cdots, 1$, find $z_h^{n-1} \in V_h^0$ such that

$$(3.23) \qquad \begin{aligned} &(-\partial_\tau z_h^n, v_h) + a_h(z_h^{n-1}, v_h) = (\nabla(\bar{u}^n - u_h^n), \nabla v_h), \\ &\forall\, v_h \in V_h^0. \end{aligned}$$

Clearly, Problem ($\mathrm{A}_{h,\tau}^*$) has a unique solution $z_h^{n-1} \in V_h^0$ for each $n = M, M-1, \cdots, 1$.

We need the following stability results for the solution z_h^n to Problem ($\mathrm{A}_{h,\tau}^*$).

Lemma 3.3 *We have*

$$(3.24) \qquad \begin{aligned} &\max_{1\le n\le M} \|z_h^{n-1}\|_{L^2(\Omega)}^2 \\ &+ \sum_{n=1}^{M} \tau \, \|\nabla z_h^{n-1}\|_{L^2(\Omega)}^2 \le C \sum_{n=1}^{M} \tau \, \|\nabla(\bar{u}^n - u_h^n)\|_{L^2(\Omega)}^2, \end{aligned}$$

$$(3.25) \qquad \sum_{n=1}^{M} \tau \, \|\partial_\tau z_h^n\|_{H^{-1}(\Omega)}^2 \le C \sum_{n=1}^{M} \tau \, \|\nabla(\bar{u}^n - u_h^n)\|_{L^2(\Omega)}^2.$$

Proof. (3.24) can be proved by taking $v_h = z_h^{n-1}$ in (3.23) and applying the standard arguments (cf. Chen-Hoffmann [6] and Hoffmann-Zou [13]). We only show (3.25) below.

Let $Q_h : H_0^1(\Omega) \mapsto V_h^0$ be the standard L^2 projection defined by

$$(Q_h\phi,\ \phi_h) = (\phi,\ \phi_h), \quad \forall\, \phi \in H_0^1(\Omega),\ \phi_h \in V_h^0.$$

We know Q_h is H^1-stable (cf. Bramble-Xu [4]), i.e.

$$\|Q_h\phi\|_{H^1(\Omega)} \le C\, \|\phi\|_{H^1(\Omega)}, \quad \forall\, v \in H_0^1(\Omega).$$

Using this and the definition of z_h^n in (3.23), we obtain

$$\begin{aligned}
&\|\partial_\tau z_h^n\|_{H^{-1}(\Omega)} \\
&= \sup_{\substack{\phi\in H_0^1(\Omega)\\ \phi\neq 0}} \frac{(\partial_\tau z_h^n, \phi)}{\|\phi\|_{H^1(\Omega)}} = \sup_{\substack{\phi\in H_0^1(\Omega)\\ \phi\neq 0}} \frac{(\partial_\tau z_h^n, Q_h\phi)}{\|\phi\|_{H^1(\Omega)}} \\
&\le C\Big(\|\nabla z_h^{n-1}\|_{L^2(\Omega)} + \|\nabla(\bar u^n - u_h^n)\|_{L^2(\Omega)}\Big) \frac{\|\nabla Q_h\phi\|_{L^2(\Omega)}}{\|\phi\|_{H^1(\Omega)}} \\
&\le C\Big(\|\nabla z_h^{n-1}\|_{L^2(\Omega)} + \|\nabla(\bar u^n - u_h^n)\|_{L^2(\Omega)}\Big).
\end{aligned}$$

This with (3.24) gives (3.25). □

Proof of Theorem 3.3. Let

$$\mathcal{E}(h,\tau) = \sum_{n=1}^{M} \tau \|\nabla(\bar u^n - u_h^n)\|^2_{L^2(\Omega)}.$$

By taking $v_h = \tau(P_h\bar u^n - u_h^n) \in V_h^0$ in (3.23) and then summing the resultant equations over n, we get

$$\begin{aligned}
\mathcal{E}(h,\tau) &= \sum_{n=1}^{M} \tau(\nabla(\bar u^n - u_h^n), \nabla(\bar u^n - P_h\bar u^n)) \\
&\quad + \sum_{n=1}^{M} \tau(\nabla(\bar u^n - u_h^n), \nabla(P_h\bar u^n - u_h^n)) \\
&= \sum_{n=1}^{M} \tau(\nabla(\bar u^n - u_h^n), \nabla(\bar u^n - P_h\bar u^n)) \\
&\quad + \sum_{n=1}^{M} \tau(-\partial_\tau z_h^n, P_h\bar u^n - \bar u^n) \\
&\quad + \sum_{n=1}^{M} \tau(-\partial_\tau z_h^n, \bar u^n - u^n)
\end{aligned}$$

$$
\begin{aligned}
&+\sum_{n=1}^{M}\tau(-\partial_\tau z_h^n, u^n-u_h^n)\\
&+\sum_{n=1}^{M}\tau\Big[a(z_h^{n-1},\bar u^n)-a_h(z_h^{n-1},u_h^n)\Big]
\end{aligned}
$$

$$\text{(3.26)}\qquad \equiv: \text{(VI)}_1+\cdots+\text{(VI)}_5,$$

where we have used (3.16) for obtaining (VI)_5.

Similarly to the proof for L^2-error estimate, we have

$$
\begin{aligned}
\text{(IV)}_4+\text{(IV)}_5 \;=\; &(u_0-\Pi_h u_0, z_h^0)+\sum_{n=1}^{M}\tau(\bar f^n-f^n, z_h^{n-1})\\
&+\sum_{n=1}^{M}\tau\left[\langle \bar g^n, z_h^{n-1}\rangle-\langle \bar g_h^n, z_h^{n-1}\rangle_h\right]
\end{aligned}
$$

$$\text{(3.27)}\qquad \equiv: \text{(VII)}_1+\text{(VII)}_2+\text{(VII)}_3.$$

Next, we are going to analyse the terms (VI)_i and (VII)_i, $i=1,2,3$.

By Lemmas 2.1 and 3.3, we immediately derive

$$
\begin{aligned}
\Big|\text{(VII)}_1+\text{(VII)}_2\Big| &\le C\,(\tau+h|\log h|^{1/2})\,(\mathcal{E}(h,\tau))^{1/2}\\
&\times\Big[\|u_0\|_{H^1(\Omega)}^2+\int_0^T\|f_t\|_{L^2(\Omega)}^2dt\Big]^{1/2},
\end{aligned}
$$

while by Lemma 2.2, we obtain

$$
\begin{aligned}
\Big|\text{(VII)}_3\Big| &\le Ch^{3/2}\sum_{n=1}^{M}\tau\|\bar g^n\|_{H^2(\Gamma)}\|z_h^{n-1}\|_{H^1(\Omega)}\\
&\le Ch^{3/2}\,(\mathcal{E}(h,\tau))^{1/2}\Big[\int_0^T\|g\|_{H^2(\Gamma)}^2dt\Big]^{1/2}.
\end{aligned}
$$

For three terms (VI)_i, $i=1,2,3$, we have by (3.18)

$$\Big|\text{(VI)}_1\Big|\le C\,(\tau+h|\log h|^{1/2})\,(\mathcal{E}(h,\tau))^{1/2}\Big[\int_0^T\|u\|_X^2dt\Big]^{1/2},$$

and by Lemma 3.3,

$$
\begin{aligned}
\Big|\text{(VI)}_2\Big| &\le \sum_{n=1}^{M}\tau\|\partial_\tau z_h^n\|_{H^{-1}(\Omega)}\|P_h\bar u^n-\bar u^n\|_{H^1(\Omega)}\\
&\le Ch\,|\log h|^{1/2}\,(\mathcal{E}(h,\tau))^{1/2}\Big[\int_0^T\|u\|_X^2dt\Big]^{1/2},
\end{aligned}
$$

and

$$\Big|(\mathrm{VI})_3\Big| \le \sum_{n=1}^{M} \tau \|\partial_\tau z_h^n\|_{H^{-1}(\Omega)} \|\bar{u}^n - u^n\|_{H^1(\Omega)}$$
$$\le C\,\tau\,(\mathcal{E}(h,\tau))^{1/2} \Big[\int_0^T \|u_t\|_Y^2 dt\Big]^{1/2}.$$

Finally, by simple calculations we have

$$\|\nabla(u-u_{h,\tau})\|_{L^2(Q_T)} \le C\tau\Big[\int_0^T \|u_t\|_Y^2 dt\Big]^{1/2} + \Big[\sum_{n=1}^{M} \tau \|\nabla(\bar{u}^n - u_h^n)\|^2_{L^2(\Omega)}\Big]^{1/2},$$

then Theorem 3.3 follows immediately from this inequality, (3.26), (3.27) and the previous estimates for $(\mathrm{VI})_i$ and $(\mathrm{VII})_i$, $i = 1,2,3$. □

4. Concluding remarks

In this paper, we have proposed some finite element methods to solve the second order elliptic and parabolic interface problems in two dimensions. The nearly optimal error estimates both in energy-norm and L^2-norm are proved under reasonable regularity assumptions on the original solutions. In fact, the energy-norm error estimate can be made optimal (cf. Remark 2.4). We note that the methods proposed in the paper can be easily extended to treat the following more general interface problems of the self-adjoint elliptic type

$$-\sum_{i,j=1}^{2} \frac{\partial}{\partial x_i}\Big(a_{ij}(x)\frac{\partial u}{\partial x_j}\Big) + a(x)u + f(x) = 0$$

and of the self-adjoint parabolic type

$$\frac{\partial u}{\partial t} - \sum_{i,j=1}^{2} \frac{\partial}{\partial x_i}\Big(a_{ij}(x)\frac{\partial u}{\partial x_j}\Big) + a(x)u + f(x,t) = 0.$$

The correposponding interface conditions are

$$[u] = 0, \quad \Big[p\frac{du}{dN} + \sigma u\Big] = g \quad \text{across the interface } \Gamma.$$

Here the functions p and σ are known and depend on x in the elliptic case and x and t in the parabolic case. The symbol du/dN denotes $\sum_{ij} a_{ij}(\partial u/\partial x_j) \times \cos(\mathbf{n}, x_i)$. The classical solvability of the above interface problems was studied carefully in Ladyzhenskaya et al. [17].

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