Regularization and convergence for ill-posed backward evolution equations in Banach spaces

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Abstract

   This work is concerned with a mathematical study of ill-posed backward evolution equations associated with densely defined linear differential operators in Banach spaces. A general approach is presented to investigate the convergence and stability of a class of regularized solutions for ill-posed backward evolution equations associated with sectorial or half-strip operators. Generalized concepts of qualification pairs and index functions are introduced to characterize the explicit convergence rates of the concerned regularized solutions. Applications of our results to general backward evolution equations are also investigated.
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1. Introduction

For a terminal time T and a given data f at the time $t = T$, we shall consider the backward evolution equations of the form:

$$\begin{cases} u'(t) + Au(t) = 0, & 0 < t < T, \\ u(T) = f, \end{cases} \tag{1.1}$$

where X is an infinite dimensional reflexive Banach space with a norm $\| \cdot \|$, and $A : \mathcal{D}(A) \subset X \to X$ is a densely defined (unbounded) operator such that $-A$ generates a uniformly stable analytic semigroup $(e^{-tA})_{t \geq 0}$. Since $(e^{-tA})^{-1}$ for $t > 0$ is an unbounded operator for general practical problems, the backward evolution equation (1.1) is mostly ill-posed. More precisely, when the exact terminal data f is replaced by a noisy data f^δ with a noise level δ, namely

$$\| f^\delta - f \| \leq \delta, \tag{1.2}$$

the solutions of the system (1.1) may not exist, and even if the solutions exist, the solution may be far from the exact solution u. In this work we shall often write the exact initial value for the backward evolution equation (1.1) with noise-free data f as $x^\dagger = u(0)$.

A general methodology to overcome the ill-posedness of inverse problems is to seek approximate solutions by solving some "well-posed" neighboring problems; see [3,4,11,15,22,23,25, 29,33,34,37,39] for various inverse problems associated with the parabolic equation of the form $u'(t) + Au(t) = g$. The most popular methodology is to approximate an inverse problem by the output least-squares formulation with an appropriate Tikhonov regularization. The classical regularization theory [13,14] ensures the convergence or even the convergence rate of the regularized solution by a Tikhonov regularization under a general source condition. The classical regularization theory has been widely studied for different inverse problems associated with the equation of the form $u'(t) + Au(t) = g$, with A being a second order self-adjoint and coercive elliptic operator; see, e.g., [13,15,24,25,30,37,39] and the references therein. But the crucial source condition involved in the classical theory requires the existence of a small source function with certain desired regularities, and is rather restrictive in general. In fact, the source condition has still not been well verified for the backward evolution system (1.1), even when the operator A is self-adjoint and coercive. So the classical regularization theory is actually quite limited in its applicability. This is the main motivation of the current work, namely to explore another basic methodology for approximating the ill-posed backward evolution system (1.1). As we demonstrate in the rest of the paper, this regularization theory applies to a much more general class of operators than the self-adjoint and coercive ones. The main idea of this alternative strategy is to construct a family of regularized PDEs of the form

$$\begin{cases} v'_\alpha(t) + A_\alpha v_\alpha(t) = 0, & 0 < t < T, \\ v_\alpha(T) = f, \end{cases} \tag{1.3}$$

where $\{A_\alpha\}_{\alpha > 0}$ is a family of perturbations of A such that for each $\alpha > 0$, A_α generates a C_0-semigroup $(e^{tA_\alpha})_{t \geq 0}$. Then we use this semigroup to construct a family of regularizing operators

$$\left\{ Q_{\alpha,t} : \alpha > 0, t \in [0, T] \right\} \subset \mathcal{L}(X)$$

such that the family of regularized solutions $\{u_\alpha\}_{\alpha>0}$, given by $u_\alpha(t) := Q_{\alpha,t} f$ for $t \in [0, T]$, approximates the solution of (1.1) as $\alpha \to 0^+$. For the noisy terminal data f^δ, the corresponding regularized solutions are given by $u_{\alpha,\delta}(t) := Q_{\alpha,t} f^\delta$. One important method for stabilizing (1.1) is the method of quasi-reversibility [28], where A_α was taken to be $A - \alpha A^2$. When $-A$ generates a uniformly bounded analytic semigroup of the angle β with $\beta \in (0, \frac{\pi}{4})$, we may take $A_\alpha = A - \alpha A^b$ with $1 < b < \frac{\pi}{\pi - 2\beta}$ [22]. One can also consider $A_\alpha = A(1 + \alpha A)^{-1}$ [23,34]. On the other hand, a modified quasi-reversibility was proposed in [3], based on the perturbations $A_\alpha = -(pT)^{-1} \ln(\alpha + e^{-pTA})$ for $\alpha > 0$ and $p \geq 1$, where A is a self-adjoint and coercive unbounded linear operator in a Hilbert space X. These perturbations may result in better approximations, compared with others (see [3]). Although the convergence behavior of regularizing operators and their regularized solutions for ill-posed backward evolution equations have been widely studied, only special cases of perturbations were considered in [3,22,23,34]. On the other hand, the investigations in [3,29,33] rely heavily on functional calculus for self-adjoint operators in Hilbert spaces, whose corresponding generalizations in Banach spaces are rarely studied. Moreover, the results in many existing literature (see, e.g., [22,23,34]) are mostly qualitative instead of quantitative. That is, these results do not provide explicit convergence rates of the regularized solutions in terms of regularization parameter α as $\alpha \to 0^+$. Even when special cases were investigated, the requirements on the true solutions are quite restrictive (see, e.g., [3]). In this work, we propose a systematic approach to study the explicit convergence rate of the regularized solutions for ill-posed backward evolution equations associated with general sectorial or half-strip operators, which is a much more general class of differential operators than self-adjoint or coercive operators, and in general Banach spaces. We will also establish general connections between the a prior estimate of the exact initial data $u(0)$ of the backward evolution equations and the explicit convergence rate of the corresponding regularized solutions.

One of the major tools we shall adopt in our analysis is the well-established theory of the functional calculus for (unbounded) operators, which, however, are rarely applied in the study of regularizations for inverse problems of evolutional PDEs in Banach spaces. Roughly speaking, the functional calculus for a (possibly unbounded) operator B in a Banach space X enables us to associate a closed operator $f(B) : \mathcal{D}(f(B)) \subset X \to X$ to each function f from an algebra of functions defined on some domain of the complex plane. The theory of functional calculus has been utilized to investigate the forward problems (cf. [16] or references therein). One of the main issues for functional calculus is the boundedness, i.e., to verify for which operators B and functions f the resulting operators $f(B)$ are bounded. It is a very useful tool in understanding phenomena around the sectorial and strip operators, e.g., in determining the domain of fractional powers of a partial differential operator and connecting the regularity of parabolic evolution equations with certain estimates in control theory; see [17,26,27] for more details. As we shall demonstrate, the boundedness of H^∞-functional calculus is also a very effective tool in our subsequent characterization of explicit convergence rates of regularized solutions for ill-posed backward evolution equations associated with general sectorial or half-strip operators in Banach spaces.

Another important concept in our analysis is qualification and index functions. The qualification for a regularization method was introduced for the special case that the qualification is only a real number (see, e.g., [13, Chap. 4]). More recently, the qualification concept was generalized in [29] to index functions in the context of variable Hilbert scales in order to express prior smoothness of unknown solutions, and the interplay between the qualification of regularization method and associated index functions. It is worth mentioning that those concepts were essentially based on the spectral theory and functions of linear self-adjoint operators in Hilbert space.

To investigate the convergence rates of the Tikhonov-type regularization of ill-posed problems in our current Banach setting, one may use the so-called distance functions, which can help quantify the violation of a reference source condition [21]. We refer to [18,20,33] for some recent studies using these concepts. In this work, we will generalize the concepts of qualifications and index functions for ill-posed linear problems in the Banach setting when the driven operators are either of sectorial or half-strip type. We propose the effective concept of qualification pairs and index functions to characterize regularization methods and prior smoothness of the unknown initial values, respectively. The interplay between these two concepts will be investigated, and then applied to derive explicit convergence rates for regularized solutions of the backward evolution equation (1.1) in our general setting.

The remainder of this work is arranged as follows. In section 2 we present some necessary notations, definitions as well as a general framework of the abstract functional calculus. In section 3, we establish our major results for sectorial operators. Subsection 3.1 is devoted to some preliminary definitions, notations and results about the functional calculus of sectorial operators. Then a convergence result for the regularizing operators of the backward evolution equation (1.1) is derived in subsection 3.2. In subsection 3.3, we will introduce the concepts of qualification pairs and index functions for the regularization methods associated with sectorial operators, investigate their properties, and then use these properties to characterize the explicit convergence rates for regularized solutions. Finally, these results will be applied to a specific family of regularization methods for the backward evolution equation (1.1). Subsection 3.4 shall recall some important classes of sectorial operators for which our results are applicable. The primary concern of section 4 is for the case when A is a half-strip operator, whose definition and basic properties are discussed in subsection 4.1. The operator $\mathcal{A} := e^{-TA}$ and its functional calculus are the main subjects of subsection 4.2, and the half-strip operators in Hilbert spaces are further studied in subsection 4.3. In subsection 4.4 we will introduce qualification pairs and index functions for the regularization methods governed by $\mathcal{A}$, and establish explicit convergence results for those regularization methods. With the help of these results, we shall be able to investigate the explicit convergence rates for the families of regularized solutions for the backward evolution equations (1.1) in section 4.5. Finally, our new general results will be applied in section 4.6 to a specific family of regularizing operators for approximating the backward evolution equations.

2. Preliminaries

We shall now present some notations and definitions that are used in the subsequent sections. By $\mathcal{L}(X)$ we denote the Banach space of all linear bounded operators on X. The spectrum, point spectrum, domain and range for a general operator A are denoted by $\sigma(A)$, $P\sigma(A)$, $\mathcal{D}(A)$ and $\mathcal{R}(A)$, respectively. Its resolvent is $R(\lambda, A)$ for $\lambda \in \rho(A) := \mathbb{C} \backslash \sigma(A)$. $A' : \mathcal{D}(A') \subset X' \to X'$ stands for the adjoint operator of A, where X' is the dual space of X.

We shall frequently use the following four regions in complex planes:

$$St_\theta := \begin{cases} \{z \in \mathbb{C} \mid |\Im m\, z| < \theta\}, & \theta \in (0, \pi], \\ \mathbb{R}, & \theta = 0; \end{cases}$$

$$\Sigma_\theta := \{e^z \mid z \in St_\theta\} = \begin{cases} \{z \neq 0 \mid |\arg z| < \theta\}, & \theta \in (0, \pi], \\ (0, \infty), & \theta = 0; \end{cases}$$

$$\Sigma_{\theta,b} := \Sigma_\theta \cap \{z \in \mathbb{C} \mid |z| < b\}, \quad \theta \in (0, \pi], \, b > 0;$$

$$H_{a,\theta} := \begin{cases} \{z \in \mathbb{C} \mid \Re e\, z > a \text{ and } |\Im m\, z| < \theta\} & \text{if } a, \theta > 0, \\ (a, \infty) & \text{if } a > 0, \theta = 0. \end{cases}$$

Obviously, St_θ and Σ_θ are respectively a horizontal strip of width 2θ and a horizontal sector of angle 2θ, while $\Sigma_{\theta,b}$ and $H_{a,\theta}$ are the intersection between Σ_θ and a circle centered at origin with radius b, and the intersection between a horizontal strip St_θ and an open half-plane $\{z \in \mathbb{C} \mid \Re e\, z > a\}$ respectively. For a domain $\mathcal{O} \subset \mathbb{C}$, we denote by $H(\mathcal{O})$ the algebra of holomorphic functions on $\mathcal{O}$, and by $H^\infty(\mathcal{O})$ the subalgebra of $H(\mathcal{O})$ consisting of all bounded holomorphic functions. For a complex function f over a domain $\mathcal{O}$, $\|f\|_\infty$ represents the supremum norm of f over $\mathcal{O}$. A complex function is always viewed to take its value at the principal branch. For a complex function $f \in H(\mathcal{O})$ and a subset $U \subset \mathcal{O}$, we write $f|_U$ for the restriction of f on U.

Our study and analysis in all the sections that follow involve several different classes of operators, including sectorial and half-strip operators. For the sake of clarity and brevity, we present a general framework to describe the various operations of operators, i.e., the common idea shared by many functional calculus constructions. Suppose that we are given an operator A on X and an algebra $\mathcal{E}$ consisting of complex functions defined on a region containing $\sigma(A)$, and a homomorphism

$$\Phi : \mathcal{E} \to \mathcal{L}(X),$$

then we can define $f(A) := \Phi(f)$ for each $f \in \mathcal{E}$. The constructions of homomorphisms in practical problems are mostly based on Cauchy-type integrals. Assume further that we are given a commutative algebra $\mathcal{M}$ containing $\mathcal{E}$ and $\mathbf{1}$, then we say that the triple $(\mathcal{M}, \mathcal{E}, \Phi)$ is an *(abstract) functional calculus*. A function e in $\mathcal{E}$ with $e(A)$ being injective is called a *regularizer*. The collection of all regularizers is denoted by $\mathrm{Reg}(\mathcal{E})$. For any $f \in \mathcal{M}$, if there is a regularizer e such that $ef \in \mathcal{E}$, then we say that f is *regularizable*, and e is a regularizer of f. If $\mathrm{Reg}(\mathcal{E})$ is not empty and e is a regularizer of $f \in \mathcal{M}$, then we can define

$$f(A) := e(A)^{-1}(ef)(A) = \Phi(e)^{-1}\Phi(ef).$$

One can further show that this is well-defined, i.e., $f(A)$ does not depend on the choice of regularizers e. In this case, we say that $(\mathcal{M}, \mathcal{E}, \Phi)$ is *proper*. The set of all regularizable functions in $\mathcal{M}$ are called the *domain* of the triple $(\mathcal{M}, \mathcal{E}, \Phi)$, and denoted by $\mathcal{M}_A$. We next collect some important properties of a proper functional calculus.

Theorem 2.1 *([17, proposition 1.2.2, corollary 1.2.3.]). Let $(\mathcal{M}, \mathcal{E}, \Phi)$ be a proper abstract functional calculus over the Banach space X with a domain $\mathcal{M}_A$. Let $e \in \mathcal{E}$ and $f, g \in \mathcal{M}_A$. Then the following assertions hold.*

(a) *If $T \in \mathcal{L}(X)$ commutes with each $\Phi(e)$, then it commutes with $\Phi(h)$ for every $h \in \mathcal{M}_A$.*
(b) *One has $1 \in \mathcal{M}_A$ and $\Phi(\mathbf{1}) = I$.*
(c) *The inclusions*

$$\Phi(f) + \Phi(g) \subset \Phi(f + g), \quad \Phi(f)\Phi(g) \subset \Phi(fg)$$

hold with $\mathcal{D}(\Phi(f)\Phi(g)) = \mathcal{D}(\Phi(fg)) \cap \mathcal{D}(\Phi(g))$. These inclusions are actually identities for $\Phi(g) \in \mathcal{L}(X)$.

For an abstract functional calculus $(\mathcal{M}, \mathcal{E}, \Phi)$ of operator A with a domain $\mathcal{M}_A$, we may assign a family of operators $\{f_\alpha(A)\}_{\alpha \in I}$ for a family of functions $\{f_\alpha\}_{\alpha \in I}$ in $\mathcal{M}_A$, where $I = (0, \infty)$ or $I = (0, a]$ for some $a > 0$. Then one can define a regularized problem for each $\alpha \in I$:

$$\begin{cases} v'_\alpha(t) + f_\alpha(A)v_\alpha(t) = 0, & 0 < t < T, \\ v_\alpha(T) = f. \end{cases} \tag{2.4}$$

Based on certain construction methods and $\{f_\alpha\}_{\alpha \in I}$, we will express the family of regularizing operators $\{Q_{\alpha,t} : \alpha \in I, t \in [0, T]\}$ in terms of $\{q_{\alpha,t}(A) : \alpha \in I, t \in [0, T]\}$, where $\{q_{\alpha,t} : \alpha \in I, t \in [0, T]\}$ is a family of functions in $\mathcal{M}_A$. In subsequent sections we will use the concept of qualification pairs to characterize the convergence behavior of the regularizing operators and their regularized solutions. On the other hand, some subspaces X_φ of X can be given through an appropriate function $\varphi \in \mathcal{M}_A$, which will be called an index function. If $u(0) \in X_\varphi$, then the study of the convergence behavior of $q_{\alpha,t}(A)f^\delta$ for $\alpha, \delta \to 0^+$ can be reduced to that of the interplay between qualification pairs and the function φ. Actually, the resulting convergence rates will be written in terms of qualification pairs and index functions explicitly. As it will be shown later, the qualification pair depends on two ingredients: the construction method of regularizations, the perturbation functions $\{f_\alpha\}_{\alpha \in I}$, while the condition $u(0) \in X_\varphi$ indicates the smoothness of $u(0)$. Since all these facts are related to the operator A, we shall pay attention to its classification and split the rest of the work into two parts. In section 3, we investigate the case when A is a sectorial operator that is widely used in the theory of PDEs and covers a large group of differential operators. Then in section 4 we will introduce the concept of half-strip operators. For these operators, we shall present another basic construction method that can help us improve the convergence rates of the regularized solutions essentially.

3. Convergence of regularizations for sectorial operators

3.1. Functional calculus for sectorial operators

A closed operator A is said to be a sectorial operator with angle $\theta \in [0, \pi)$ (in short, $A \in \text{Sect}(\theta)$) if $\sigma(A) \subset \overline{\Sigma_\theta}$, and for each $\omega \in (\theta, \pi)$,

$$\sup\left\{ |\lambda| \|R(\lambda, A)\| \mid \lambda \in \mathbb{C} \backslash \Sigma_\omega \right\} < \infty.$$

There is a natural holomorphic functional calculus associated with this sectorial operator A. One can use the Cauchy formula to define an algebra homomorphism by setting

$$\Phi(f) := \frac{1}{2\pi i} \int_\Gamma f(z) R(z, A)\,dz, \tag{3.5}$$

for a holomorphic function f from the algebra $H_0^\infty(\Sigma_\omega)$, which is defined by

$$H_0^\infty(\Sigma_\omega) := \left\{ f \in H^\infty(\Sigma_\omega) \mid |f(z)| \leq C \min\{|z|^\alpha, |z|^{-\alpha}\} \ \forall\, z \in \Sigma_\omega \ \text{for some } \alpha > 0, \ C \geq 0 \right\}$$

for $\theta < \omega < \pi$. The contour Γ in (3.5) is the positively oriented boundary of a smaller sector $\Sigma_{\omega'}$ with $\omega' \in (\theta, \omega)$, and it is easy to see that this definition is independent of the actual choice of $\omega' \in (\theta, \omega)$, using the standard argument involving the Cauchy's theorem. If we define

$$\mathcal{E}(\Sigma_\omega) = H_0^\infty(\Sigma_\omega) \oplus \langle \frac{1}{1+z} \rangle \oplus \langle \mathbf{1} \rangle,$$

and set $\Phi((1+z)^{-1}) = -R(-1, A)$ and $\Phi(\mathbf{1}) = I$, then $(H(\Sigma_\omega), \mathcal{E}(\Sigma_\omega), \Phi)$ is an abstract functional calculus. Moreover, if A is injective, then the function $e := z(1+z)^{-2}$ is a regularizer (cf. [17, lemma 2.3.6]). Following section 2, we denote by $\mathcal{M}_A$ the domain of the triple $(H(\Sigma_\omega), \mathcal{E}(\Sigma_\omega), \Phi)$, with $\mathcal{M} = H(\Sigma_\omega)$, and extend the homomorphism to $\mathcal{M}_A$. We remark that $\mathcal{M}_A(\Sigma_\omega)$ here contains the collection of all functions f that are regularized by powers of e, i.e., $e^n(z)f(z) \in H_0^\infty(\Sigma_\omega)$ for some $n \in \mathbb{N}$, and this collection will be denoted by $\mathcal{B}(\Sigma_\omega)$.

Next we collect some basic results about the fractional powers of sectorial operators (cf., e.g., [17, proposition 3.1.1 and corollary 3.1.5]).

Proposition 3.2. *Let* $A : \mathcal{D}(A) \subset X \to X$ *be a sectorial operator with angle* $\theta \in (0, \pi)$, *then*

(1) *(First law of exponents) For each* $\beta_1, \beta_2 > 0$, $A^{\beta_1} A^{\beta_2} = A^{\beta_1 + \beta_2}$.
(2) *(Second law of exponents) For* $\beta_1 \in (0, \pi/\theta)$ *and* $\beta_2 > 0$, $(A^{\beta_1})^{\beta_2} = A^{\beta_1 \beta_2}$.

Of particular importance in the theory of functional calculus is the so-called convergence lemma, which is crucial to our subsequent analysis (cf. e.g. [17, proposition 5.1.4.]).

Lemma 3.3. *Let* $A \in \mathrm{Sect}(\theta)$ *and* (f_α) *a net of functions from* $H^\infty(\Sigma_\omega)$ *with* $\omega \in (\theta, \pi)$. *Suppose* $\sup_\alpha \|f_\alpha\|_\infty < \infty$ *and that the limit* $f(z) = \lim_\alpha f_\alpha(z)$ *exists pointwise on* Σ_ω. *Then*

$$f_\alpha(A)x \to f(A)x \quad \forall x \in \mathcal{D}(A) \cap \mathcal{R}(A).$$

Moreover, the following assertions hold

(a) *If* A *is injective,* $f_\alpha(A) \in \mathcal{L}(X)$ *for all* α, *and* $f_\alpha(A) \to T \in \mathcal{L}(X)$ *strongly, then* $f(A) = T$.
(b) *If* A *is densely defined with a dense range, and* $\sup_\alpha \|f_\alpha(A)\| < \infty$, *then* $f(A) \in \mathcal{L}(X)$ *and*
   $f_\alpha(A) \to f(A)$ *strongly.*

In subsection 4.2 we shall deal with functional calculus for bounded sectorial operators, hence we present the functional calculus and its properties below. For a bounded sectorial operator A such that $\sigma(A) \subset \overline{\Sigma_{\theta,a}}$ for some $\theta \in (0, \pi)$ and $a > 0$ (it is written as $A \in \mathrm{Sect}(\theta, a)$), it is clear that the behavior of f at infinity is irrelevant in order for the expression $f(A)$ to make sense. Let us define the function space

$$H_0^\infty(\Sigma_{\omega,b}) := \left\{ f \in H^\infty(\Sigma_{\omega,b}) \mid |f(z)| \leq C|z|^\alpha \ \forall z \in \Sigma_{\omega,b} \text{ for some } \alpha > 0, \ C \geq 0 \right\}$$

for $\theta < \omega < \pi$ and $b > a$. If $A \in \mathrm{Sect}(\theta, a)$ and $f \in H_0^\infty(\Sigma_{\omega,b})$, then we define

$$f(A) = \Phi(A) := \frac{1}{2\pi i} \int_\Gamma f(z) R(z, A) dz,$$

where Γ is the positively oriented boundary of the truncated sector $\Sigma_{\omega',a'}$ with $\omega' \in (\theta, \omega)$ and $a' \in (a, b)$. We see that this definition does not depend on the choices of $\omega' \in (\theta, \omega)$ and

$a' \in (a, b)$. Further, we can check that $(H(\Sigma_{\omega,b}), \mathcal{E}(\Sigma_{\omega,b}), \Phi)$ is a proper functional calculus for $\mathcal{E}(\Sigma_{\omega,b}) = H_0^\infty(\Sigma_{\omega,b})$ if A is injective. If A is not injective, then we set $\mathcal{E}(\Sigma_{\omega,b}) = H_0^\infty(\Sigma_{\omega,b}) \oplus \langle \frac{1}{1+z} \rangle$ and define the primary calculus by $\Phi(f + \frac{c}{1+z}) = \Phi(f) - cR(-1, A)$ for every $f \in H_0^\infty(\Sigma_{\omega,b})$ and $c \in \mathbb{C}$. In either case, the triple $(H(\Sigma_{\omega,b}), \mathcal{E}(\Sigma_{\omega,b}), \Phi)$ is a proper functional calculus (cf. [17, Section 2.5] for details). As usual, we write the collection of all regularizable functions in $H(\Sigma_{\omega,b})$ as $\mathcal{M}_A(\Sigma_{\omega,b})$.

3.2. Existence and convergence of regularizing operators

In this whole section 3, we establish all our convergence results for the regularizations of sectorial operators, and investigate the convergence for half-strip operators in Section 4. We start with an exact classification of all sectorial operators A of our interest.

(**H1**) A is an injective and densely defined sectorial operator with angle $0 \leq \theta < \frac{\pi}{2}$, and has a bounded H^∞-calculus over $\Sigma_{\theta*}$ for some $\frac{\pi}{2} > \theta^* > \theta$, i.e., there exists a constant $C_s \geq 1$ such that

$$\| f(A) \| \leq C_s \| f \|_\infty \qquad \forall f \in H^\infty(\Sigma_{\theta*}).$$

One can easily see that for such sectorial operator A, the triple $(H(\Sigma_{\theta*}), \mathcal{E}(\Sigma_{\theta*}), \Phi)$ as defined in subsection 3.1 is a proper abstract functional calculus with some domain $\mathcal{M}_A(\Sigma_{\theta_A})$ (cf. e.g. [17]). For the sake of brevity, we write

$$\mathcal{M}_A = \mathcal{M}_A(\Sigma_{\theta*})$$

throughout the rest of this section if there is no confusion caused.

We next introduce an important concept that is frequently used in our subsequent analysis.

Definition 3.4. A function $\varphi \in \mathcal{M}_A$ is called an index function if $\varphi(A)$ is bounded and injective.

Obviously, $\varphi(A)^{-1}$ is well defined in the range of $\varphi(A)$, usually unbounded and closed. Thus, we can assign a norm

$$\| x \|_\varphi := \| \varphi(A)^{-1} x \| \qquad \forall x \in \mathcal{D}(\varphi(A)^{-1}). \tag{3.6}$$

The completion of $\mathcal{D}(\varphi(A)^{-1})$ under this norm is denoted by X_φ. As $\varphi(A)^{-1}$ is a closed operator, X_φ is complete with the graph norm induced by $\varphi(A)^{-1}$. Since $\varphi(A)$ is bounded, we know that the norm (3.6) is equivalent to the graph norm. Thus we have $x \in X_\varphi$ if and only if

$$x = \varphi(A) y$$

for some $y \in X$. A simple case of the index function is $\varphi(z) = (1 + z)^{-\beta}$ with $\beta > 0$, which induces the fractional power space (see, e.g., [17, Proposition 3.1.9.]).

In this work we shall follow the following definition of regularizing operators that adapts the standard concept on regularizing operators for our backward evolution equation (1.1) (cf. [30, 35]).

Definition 3.5. Let I be an index set with $0 \in \overline{I}$. A set $\{Q_{\alpha,t} : \alpha \in I, t \in [0, T]\} \subset \mathcal{L}(X)$ is called a family of regularizing operators for the backward evolution equation (1.1) if for each solution $u(t)$ with a data f at the terminal time, and for each $\delta > 0$, there exists an a priori regularization parameter choice $\alpha(\delta)$ such that $\alpha(\delta) \to 0$ as $\delta \to 0$, and $\|Q_{\alpha(\delta),t} f^\delta - u(t)\| \to 0$ as $\delta \to 0$ for each $t \in [0, T]$ whenever f^δ satisfies (1.2).

There are different approaches to construct regularizing operators, such as quasi-reversibility and modified quasi-reversibility methods [3,28]. In this section we shall focus only on regularizing operators that are constructed by applying the quasi-reversibility method [28] for the backward evolution equation (1.1). In view of assumption (**H1**), we can assign a family of closed operators $\{f_\alpha(A)\}$ for each family of functions $\{f_\alpha\}_{\alpha \in I}$ in $\mathcal{M}_A$, with $I := (0, \infty)$ or $I := (0, a]$ for some $a > 0$. For each $\alpha \in I$, we approximate the backward evolution equation (1.1) by

$$\begin{cases} u'(t) + f_\alpha(A)u(t) = 0, & t \in [0, T), \\ u(T) = f^\delta. \end{cases} \tag{3.7}$$

It is well-known that if $f_\alpha(A)$ generates a C_0-semigroup $(\exp(t f_\alpha(A)))_{t \geq 0}$, then the (mild) solution for (3.7) can be rewritten as

$$u_{\alpha,\delta}(t) = R_{\alpha,t} f^\delta := \exp((T - t) f_\alpha(A)) f^\delta \tag{3.8}$$

(see, e.g., [12, chapter 2, section 6] or [31, section 4.1]). We will investigate for what family of operators $\{f_\alpha\}_{\alpha \in I_s}$ the resulting operators $\{R_{\alpha,t} : \alpha \in I, \ t \in [0, T]\}$ is a family of regularizing operators. We first present a sufficient condition to ensure function (3.8) is well-defined, and also a (mild) solution to (3.7) (cf. [2, Proposition 2.5]).

Lemma 3.6. *For any $f \in \mathcal{M}_A$ satisfying*

$$\mathfrak{Re}(f(z)) \leq \omega_0 \quad \forall z \in \Sigma_{\theta*} \tag{3.9}$$

with some $\omega_0 \in \mathbb{R}$, $(\exp(t f(A)))_{t \geq 0}$ is a C_0-semigroup with generator $f(A)$, and $\|\exp(t f(A))\| \leq C_s e^{t \omega_0}$ for $t \geq 0$.

Theorem 3.7. *Let $\{f_\alpha\}_{\alpha \in I}$ be a family of functions in $\mathcal{M}_A$, and we assume*
(1) $f_\alpha(z) \to z \ \forall z \in \Sigma_{\theta*}$ as $\alpha \to +0$.
(2) There exists a constant $M > 0$ such that

$$\sup_{\alpha \in I_s} \sup_{z \in \Sigma_{\theta*}} \mathfrak{Re}(f_\alpha(z) - z) \leq M.$$

(3) There exists a continuous and decreasing function $\omega : I \to (0, +\infty)$ such that $\omega(\alpha) \to +\infty$ as $\alpha \to +0$, and

$$\sup_{z \in \Sigma_{\theta*}} \mathfrak{Re} f_\alpha(z) \leq \omega(\alpha) \quad \alpha \in I.$$

Then $R_{\alpha,t} := \exp((T - t) f_\alpha(A))$ for $\alpha \in I$, $t \in [0, T]$ is a family of regularizing operators for the backward evolution equation (1.1).

Proof. It follows immediately from Lemma 3.6 that

$$\| \exp(t f_\alpha(A)) \| \leq C_s e^{t\omega(\alpha)} \quad \forall t \geq 0. \tag{3.10}$$

Let u be the solution to the backward evolution equation (1.1), then we know that for all $\alpha \in I$,

$$\| u(T) - R_{\alpha,T} f^\delta \| = \| f - f^\delta \| \to 0 \quad \text{as} \quad \delta \to +0.$$

For a fixed $t \in [0, T)$, we have

$$\| u(t) - R_{\alpha,t} f^\delta \| \leq \| R_{\alpha,t} f^\delta - R_{\alpha,t} f \| + \| R_{\alpha,t} f - u(t) \| := (\mathbf{I})_1 + (\mathbf{I})_2.$$

By (3.10), we see that

$$(\mathbf{I})_1 \leq \| \exp((T-t) f_\alpha(A)) \| \| f^\delta - f \| \leq C_s e^{(T-t)\omega(\alpha)} \delta.$$

As $f = e^{-AT} x^\dagger$, it follows from the semigroup property of $(e^{-tA})_{t\geq 0}$ and Theorem 2.1 (c) that

$$(\mathbf{I})_2 = \| (\exp((T-t)(f_\alpha(A) - A)) - I) e^{-tA} x^\dagger \|.$$

The condition **(2)** and assumption **(H1)** imply that

$$\sup_{\alpha > 0} \| \exp((T-t)(f_\alpha(\cdot) - \cdot)) \|_\infty \leq C_s e^{M_s(T-t)},$$

which, together with the convergence in Lemma 3.3, yields that

$$(\mathbf{I})_2 = \| \exp((T-t)(f_\alpha(A) - A)) e^{-tA} x^\dagger - e^{-tA} x^\dagger \| \to 0^+ \quad \text{as} \quad \alpha \to +0. \tag{3.11}$$

Now if we choose

$$\alpha(\delta) = \omega^{-1}\left(\frac{1}{T} \ln\left(\frac{1}{\sqrt{\delta}}\right)\right), \quad 0 < \delta < 1,$$

then it is easy to see that $\alpha(\delta) \to 0$ as $\delta \to +0$, and

$$(\mathbf{I})_1 \leq C_s \sqrt{\delta} \to 0 \quad \text{as} \quad \delta \to +0. \tag{3.12}$$

Therefore, the combination of the estimate (3.11) with (3.12) yields

$$\| R_{\alpha,t} f^\delta - u(t) \| \to 0 \quad \text{as} \quad \delta \to +0. \quad \square$$

3.3. Explicit convergence rates of regularized solutions

The goal of this subsection is to establish the explicit convergence rates of regularized solutions to the backward evolution equation (1.1). We shall develop a systematic approach for the study within the framework of the functional calculus driven by an operator A. More precisely, we are interested in the convergence behaviors of some specific family of operators $\{g_\alpha(A)\}_{\alpha \in I}$, which could be seen as approximations of the identity operator I in X, where $\{g_\alpha\}_{\alpha \in I}$ is a collection of functions in $\mathcal{M}_A$.

The concepts of qualification and index function for general regularization methods governed by standard self-adjoint and coercive operators in Hilbert spaces have been widely studied, which appear to be very effective to help characterize the convergence rate for linear ill-posed problems (see e.g. [29]). These concepts can not apply to the study of linear ill-posed problems in Banach spaces driven by sectorial or half-strip operators, whose studies are clearly more difficult and technical than self-adjoint and coercive operators in Hilbert spaces. For the purpose, we propose the much more general new concepts of qualification pairs and index functions. These concepts can be seen as the generalizations of the existing ones, but there are essential differences due to the obvious distinction between self-adjoint and sectorial operators.

Definition 3.8. A pair of functions $(\rho_c, \rho_r) \in \mathcal{M}_A \times C(\mathbb{R}^+)$ is called a qualification pair if $\rho_c(z) \neq 0$ for each $z \in \Sigma_{\theta*}$, and $\rho_r : (0, +\infty) \to \mathbb{R}^+$ is a decreasing function satisfying $\rho_r(t) \to 0$ as $t \to +\infty$.

Definition 3.9. A family of functions $\{g_\alpha\}_{\alpha \in I}$ in $\mathcal{M}_A$ is called a regularization associated with a qualification pair (ρ_c, ρ_r) if there exists a positive constant R_1 such that

$$\sup_{z \in \Sigma_{\theta*}} |1 - g_\alpha(z)| \leq R_1 \quad \forall \alpha \in I, \tag{3.13}$$

$$\sup_{z \in \Sigma_{\theta*}} |(1 - g_\alpha(z))\rho_c(z)| \leq R_1 \rho_r(\alpha^{-1}) \quad \forall \alpha \in I. \tag{3.14}$$

Next, we show that $\{g_\alpha(A)\}_{\alpha \in I}$ is indeed a family of approximations of the identity on X.

Proposition 3.10. *If $\{g_\alpha\}_{\alpha \in I}$ is a regularization associated with a qualification pair (ρ_c, ρ_r), then $g_\alpha(A) \to I$ strongly as $\alpha \to 0^+$.*

Proof. For a fixed $z_0 \in \Sigma_{\theta*}$, we can see that

$$1 - g_\alpha(z_0) = \big((1 - g(z_0))\rho_c(z_0)\big) \cdot \frac{1}{\rho_c(z_0)} \leq \frac{1}{\rho_c(z_0)} \sup_{z \in \Sigma_{\theta*}} |(1 - g_\alpha(z))\rho_c(z)|.$$

Then it follows from (3.14) and the decay property of ρ_r that $\lim_{\alpha \to 0^+} (g_\alpha(z_0) - 1) = 0$. In view of (3.13), Lemma 3.3(**b**) is now applicable, then the desired result follows immediately. $\square$

Example 3.11. We present two examples of regularizations associated with some associated qualification pairs. Let $g_\alpha(z) = e^{-\alpha z^b}$ for $\alpha \in (0, +\infty)$ with $b > 1$ and $\omega \in (0, \frac{\pi}{2b})$, then we can claim that $\{g_\alpha\}_{\alpha > 0}$ is a regularization with qualification pair $(1/z^\beta, 1/\alpha)$ for any $0 < \beta \leq b$. To see this, it suffices to show that

$$\sup_{z \in \Sigma_\omega} |\frac{e^{-\alpha z^b} - 1}{\alpha z^b}| < \infty \quad \forall \alpha > 0. \tag{3.15}$$

Setting $\lambda = \alpha z^b$, we get $\lambda \in \Sigma_{b\omega}$ with $b\omega < \frac{\pi}{2}$. If $|\lambda| \le R$ for some prescribed bound $R > 0$, then there exists a constant $C > 0$ such that

$$|e^{-\lambda} - 1| < C|\lambda|.$$

On the other hand, it is easy to check that

$$\lim_{|\lambda| \to \infty,\, \lambda \in \Sigma_{b\omega}} \frac{|e^{-\lambda} - 1|}{|\lambda|} = 0.$$

Thus the desired inequality (3.15) is true, so is our claim. A second example is the family given by $\{g_\alpha(z)\}_{\alpha > 0} := \{e^{-\frac{\alpha z^2}{1 + \alpha z}}\}_{\alpha > 0}$. In a quite similar way, we can verify that this is a regularization with qualification pair $(1/z^2, 1/\alpha)$, provided that $\omega \in (0, \frac{\pi}{4})$.

From Proposition 3.10 we note that $\lim_{\alpha \to 0} \|g_\alpha(A)x - x\| = 0$ for any $x \in X$. But, the decay rate of the error $\|g_\alpha(A)x - x\|$ as $\alpha \to +0$ depends largely on the choice of $x \in X$, and the decay can be arbitrarily slow. Next, we shall analyze the decay rate exploiting a prior smoothness condition on x in terms of index functions.

Definition 3.12. An index function φ is said to be *proper* if there exists some $c_\varphi > 0$ such that

$$|\varphi(z)| \le c_\varphi\, \varphi(|z|) \quad \forall z \in \Sigma_\omega, \tag{3.16}$$

and $\varphi|_{\mathbb{R}^+}$ is a non-negative, real-valued and decreasing function satisfying $\varphi(|z|) \to 0$ as $|z| \to \infty$.

Definition 3.13. The qualification pair (ρ_c, ρ_r) is said to cover a proper index function φ if there exists a constant $c' > 0$ such that

$$\sup_{z \in \Sigma_{\theta^*, \alpha^{-1}}} \left| \frac{\varphi(z)}{\rho_c(z)} \right| \le c'\, \frac{\varphi(\alpha^{-1})}{\rho_r(\alpha^{-1})} \quad \forall \alpha \in I. \tag{3.17}$$

Then we say that φ is covered by (ρ_c, ρ_r) with constant $c' > 0$.

Proposition 3.14. *Let $\{g_\alpha\}_{\alpha \in I}$ be a regularization associated with the qualification pair (ρ_c, ρ_r), and φ a proper index function covered by (ρ_c, ρ_r) with constant $c' > 0$, then it holds that*

$$\|(1 - g_\alpha(A))x\| \le C_{\varphi,s}\, \varphi(\alpha^{-1})\|x\|_\varphi \quad \forall x \in X_\varphi, \tag{3.18}$$

with $C_{\varphi,s} := C_s R_1(c' + c_\varphi)$, where the constants C_s, R_1, c_φ and c' are from (H1), (3.13), (3.16) and (3.17), respectively.

Proof. Fixing $\alpha \in I_s$, we study the function $z \to (1 - g_\alpha(z))\varphi(z)$. From (3.13), (3.16) and the monotonicity of $\varphi|_{\mathbb{R}^+}$ it follows that

$$\sup_{z \in \Sigma_{\theta^*} \setminus \Sigma_{\theta^*, \alpha^{-1}}} |(1 - g_\alpha(z))\varphi(z)| \leq R_1 c_\varphi \varphi(\alpha^{-1}). \tag{3.19}$$

It remains to get an estimate over $\Sigma_{\theta^*, \alpha^{-1}}$. Using (3.17), we obtain

$$\sup_{z \in \Sigma_{\theta^*, \alpha^{-1}}} |(1 - g_\alpha(z))\varphi(z)|$$

$$= \sup_{z \in \Sigma_{\theta^*, \alpha^{-1}}} \left| (1 - g_\alpha(z))\rho_c(z)\frac{\varphi(z)}{\rho_c(z)} \right|$$

$$\leq \sup_{z \in \Sigma_{\theta^*, \alpha^{-1}}} |(1 - g_\alpha(z))\rho_c(z)| \sup_{z \in \Sigma_{\theta^*, \alpha^{-1}}} \left| \frac{\varphi(z)}{\rho_c(z)} \right|$$

$$\leq R_1 c' \rho_r(\alpha^{-1}) \frac{\varphi(\alpha^{-1})}{\rho_r(\alpha^{-1})} = R_1 c' \varphi_c(\alpha^{-1}).$$

Combining this with (3.19) yields

$$\sup_{z \in \Sigma_{\theta^*}} |(1 - g_\alpha(z))\varphi(z)| \leq R_1 (c' + c_\varphi)\varphi(\alpha^{-1}). \tag{3.20}$$

Given $x \in X_\varphi$, there exists $y \in X$ such that $x = \varphi(A)y$, and $\|y\| = \|x\|_\varphi$. Then it follows directly from (3.20), **(H1)** and Theorem 2.1 (c) that

$$\|(I - g_\alpha(A))x\| = \|(I - g_\alpha(A))\varphi(A)y\|$$

$$\leq \|(I - g_\alpha(A))\varphi(A)\| \|y\|$$

$$\leq C_s \sup_{z \in \Sigma_{\theta^*}} |(1 - g_\alpha(z))\varphi(z)| \|y\|$$

$$\leq C_{\varphi,s} \varphi(\alpha^{-1}) \|x\|_\varphi. \quad \square$$

Theorem 3.15. *Let $\{f_\alpha\}_{\alpha \in I}$ be a family of functions in $\mathcal{M}_A$ satisfying the following conditions:*
(i) *There exist constants $C_f > 0$ and $p > 0$ such that*

$$\sup_{z \in \Sigma_{\theta^*}} \mathfrak{Re}\, f_\alpha(z) \leq \frac{C_f}{\alpha^p} \quad \forall \alpha \in I.$$

(ii) *The inclusion*

$$\{z - f_\alpha(z) \mid z \in \Sigma_{\theta^*}\} \subset \overline{\Sigma_{\omega'}} \quad \forall \alpha \in I,$$

and the inequality

$$\sup_{z \in \Sigma_{\theta *}} |(z - f_\alpha(z))\rho_c(z)| \le R_g \rho_r(\alpha^{-1}) \quad \forall \alpha \in I$$

hold for some $0 < \omega' < \frac{\pi}{2}$, $R_g > 0$ and a qualification pair (ρ_c, ρ_r).

(iii) *There exists a proper index function φ covered by (ρ_c, ρ_r).*

Then for each $t \in (0, T]$, $\{e^{(f_\alpha(z)-z)t}\}_{\alpha \in I}$ is a regularization associated with the qualification pair $(\rho_c, c_0 t \rho_r)$ for some constant $c_0 > 0$. And for given $x^\dagger = u(0) \in X_\varphi$ satisfying $\|x^\dagger\|_\varphi \le Q$, there exists some constant $C_{\mathcal{M}} > 0$ such that

$$\|R_{\alpha, t} f^\delta - u(t)\| \le C_{\mathcal{M}} \left(\delta\, e^{\frac{C_f(T-t)}{\alpha^p}} + (T-t)\, Q\, \varphi(\alpha^{-1}) \right) \quad \forall \alpha \in I,\ t \in [0, T]. \tag{3.21}$$

Furthermore, under the a priori parameter choice

$$\alpha(\delta) = \left[\frac{1}{\log(1/\delta)} \right]^\kappa \quad \text{with} \quad 0 < \kappa < \frac{1}{p\, C_f\, (T-t)}, \tag{3.22}$$

the following estimate of the convergence rate holds for $\eta := 1 - \kappa\, p\, C_f\, (T-t)$,

$$\|R_{\alpha(\delta), t} f^\delta - u(t)\| = \mathcal{O}\left(\delta^\eta + (T-t)\, Q\, \varphi\left(\left[\frac{1}{\log(1/\delta)} \right]^{-\kappa} \right) \right) \quad \text{as} \quad \delta \to 0. \tag{3.23}$$

Proof. Noting that function $h_\alpha(z) := z - f_\alpha(z)$ maps the sector $\Sigma_{\theta *}$ into $\overline{\Sigma_{\omega'}}$ for all $\alpha \in I$, we have $\sup_{z \in \Sigma_{\theta *}} |e^{-t h_\alpha(z)}| \le 1$ for all $\alpha \in I$ and $t \ge 0$. Furthermore, we can verify from the reasoning in Example 3.11 that $|e^{-\lambda} - 1|/|\lambda| \le C_\Lambda$ for all $\lambda \in \Sigma_{\omega'}$. Then, using condition **(ii)** we obtain

$$\sup_{z \in \Sigma_{\theta *}} \left(|e^{-t h_\alpha(z)} - 1||\rho_c(z)| \right) \le C_\Lambda t \sup_{z \in \Sigma_{\theta *}} |(z - f_\alpha(z))\rho_c(z)| \le C_\Lambda R_g t \rho_r(\alpha^{-1}) \tag{3.24}$$

for all $t \in [0, T]$ and $\alpha \in I$. Thus the family of operators $\{e^{(T-t)(f_\alpha(z)-z)}\}_{\alpha \in I}$ is a regularization associated with the qualification pair $(\rho_c, c_0(T-t)\rho_r)$ for $c_0 = \max\{C_\Lambda R_g, 1\}$ by definition.

The estimate (3.21) is obvious for $t = T$. It remains to consider the case with $t \in [0, T)$. For a fixed $t \in [0, T)$, let $(\mathbf{I})_1$ and $(\mathbf{I})_2$ be the two terms used in the proof of Theorem 3.7, i.e.,

$$(\mathbf{I})_1 := \|R_{\alpha, t}(f^\delta - f)\|, \quad (\mathbf{I})_2 := \|R_{\alpha, t} - u(t)\|.$$

Using condition **(i)**, it is easy to see that $(\mathbf{I})_1$ can be bounded by $C_s \delta e^{\frac{C_f(T-t)}{\alpha^p}}$. As e^{-At} commutes with $\exp((T-t)h_\alpha(A))$ and $\|e^{-tA}\| \le C_s$, we can apply Proposition 3.14 to obtain

$$(\mathbf{I})_2 = \left\| \left(\exp(-(T-t)h_\alpha(A)) - I \right) e^{-tA} x^\dagger \right\|$$

$$= \left\| e^{-tA} \left(\exp(-(T-t)h_\alpha(A)) - I \right) x^\dagger \right\|$$

$$\le (T-t) c_0 C_{\varphi, s} C_s \varphi(\alpha^{-1}) \|x^\dagger\|_\varphi$$

for some constant $C_{\varphi, s} > 0$. Using the above estimates for $(\mathbf{I})_1$ and $(\mathbf{I})_2$, we immediately derive

$$\|R_{\alpha,t} f^\delta - u(t)\| \leq (\mathbf{I})_1 + (\mathbf{I})_2 \leq C_s \delta e^{\frac{C_f(T-t)}{\alpha^p}} + (T-t)c_0 C_{\varphi,s} C_s \varphi(\alpha^{-1}) \|x^\dagger\|_\varphi,$$

which gives (3.21). And the estimate (3.23) of convergence rate follows directly with the a priori choice (3.22) for the parameter $\alpha = \alpha(\delta)$.   □

Corollary 3.16. *Under the same assumptions as in Theorem 3.15 except that* (**i**) *is replaced by*

(**i′**) $\sup_{z \in \Sigma_\theta} \Re f_\alpha(z) \leq 0 \quad \forall \alpha \in I$,

for any $t \in (0, T]$, $\{e^{(f_\alpha(z)-z)t}\}_{\alpha \in I_s}$ is a regularization associated with the qualification pair $(\rho_c, c_0 t \rho_r)$ for some positive constant c_0. In addition, for given $x^\dagger = u(0) \in X_\varphi$ satisfying $\|x^\dagger\|_\varphi \leq Q$, we have the error estimate

$$\|R_{\alpha,t} f^\delta - u(t)\| \leq C_{\mathcal{M}} \left(\delta + (T-t) \varphi(\alpha^{-1}) Q \right) \quad \forall \alpha \in I, \ t \in [0, T] \tag{3.25}$$

for some constant $C_{\mathcal{M}} > 0$. Furthermore, under the parameter choice

$$\alpha(\delta) = \left[\varphi^{-1}(\delta) \right]^{-1}, \tag{3.26}$$

we have the following estimate of the convergence rate for all $0 \leq t \leq T$,

$$\|R_{\alpha(\delta),t} f^\delta - u(t)\| = \mathcal{O}(\delta) \quad as \quad \delta \to +0. \tag{3.27}$$

It is easy to see that the optimal rate in (3.27) indicates that the modified condition (**i′**) implies the well-posedness of the backward evolution equation (1.1). This is clearly unreasonable, and implies that condition (**i′**) could not be satisfied in practical applications.

We end this section with an application of the main result, i.e., approximating the ill-posed backward equation (1.1) by the following regularized problem

$$\begin{cases} u'(t) + (A - \epsilon A^b)u(t) = 0, & 0 < t < T, \\ u(T) = f^\delta, \end{cases} \tag{3.28}$$

where A is a densely defined sectorial operator with bounded H^∞-calculus on a sector $\omega_H \leq \frac{\pi}{4}$ and $1 < b < \frac{\pi}{\pi - 2\omega_H}$. Setting

$$f_\alpha(z) := z - \alpha^b z^b,$$

then $f_\alpha(A) = A - \epsilon A^b$ if $\alpha = \epsilon^{1/b}$. So it suffices to study the convergence behaviors of $f_\alpha(A)$. It can be verified that $(\exp(t f_\alpha(A)))_{t \geq 0}$ is an analytic semigroup for each $\alpha > 0$, and $R_{\alpha,t} := \exp((T-t) f_\alpha(A))$ for $\alpha > 0$, $t \in [0, T]$ is indeed a family of regularizing operators to (1.1). Next we shall present an explicit convergence rate of the family $\{R_{\alpha,t}; \alpha > 0, t \in [0, T]\}$. An elementary computation yields that condition (**i**) in Theorem 3.15 holds with $p = \frac{1}{b-1}$ and

$$C_f = \left(1 - \frac{1}{b} \right) \left(\frac{\cos \omega_H}{b \cos(b \omega_H)} \right)^{\frac{1}{b-1}}.$$

In addition, it is readily checked that $z - f_\alpha(z)$ maps the sector Σ_{ω_H} into $\Sigma_{b\omega_H}$ with $b\omega_H < \pi/2$ for all $\alpha > 0$. Obviously, condition **(ii)** in Theorem 3.15 is satisfied by the qualification pair (z^{-b}, α^{-b}), which covers any proper index function φ such that the function $t \mapsto t^b\varphi(t)$ is increasing on $(0, \infty)$. In particular, if we choose the index function $\varphi(z) = (1 + z)^{-\beta}$ with $0 < \beta < b$, we have

$$\sup_{z \in \Sigma_{\omega_H, \alpha^{-1}}} \left| \frac{\varphi(z)}{\rho_c(z)} \right| = \sup_{z \in \Sigma_{\omega_H, \alpha^{-1}}} \left| \frac{z^b}{(1+z)^\beta} \right| \lesssim \sup_{t \in (0, \alpha^{-1})} \frac{t^b}{(1+t)^\beta} = \frac{\alpha^{-b}}{(1+\alpha^{-1})^\beta}.$$

Thus we can apply Theorem 3.15. If $u(0) \in X_\varphi$, then for each $t \in [0, T)$, using a priori parameter choice $\alpha(\delta) = \left[\frac{1}{\log(1/\delta)} \right]^\kappa$ with an appropriate exponent $\kappa > 0$ based on the formula (3.22), we obtain the logarithmic convergence rate

$$\| R_{\alpha(\delta), t} f^\delta - u(t) \| = \mathcal{O}\left(\left[\frac{1}{\log(1/\delta)} \right]^\zeta \right) \quad \text{as} \quad \delta \to +0$$

for some exponent $\zeta > 0$, which depends on β and κ. It is worth mentioning that $u(0) \in X_\varphi$ if and only if $u(0) \in \mathcal{D}(A^\beta)$.

3.4. Examples of sectorial operators with a bounded H^∞-calculus

The results in subsections 3.2–3.3 were established under assumption **(H1)** for the sectorial operator A. Now we present some examples that fulfill **(H1)**, i.e., some important classes of sectorial operators with bounded H^∞-calculus. For a general sectorial operator A, let $\omega_H(A)$ stand for the minimum of all ω such that A has a bounded H^∞-calculus over Σ_ω. Obviously, $\omega_H(A)$ is larger than the sectorial angle of A [26].

Let us consider Laplacian $A = -\Delta$ defined on $L_p(\mathbb{R}^n)$ $(1 < p < \infty)$ as a Fourier multiplier operator, i.e.,

$$A = \mathcal{F}^{-1} M_m \mathcal{F},$$

where $\mathcal{F}$ denotes the Fourier transform, and M_m is the multiplication operator with a multiplier function $m(u) = |u|^2$. Then for each function f in $H_0^\infty(\Sigma_\sigma)$ with $\sigma > 0$, the operator $f(A)$ is identical to the Fourier multiplier operator $\mathcal{F}^{-1} M_{f \circ m} \mathcal{F}$. With the help of Marcinkiewicz's multiplier theorem, one can establish the boundedness of $f(A)$. Moreover, if one considers Laplacian $-\Delta$ defined on $L_p(\mathbb{R}^n, X)$, then $-\Delta$ has a bounded H^∞-calculus over any $H_0^\infty(\Sigma_\sigma)$ with $\sigma > 0$ whenever X is a UMD-space (cf. [17, Proposition 8.3.4.]).

Now we move to investigate the second order elliptic operators on bounded domains $\Omega \subset \mathbb{R}^n$ with $n \geq 1$. Let $L_p(\Omega, \mu)$ denote the usual L^p-space on a measure space (Ω, μ). It was shown in [26, corollary 5.2] that if an operator $-A : \mathcal{D}(A) \subset L_p(\Omega, \mu) \to L_p(\Omega, \mu)$ $(1 < p < \infty)$ generates an analytic contractive and positive semigroup on $L_p(\Omega, \mu)$, then $\omega_H(A) < \frac{\pi}{2}$. In particular, we may consider the second order elliptic differential operator on a bounded and smooth domain $\Omega \subset \mathbb{R}^n$ $(n \geq 1)$ of the form:

$$(Au)(\xi) = -\sum_{j,k=1}^n \partial_j(a_{j,k}(\xi)\partial_k u(\xi)) + \sum_{j=1}^n b_j(\xi)\partial_j u(\xi) + c(\xi)u(\xi), \quad \xi \in \Omega,$$

where $c \in C(\overline{\Omega})$, $a_{j,k} \in C^1(\overline{\Omega})$ and $b_j \in C(\overline{\Omega})$ for $j, k = 1, \cdots, n$. We denote its $L^p(\Omega)$-realization under Dirichlet boundary condition by A_p, whose domain is $\mathcal{D}(A_p) := W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$. It is known that if A_p satisfies

$$\|(\lambda + A_p)u\|_{L_p(\Omega)} \geq \lambda \|u\|_{L_p(\Omega)} \quad \forall u \in \mathcal{D}(A_p) \tag{3.29}$$

for all $\lambda > 0$, then $-A_p$ generates a contraction semigroup $(e^{-tA_p})_{t \geq 0}$ (cf. [31, Section 1.4]). Positivity of the semigroup can be deduced from the maximal principle. Moreover, if A_p is a sectorial operator with angle less than $\frac{\pi}{2}$, then $-A_p$ generates an analytic semigroup $(e^{-tA_p})_{t \geq 0}$. It was shown in [10, theorem 4] and [38, chapter 2, section 6.2] that for $b_j = 0$, the operator A_p is sectorial and its induced semigroup is positive, hence A_p possesses a bounded H^∞-calculus over a sector Σ_σ with some $\sigma \in (0, \frac{\pi}{2})$. Using the perturbation theorem for H^∞-calculus, one can show that A_p still has a bounded H^∞-calculus if $\|b_j\|_{L^\infty}$ is small enough for all $j = 1, 2, \cdots n$ [8]. For more elliptic operators with bounded H^∞-calculus, we refer to [36] and references therein for more detail.

Although the results above have already covered a large class of elliptic differential operators, bounded H^∞-calculus can also be expected for some Λ-elliptic pseudo-differential operators. Let us consider the system of the form

$$(Au)(\xi) = \sum_{|\alpha| \leq 2m} a_\alpha (D^\alpha u)(\xi), \quad \xi \in \Omega \subset \mathbb{R}^n,$$

where $D^\alpha := (-i\partial_1)^{\alpha_1} \cdots (-i\partial_n)^{\alpha_n}$ and the $\mathbb{C}^{N \times N}$-valued symbol $a(\xi_1, \xi_2) = \sum_{|\alpha|=2m} a_\alpha(\xi_1)\xi_2^\alpha$ is elliptic in the sense that

$$a_\alpha \in L_\infty(\Omega), \quad \sigma(a(\xi_1, \xi_2)) \subset \overline{\Sigma_\omega} \backslash \{0\}, \quad \|a(\xi_1, \xi_2)\| \leq M$$

for all $\xi_1 \in \mathbb{R}^n$ and $|\xi_2| = 1$ with $\omega \in (0, \pi)$ and $M > 0$. The boundedness of H^∞-calculus for A was studied extensively. For $\Omega = \mathbb{R}^n$, one may refer to [1,27] for the Hölder continuous coefficients, and [9,19] for continuous coefficients and *VMO*-coefficients.

4. Convergence of regularizations for half-strip operators

We studied in section 3 the convergence of the regularizing operators constructed by the quasi-reversibility method. But as we have seen from section 3, the quasi-reversibility method does not take advantage of the semigroup effectively. In this section, we consider the regularizing operators constructed by the modified quasi-reversibility method for the class of *half-strip operators*. As we will see, this will help us greatly improve our previous convergence results. In practical applications, some elliptic operators belong to both sectorial and half-strip classes. We will demonstrate (see sections 3.3 and 4.6), the quasi-reversibility method may guarantee only the logarithmic-type convergence rates for regularized solutions while the modified quasi-reversibility method can achieve the Hölder-type convergence rates, under similar assumptions on the exact initial value $u(0)$.

4.1. Definitions of half-strip operators and their functional calculus

In this subsection, we introduce a class of unbounded operators, and establish its abstract functional calculus, which is used later as our major analysis tool. We were not aware of the discussion about the notion of functional calculus on half-strips in the literature except that it was briefly mentioned in [5,6] for the properties of the sum of operator logarithms. We intend to provide a more detailed study of the half-strip operators.

Definition 4.17. A densely defined closed operator $A : \mathcal{D}(A) \subset X \to X$ is called half-strip with a height $\omega \in [0, +\infty)$ and a parameter $a > 0$, if the following hold

(1) $\sigma(A) \subset \overline{H}_{a,\omega}$.
(2) $M_1(A, \omega') = \sup\{\|R(\lambda, A)\|; \ |\mathfrak{Im}\,\lambda| \geq \omega'\} < \infty$ for each $\omega' > \omega$.
(3) $M_2(A, a') = \sup\{\|R(\lambda, A)\|; \ \mathfrak{Re}\,\lambda \leq a'\} < \infty$ for each $a' \in (0, a)$.

We write the collection of all half-strip operators on X with a height ω and a parameter a by $H(a, \omega)$. For the sake of exposition, we consider only the parameter $a > 0$ in the definition of $H(a, \omega)$, but similar results to what we obtain below still hold for general $a \in \mathbb{R}$.

Now, we are going to define a functional calculus for a half-strip operator $A \in H(a, \omega)$. For $\theta > \omega$ and $0 < a' < a$, we write

$$\mathcal{E}(H_{a',\theta}) := \{f \in H^\infty(H_{a',\theta}) \,|\, |f(z)| = \mathcal{O}(|\mathfrak{Re}\,z|^{-\alpha}) \ (|z| \to +\infty) \ \text{for some} \ \alpha > 1\},$$

then we define the operator $f(A)$ by a *Dunford–Riesz integral*

$$f(A) := \frac{1}{2\pi i} \int_{\Gamma_{b,\theta_0}} f(z) R(z; A) dz \tag{4.30}$$

for each $f \in \mathcal{E}(H_{a',\theta})$, where the integral contour

$$\Gamma_{b,\theta_0} := \{\theta_0 + i\rho; \rho \in [b, +\infty)\} \cup \{b + it; -\theta_0 \leq t \leq \theta_0\} \cup \{-\theta_0 + i\rho; \rho \in [b, +\infty)\} \tag{4.31}$$

is positively oriented with $a' < b < a$ and $\omega < \theta_0 < \theta$. It is easy to see that the integral (4.30) exists, and $f(A) \in \mathcal{L}(X)$. An standard argument involving the Cauchy's theorem shows that the definition of $f(A)$ is independent of the actual choice of b and θ_0. The following result can be obtained by some standard arguments with natural modifications (see, e.g., [17, lemma 2.3.1 and proposition 4.2.1]).

Proposition 4.18. *For $A \in H(a, \omega)$, $\theta > \omega$ and $0 < a' < a$, the following assertions hold*

(a) *The mapping $\Phi_A := (f \mapsto f(A)) : \mathcal{E}(H_{a',\theta}) \longrightarrow \mathcal{L}(X)$ defined above is a homomorphism of algebras.*
(b) *for any $f \in \mathcal{E}(H_{a',\theta})$,*
   (1) *$(f(z)(\lambda - z)^{-1})(A) = R(\lambda, A)f(A)$ for $\lambda \notin H_{a',\theta}$;*
   (2) *if B is a closed operator commuting with the resolvents of A, then B commutes with $f(A)$. In particular, $f(A)$ commutes with both A and $R(\lambda, A)$ for any $\lambda \notin H_{a',\theta}$.*
(c) *For any two $\lambda, \mu \notin H_{a',\theta}$, $((\lambda - z)^{-1}(\mu - z)^{-1})(A) = R(\lambda, A)R(\mu, A)$.*

We can easily see that the triple $(H(H_{a',\theta}), \mathcal{E}(H_{a',\theta}), \Phi_A)$ is an abstract functional calculus. From Proposition 4.18 (c) it follows that $z^{-2}(A)$ is bounded and injective, hence the set of regularizers for $(H(H_{a',\theta}), \mathcal{E}(H_{a',\theta}), \Phi_A)$ is not empty, so is the domain of the triple $(H(H_{a',\theta}), \mathcal{E}(H_{a',\theta}), \Phi_A)$. The set of all regularizable functions in $(H(H_{a',\theta}), \mathcal{E}(H_{a',\theta}), \Phi_A)$ is denoted by $\mathcal{M}_A(H_{a',\theta})$.

We end this subsection with some basic results from functional calculus for a half-strip operator, and the proof can be in a quite similar manner to the one in [17, lemma 4.2.3 and proposition 5.1.7] for strip operators.

Proposition 4.19. *Assume $A \in H(a,\omega)$, $0 < a' < a$, and $\theta > \omega$.*
(a) If $f \in H(H_{a',\theta})$ is regularly decaying at ∞, i.e., $|f(z)| \le C|z|^{\alpha}$ as $|z| \to \infty$ for some $C \ge 0$ and $\alpha > 0$, then f belongs to $\mathcal{M}_A(H_{a',\theta})$, and hence $f(A)$ is well-defined.
(b) Let $T > 0$ with $T\omega \in [0, \pi)$, and $\lambda \in \mathbb{C}\backslash\{0\}$ such that $|\arg\lambda| \in (T\omega, \pi]$, then $(\lambda - e^{-zT})^{-1} \in H^{\infty}(H_{a,\omega})$, and $(\lambda - e^{-zT})(A)$ is injective. Moreover, e^{-TA} is injective and $(\mathbb{C}\backslash\Sigma_{T\omega}) \cap P\sigma(e^{-AT}) = \emptyset$.
(c) Let $\{f_n\}_{n=1}^{\infty}$ be a net of holomorphic functions on $H_{a',\theta}$ such that f_n converges to f pointwise over $H_{a',\theta}$ as $n \to \infty$.
  (1) If $\beta > 1$ and

$$\sup_{n \ge 1} \sup_{z \in H_{a',\theta}} |f_n(z)|(1 + |\mathfrak{Re}\, z|^{\beta}) < \infty,$$

then $f \in \mathcal{E}(H_{a',\theta})$ and $f_n(A) \to f(A)$ strongly as $n \to \infty$.
  (2) If $\{f_n\}_{n=1}^{\infty} \subset H^{\infty}(H_{a',\theta})$, $f_n(A) \in \mathcal{L}(X)$ for all $n \ge 1$, and

$$\sup_{n \ge 1} \sup_{z \in H_{a',\theta}} |f_n(z)| < \infty, \qquad \sup_{n \ge 1} \|f_n(A)\| < \infty,$$

then $f_n(A)$ converges strongly to $f(A)$ as $n \to \infty$.

In the sequel, we study the connections between a half-strip operator and a bounded sectorial operator. For this purpose, we end this subsection by a composition rule, whose proof is similar to the one for the composition rule between sectorial operators and strip operators (see, e.g., [17, Theorem 4.2.4]).

Theorem 4.20 *(Composition rule). Let $A \in H(a,\omega)$ and $g \in \mathcal{M}(H_{a',\theta})$ with $a' \in (0,a)$ and $\theta > \omega$. Assume that $g(A) \in \mathrm{Sect}(\varphi, b)$ for some $\varphi \in (0, \pi)$ and $b > 0$, and for any $\varphi' \in (\varphi, \pi)$ and $b' > b$, there exist constants c and κ such that $0 < c < a$ and $\kappa > \omega$, and $g(H_{c,\kappa}) \subset \overline{\Sigma_{\varphi',b'}}$. Then $f \circ g \in \mathcal{M}_A(H_{a',\theta})$ and $(f \circ g)(A) = f(g(A))$ for all $f \in \mathcal{M}_{g(A)}(\Sigma_{\varphi',b'})$.*

4.2. Relations between half-strip and bounded sectorial operators

In this subsection, we first give a condition on the operator A, under which we demonstrate that there exists a deep connection between A and $\mathcal{A} := e^{-TA}$, and $\mathcal{A}$ is a bounded sectorial operator. Via the functional calculus of A, we shall further establish a functional calculus for $\mathcal{A}$, which is different from the one for sectorial operators in section 3. This will be crucial for us to define the qualification pairs and index functions associated with $\mathcal{A}$. As will be seen later, these

concepts are not quite the same as the ones in section 3, and will help us investigate the backward evolution equation (1.1) driven by A.

Now we define the class of half-strip operators A we shall study in this section.

(**A**) $A \in H(a, \omega)$ and A has a bounded H^∞-calculus over a half-strip H_{ω_H, a_H} with $0 < \omega \le \omega_H < \frac{\pi}{2T}$ and $0 < a_H \le a$, i.e., there exists a constant $C_h \ge 1$ such that

$$\|f(A)\| \le C_h \|f\|_\infty \quad \forall f \in H^\infty(H_{a_H, \omega_H}).$$

Proposition 4.19 implies that $\mathcal{A}$ is injective. In addition, one can show that the adjoint A' is also a half-strip operator on X'. Thus, $\mathcal{A}' = e^{-Tz}(A')$ is also injective whence $\mathcal{A}$ has a dense range in X. Moreover, as shown in the lemma below, $\mathcal{A}$ is a bounded sectorial operator.

Lemma 4.21. *It holds for each $\omega' \in (T\omega_H, \pi)$ that*

$$\sup\{|\lambda| \|R(\lambda, \mathcal{A})\|; \lambda \in \mathbb{C} \setminus \Sigma_{\omega'}\} < \infty, \tag{4.32}$$

and for each $h > e^{-a_H T}$,

$$\sup\{|\lambda| \|R(\lambda, \mathcal{A})\|; |\lambda| > h\} < \infty. \tag{4.33}$$

Moreover, $(e^{-t\mathcal{A}})_{t \ge 0}$ is a (real) analytic semigroup generated by $-\mathcal{A}$.

Proof. We choose $\lambda \ne 0 \in \mathbb{C} \setminus \Sigma_{\omega'}$. From assumption (**A**) it follows that both $\lambda - \mathcal{A} = (\lambda - e^{-zT})(A)$ and $(\lambda - e^{-zT})^{-1}(A)$ are continuous. Then Theorem 2.1 ensures that $(\lambda - e^{-zT})^{-1}(A)$ is indeed the resolvent for $\lambda - \mathcal{A}$. As $\lambda R(\lambda, \mathcal{A}) = \lambda(\lambda - e^{-zT})^{-1}(A)$, we consider the supreme norm of the function $g(z) := \lambda(\lambda - e^{-zT})^{-1}$ over H_{a_H, ω_H}. It is easy to see that

$$\|g(z)\|_\infty = \sup_{\xi \in \Sigma_{\varphi, a}} |\lambda(\lambda - \xi)^{-1}| \le \begin{cases} \dfrac{1}{\sin(|\arg\lambda| - \varphi)} & \text{if } |\arg(\lambda)| - \varphi < \pi/2, \\ 1 & \text{if } |\arg(\lambda)| - \varphi \ge \pi/2, \end{cases}$$

where $\varphi = T\omega_H$ and $a = e^{-a_H T}$. Then the boundedness of the H^∞-calculus for A implies the first estimate (4.32). The second estimate (4.33) can be proven in a quite similar manner.

Finally, for each $t \ge 0$, one knows $e^{-tz} \in H^\infty(H_{a_H, \omega_H})$. In addition, by assumption (**A**) we have $\|e^{-tA}\| \le C_h e^{-ta_H}$ for all $t \ge 0$. From Proposition 4.18(a) it follows that the semigroup property holds: $e^{-tA}e^{-sA} = e^{-(t+s)A}$ for all $s, t \ge 0$. It is readily checked that $\sup_{t \ge 0} \|e^{-tA}\| \le C_h$, and $e^{-tz} \to 1$ over H_{a_H, ω_H} as $t \to 0^+$. Thus, Proposition 4.19(c)(2) applies, so $\lim_{t \to 0^+} e^{-At} x = x$ for any $x \in X$. Then proposition 5.3 in [12, chapter 2] implies that $(e^{-tA})_{t \ge 0}$ is a C_0-semigroup.

Let us denote by $(B, D(B))$ the generator of C_0-semigroup $(e^{-tA})_{t \ge 0}$. For $x \in D(A)$, there exists $y \in X$ such that $x = A^{-1}y$. Let $\tau(z) = z^{-1}$ and $f_\alpha(z) := \frac{e^{-z\alpha} - 1}{\alpha}$ for $\alpha > 0$. Then $f_\alpha(A)x = g_\alpha(A)y$ for $\alpha > 0$, with $g_\alpha := \tau f_\alpha$. On the other hand, g_α converges to -1 over H_{a_H, ω_H} as $\alpha \to 0^+$. It is readily checked that there exists $C > 0$ such that

$$|g_\alpha(z)| \le C \quad \forall z \in H_{a_H, \omega_H}.$$

Thus, we can apply Proposition 4.19 (c) (2) to obtain that $g_\alpha(A) \to -I$ strongly as $\alpha \to +0$, whence $\lim_{t \to +0}(e^{-tA}x - x)/t = \lim_{\alpha \to 0^+} g_\alpha(A)y = -Ax$. From this it follows that $\mathcal{D}(A) \subset \mathcal{D}(B)$ and $B|_{\mathcal{D}(A)} = -A$. For $\lambda > 0$, $\lambda + A$ is bijective from X to $\mathcal{D}(A)$. Since $(e^{-tA})_{t \geq 0}$ is uniformly bounded, we have $\lambda \in \rho(B)$. So $\lambda - B$ is bijective from X to $\mathcal{D}(B)$. As $\lambda - B$ and $\lambda + A$ coincide with each other on $\mathcal{D}(A)$, we have $D(A) = D(B)$ and $-A = B$. It remains to show that A is a sectorial operator. Indeed, $\sigma(A)$ is contained in the sector Σ_γ with angle $\gamma = \arctan(\frac{\omega_H}{a_H})$. From **(A)** it follows that $\|\lambda R(\lambda, A)\| < \infty$ for $\lambda \in \mathbb{C} \backslash \Sigma_\gamma$. Therefore the C_0-semigroup generated by $-A$ is analytic (see. e.g. [12]). $\quad\square$

Lemma 4.21 implies that $\mathcal{A}$ is a bounded sectorial operator. More precisely, $\mathcal{A} \in \mathrm{Sect}(\omega', b')$ for every $\omega' \in (\omega_T, \pi)$ and $b' > b_T$, where

$$\omega_T := T\omega_H, \quad b_T := e^{-a_H T}. \tag{4.34}$$

Following the discussion at the end of subsection 3.1, we know that there exists a homomorphism $\Phi_{\mathcal{A}}$ from $\mathcal{E}(\Sigma_{\omega', b'}) \to \mathcal{L}(X)$ and the triple $(H(\Sigma_{\omega', b'}), \mathcal{E}(\Sigma_{\omega', b'}), \Phi_{\mathcal{A}})$ is an abstract functional calculus, whose domain is denoted by $\mathcal{M}_{\mathcal{A}}(\Sigma_{\omega', b'})$. Let us further define

$$\mathcal{E}_{\mathcal{A}} := \bigcup_{\pi > \omega' > \omega_T; b' > b_T} \mathcal{E}_{\mathcal{A}}(\Sigma_{\omega', b'}), \quad \mathcal{M}_{\mathcal{A}} := \bigcup_{\pi > \omega' > \omega_T; b' > b_T} \mathcal{M}_{\mathcal{A}}(\Sigma_{\omega', b'}).$$

We can see that $(\mathcal{M}_{\mathcal{A}}, \mathcal{E}_{\mathcal{A}}, \Phi_{\mathcal{A}})$ is a proper functional calculus and $f(\mathcal{A})$ is well-defined for any $f \in \mathcal{M}_{\mathcal{A}}$. We now end this subsection with the following important result, which will be frequently used in the subsequent sections.

Lemma 4.22. *If $f \in \mathcal{M}_{\mathcal{A}}$ is uniformly bounded on Σ_{ω_T, b_T}, it holds under Assumption **(A)** that*

$$f(\mathcal{A}) \in \mathcal{L}(X) \quad and \quad \|f(\mathcal{A})\| \leq C_h \|f\|_\infty,$$

*where $C_h \geq 1$ is the constant in Assumption **(A)**, and $\|f\|_\infty$ represents the sup-norm of f over Σ_{ω_T, b_T}.*

Proof. Assume $f \in \mathcal{M}_{\mathcal{A}}(\Sigma_{\omega', b'})$ for some $\omega' \in (\omega_T, \pi)$ and $b' \in (b, +\infty)$. Since the mapping $z \mapsto e^{-Tz}$ $H_{b_0, \omega_0} \mapsto \Sigma_{\omega', b'}$ is a biholomorphic mapping with $b_0 = \frac{1}{T} \log \frac{1}{b'}$, $\omega_0 = \frac{\omega'}{T}$ and $H^\infty(\Sigma_{\omega', b'}) \subset \mathcal{M}_A(\Sigma_{\omega', b'})$, the composition rule (cf. Theorem 4.20) ensures that

$$f(\mathcal{A}) = f(g(A)) = (f \circ g)(A)$$

is well-defined for $g(A) = (e^{-Tz})(A)$. As $f \circ g$ is bounded over H_{a_H, ω_H}, Assumption **(A)** implies

$$\|f(\mathcal{A})\| = \|(f \circ g)(A)\| \leq C_h \sup_{z \in H_{a_H, \omega_H}} |(f \circ g)(z)| = C_h \|f\|_\infty. \quad\square$$

Remark 4.23. The result of this subsection can be extended to *(strong) strip-type operators*. If A is a strip-type operator of height less than π, then $\exp(A)$ can be well-defined but it is not necessarily a sectorial operator. Monniaux's Theorem states that $\exp(A)$ is sectorial provided that iA generates a C_0-group and the underlying space possesses UMD property [17, Theorem 4.4.3.].

Furthermore, if A has bounded H^∞-calculus on a strip St_ω with the height $\omega < \pi$, then $\exp(A)$ is a sectorial operator and the natural calculus for it is also bounded.

To see this, consider a complex number $\lambda \neq 0$ with $|\arg \lambda| \in (\omega, \pi)$, then we need to analyze the behavior of $\lambda R(\lambda, \exp(A))$. First, it follows from [17, Theorem 1.3.2 (f)] that $(\lambda - \exp(A))^{-1} = (\lambda - \exp(z))^{-1}(A)$. Then the boundedness of the H^∞-calculus for A implies that $(\lambda - \exp(z))^{-1}(A)$ is bounded, hence a resolvent. That is, $\lambda \in \rho(\exp(A))$ and $R(\lambda, \exp(A)) = (\lambda - \exp(A))^{-1}$ is well-defined. One can further show that $|\lambda| \|R(\lambda, \exp(A))\|$ is bounded because the function $z \mapsto \lambda(\lambda - \exp(z))^{-1}$ is bounded on the strip St_ω and $\sup_{z \in St_\omega} |\lambda||(\lambda - e^z)^{-1}|$ depends on $|\arg \lambda| - \omega$. Therefore we can see that $\exp(A)$ is indeed a sectorial operator. Next, we can apply the composition rule from [17, Theorem 4.2.4], which is also valid for the case when A is strip-type and $g(A)$ is sectorial for some g. In our case, we can choose $g(z) = \exp(z)$, then we can conclude that the natural functional calculus for the sectorial operator $\exp(A)$ is bounded on some sector.

4.3. Half-strip operators in Hilbert spaces

In order to help us better understand half-strip operators, we shall study their properties in Hilbert spaces in this subsection, especially we will establish some criteria for operators in a Hilbert space to be half-strip, and then prove that the corresponding half-strip operators possess bounded H^∞-calculus over some half-strips.

We first recall some concepts. Let $\mathbb{H}$ be a complex Hilbert space with inner product $(\cdot, \cdot)_{\mathbb{H}}$, and $\omega \in [0, \frac{\pi}{2}]$. An operator $A : \mathcal{D}(A) \subset \mathbb{H} \to \mathbb{H}$ is called ω-accretive if its numerical range $W(A) := \{(Ax, x)_{\mathbb{H}}; x \in \mathcal{D}(A), \|x\| = 1\}$ is contained in $\overline{\Sigma_\omega}$. Observe that this is equivalent to

$$|\mathfrak{Im}\, (Ax, x)_{\mathbb{H}}| \leq \tan(\omega) \,\mathfrak{Re}\, (Au, u)_{\mathbb{H}} \quad \forall x \in \mathcal{D}(A).$$

If $\omega = \frac{\pi}{2}$, then A is said to be accretive. The operator A is called m-ω-accretive (resp. m-accretive) if it is ω-accretive (resp. m-accretive), and $\mathcal{R}(A + 1)$ is dense in $\mathbb{H}$. In addition, for $\omega_0 \in \mathbb{R}$, $A \geq \omega_0$ means that $A - \omega_0$ is accretive. Then such operators A have the following important properties, whose proof can be done similarly to the one for [17, proposition 7.1.2.].

Proposition 4.24. *Let A be an operator in the Hilbert space $\mathbb{H}$, $\omega \in [0, \frac{\pi}{2})$, and $a > 0$. Then $A \in H(a, \omega)$ if any one of the following assertions holds. Moreover, all the following 5 assertions are equivalent.*
(1) $-A$ generates a contraction semigroup $(S(t))_{t \geq 0}$ with growth order

$$\|S(t)\| \leq e^{-at} \quad \forall t \geq 0,$$

and iA generates a C_0-group $(T(t))_{t \in \mathbb{R}}$ such that

$$\|T(t)\| \leq e^{\omega|t|} \quad \forall t \in \mathbb{R}.$$

(2) There are self-adjoint operators B and C such that $a \leq B$, $-\omega \leq C \leq \omega$, and $A = B + iC$.
(3) The inclusion $\sigma(A) \subset \overline{H_{a,\omega}}$, and the following resolvent estimates are satisfied:

$$\|R(\lambda, A)\| \leq \frac{1}{|\mathfrak{Im}\, \lambda| - \omega} \quad \forall |\mathfrak{Im}\, \lambda| > \omega,$$

$$\|R(\lambda, A)\| \leq \frac{1}{a - \mathfrak{Re}\lambda} \quad \forall \mathfrak{Re}\lambda < a$$

(4) Both the operators $\omega \pm iA$ and $A - a$ are m-accretive.
(5) $W(A) \subset \overline{H}_{a,\omega}$ and both of $\mathcal{R}(A \pm (\omega + 1))$ and $\mathcal{R}(A + 1)$ are dense in $\mathbb{H}$.

Similarly to [17, Theorem 7.1.16], we can derive the following result.

Proposition 4.25. *If A satisfies one of the assertions in Proposition 4.24, then A belongs to $H(a', \omega)$ for each $0 < a' < a$, and there exists constant $C \geq 1$ such that*

$$\|f(A)\| \leq C\|f\|_\infty \quad \forall f \in H^\infty(H_{a',\omega}). \tag{4.35}$$

Proof. We may suppose that $f \in \mathcal{E}(H_{a',\omega})$ and reduce the result to the case when A is bounded. More precisely, let $A \in H(a', \omega') \cap \mathcal{L}(\mathbb{H})$ for some $0 < \omega' < \omega$. Then Proposition 4.24 ensures that there exists a decomposition $A = B + iC$ with B and C being self-adjoint operators such that $B \geq a'$ and $-\omega' \leq C \leq \omega'$. We shift the path of integration to $\partial H_{b,\omega}$ for $(a' + a)/2 \leq b < a$ to obtain

$$f(A) = \frac{1}{2\pi i} \int_{\partial H_{b,\omega}} f(z)R(z, A)dz$$

$$= \frac{1}{2\pi i} \int_{\gamma_1} f(z)R(z, A)dz + \frac{1}{2\pi i} \int_{\gamma_2} f(z)R(z, A)dz$$

$$= \frac{1}{2\pi i} \int_{\gamma_1} f(z)R(z, A)dz + \frac{1}{2\pi i} \int_{\gamma_2} f(z)[R(z, A) - R(z, A)']dz$$

$$+ \frac{1}{2\pi i} \int_{\gamma_2} f(z)R(\bar{z}, A')dz$$

$$=: (\mathbf{I})_1 + (\mathbf{I})_2 + (\mathbf{I})_3,$$

where we have used the fact that $R(z, A)' = R(\bar{z}, A')$ for $z \in \rho(A)$, and the oriented counterclockwise paths γ_1 and γ_2 are defined by

$$\gamma_1 = \{b + it; \ -\omega \leq t \leq \omega\} \quad \text{and} \quad \gamma_2 = \{t \pm i\omega; \ t \geq b\}.$$

Following the argument in [7, lemma 1] or [17, theorem 7.1.16] with some necessary modifications, one can estimate $(\mathbf{I})_2$ by $\|(\mathbf{I})_2\| \leq 2\|f\|_\infty$. Next we estimate $(\mathbf{I})_3$:

$$\frac{1}{2\pi i} \int_{\gamma_2} f(z)R(\bar{z}, A')dz$$

$$= \frac{1}{2\pi i} \int_{\mathfrak{Im}\, z = -\omega, \, \mathfrak{Re}z \geq a} f(z)R(z + 2i\omega, A')dz - \frac{1}{2\pi i} \int_{\mathfrak{Im}\, z = \omega, \, \mathfrak{Re}z \geq b} f(z)R(z - 2i\omega, A')dz$$

$$= \frac{1}{2\pi i} \int\limits_{[b,+\infty)} f(z)R(z + 2i\omega, A')dz - \frac{1}{2\pi i} \int\limits_{[b,+\infty)} f(z)R(z - 2i\omega, A')dz + (\mathbf{J})_1$$

$$= \frac{-2\omega}{\pi} \int\limits_{b}^{+\infty} f(t)R(t + 2i\omega, A')R(t - 2i\omega, A')dt + (\mathbf{J})_1,$$

where we have used the Cauchy's theorem and

$$(\mathbf{J})_1 := \frac{1}{2\pi i} \int\limits_{-\omega}^{0} f(b + is)R(b + is + 2i\omega, A')ds + \frac{1}{2\pi i} \int\limits_{0}^{\omega} f(b + is)R(b + is - 2i\omega, A')ds.$$

From the proof of [7, lemma 1] or [17, theorem 7.1.16] it follows readily that

$$\|(\mathbf{I})_3 - (\mathbf{J})_1\| \le \frac{2}{\sqrt{3}}\|f\|_\infty.$$

Using Proposition 4.24 (3) we have

$$\|(\mathbf{I})_1\| + \|(\mathbf{J})_1\| \le \frac{1}{2\pi} \sup_{z \in \gamma_1} \|R(z, A)\| \times (4\omega) \times \|f\|_\infty \le \frac{2\omega}{|a' - a|\pi}\|f\|_\infty.$$

The combination of the above estimates yields that (4.35) is valid for the case when A is bounded and $f \in \mathcal{E}(H_{a',\omega})$.

For the case when A is unbounded, we consider its bounded approximations

$$A_n = (1 - \frac{1}{n})[n(n + \omega + iA)^{-1}]'(A - a')[n(n + \omega + iA)^{-1}] + a', \quad n \in \mathbb{N}^+.$$

Clearly, $A_n \in H_{a',(1-\frac{1}{n})\omega}$ and $\lim_{n\to\infty} A_n x = Ax$ for each $x \in \mathcal{D}(A)$. Moreover, $\|R(\lambda, A_n)\|$ is uniformly bounded in $n \ge 1$ and $\lambda \in \partial H_{b,\omega}$. This ensures that $\lim_{n\to\infty} R(\lambda, A_n)x = R(\lambda, A)x$ for all $x \in X$, and thus $f(A_n) \to f(A)$ strongly as $n \to \infty$. Therefore (4.35) is valid when $f \in \mathcal{E}(H_{a',\omega})$. Given that $f \in H^\infty(H_{a',\omega})$, an interplay of approximation $f_n(z) := f(z)(n/(n + \omega + iz)^2)$ and Proposition 4.19 yields that $f_n(A) \to f(A)$ strongly as $n \to \infty$. Thus inequality (4.35) is still valid for such f. This completes the proof. $\quad\square$

4.4. Regularizing operators and their convergence

Regularizing operators can be generated in different approaches. In this section, we shall study a family of regularizing operators that is constructed by the modified quasi-reversibility method, and their regularized solutions. We recall the constant b_T in (4.34), then for any interval $I := (0, a] \subset (0, b_T]$ and a family of functions $\{f_\alpha\}_{\alpha \in I}$ in $\mathcal{M}_\mathcal{A}$, we can define a family of operators $\{f_\alpha(\mathcal{A})\}_{\alpha \in I}$. To approximate the ill-posed backward evolution equation (1.1), we first solve the following regularized system

$$\begin{cases} u'(t) + f_\alpha(\mathcal{A})u(t) = 0, & 0 < t < T, \\ u(T) = f^\delta \end{cases} \tag{4.36}$$

for each $\alpha \in I$. Let $x_{\alpha,\delta}$ be the exact initial value of the solutions for (4.36), i.e., $x_{\alpha,\delta} := e^{Tf_\alpha(\mathcal{A})} f^\delta$. Using $x_{\alpha,\delta}$ as an initial value, we then consider the forward evolution equation

$$\begin{cases} u'(t) + Au(t) = 0, & 0 < t < T, \\ u(0) = x_{\alpha,\delta}, \end{cases} \tag{4.37}$$

which is well-posed due to Lemma 4.21 for each $\alpha \in I$ and $\delta > 0$, and use all the solutions to construct a family of regularizing operators for the backward evolution equation (1.1). More precisely, we approximate the exact solutions of (1.1) by the regularized solutions

$$M_{\alpha,t} f^\delta := e^{-tA} x_{\alpha,\delta} = e^{-tA} (e^{Tf_\alpha(\mathcal{A})} f^\delta), \quad t \in [0, T], \, \alpha \in I. \tag{4.38}$$

To study the convergence behavior of $\{M_{\alpha,t} : \alpha \in I, \, t \in [0, T]\}$, we now introduce some definitions and notations that are similar to those used for the unbounded sectorial operators in subsection 3.3. φ is called an *index function* (for $\mathcal{A}$) if $\varphi \in H^\infty(\Sigma_{\omega_T, b_T})$ and $\varphi(\mathcal{A})$ is injective. Then we define a subspace $X_\varphi \subset X$, consisting of all elements in $\mathcal{D}(\varphi(\mathcal{A})^{-1})$, endowed with the norm:

$$\|x\|_\varphi = \|\varphi(\mathcal{A})^{-1} x\| \quad \forall x \in \mathcal{D}(\varphi(\mathcal{A})^{-1}). \tag{4.39}$$

It is easy to verify that X_φ is a Banach space.

Definition 4.26. A pair of functions $(\rho_c, \rho_r) \in \mathcal{M}_{\mathcal{A}} \times C([0, b_T])$ is called a qualification pair if it holds that $\rho_c(z) \neq 0$ for $z \in \Sigma_{\omega_T, b_T}$, $\rho_r : [0, b_T] \mapsto \mathbb{R}^+$ is an increasing function, and $\rho_r(t) \to 0$ as $t \to 0^+$.

Definition 4.27. A family $\{g_\alpha\}_{\alpha \in I_s}$ is called a regularization associated with a qualification pair (ρ_c, ρ_c) if there exists some constant $R_1 > 0$ such that

$$\sup_{\lambda \in \Sigma_{\omega_T, b_T}} |1 - zg_\alpha(z)| \leq R_1 \quad \forall \alpha \in I, \tag{4.40}$$

$$\sup_{z \in \Sigma_{\omega_T, b_T}} |(1 - zg_\alpha(z))\rho_c(z)| \leq R_1 \rho_r(\alpha) \quad \forall \alpha \in I. \tag{4.41}$$

Definition 4.28. An index function (for $\mathcal{A}$) is said to be a proper index function if there exists some $c_\varphi > 0$ such that

$$|\varphi(z)| \leq c_\varphi \varphi(|z|) \quad \forall z \in \Sigma_{\omega_T, b_T}, \tag{4.42}$$

and $\varphi|_{(0,b_T]}$ is a non-negative and increasing function satisfying $\varphi(|z|) \to 0$ as $|z| \to 0$.

There are some significant differences between the index functions for half-strip and sectorial operators. In particular, the index function for a half-strip operator is increasing while it is decreasing for a sectorial operator.

Definition 4.29. We say that the qualification (ρ_c, ρ_r) covers an index function φ if there exists a constant $c' > 0$ such that

$$\sup_{z \in \Sigma_{\omega_T, b_T}, |z| \geq \alpha} \left| \frac{\varphi(z)}{\rho_c(z)} \right| \leq c' \frac{\varphi(\alpha)}{\rho_r(\alpha)}. \tag{4.43}$$

In this case we say that φ is covered by (ρ_c, ρ_r) with constant c'.

Following the argument in the proof of Proposition 3.10 with some necessary modifications, we can show the following result.

Proposition 4.30. *Let $\{g_\alpha\}_{\alpha \in I_s}$ be a regularization associated with some qualification pair (ρ_c, ρ_r). Then, $\mathcal{A}g_\alpha(\mathcal{A})$ converges strongly to the identity I as $\alpha \to 0^+$.*

Proposition 4.31. *Let $\{g_\alpha\}_{\alpha \in I_s}$ be a regularization associated with a qualification pair (ρ_c, ρ_r), and φ a proper index function covered by (ρ_c, ρ_r) with constant $c' > 0$. Then, we have the following convergence rate*

$$\|(I - \mathcal{A}g_\alpha(\mathcal{A}))x\| \leq C_{\varphi, h}\varphi(\alpha)\|x\|_\varphi \quad \forall x \in X_\varphi, \tag{4.44}$$

with $C_{\varphi, h} := C_h R_1(c' + c_\varphi)$, where the constants R_1, c_φ and c' are from (4.40), (4.42) and (4.43).

Proof. We can derive the result by a reasoning similar to that used in Proposition 3.14. So we give only a brief outline of the proof below. For any $x \in X_\varphi$, there exists $y \in X$ such that $x = \varphi(\mathcal{A})y$ and $\|y\| = \|x\|_\varphi$. It is easy to see that for each $\alpha \in I$,

$$\|(I - \mathcal{A}g_\alpha(\mathcal{A}))\varphi(\mathcal{A})y\|$$

$$\leq C_h \sup_{z \in \Sigma_{\omega_T, b_T}, |z| \geq \alpha} |(1 - zg(z))\varphi(z)| \|y\| + C_h \sup_{z \in \Sigma_{\omega_T, \alpha},} |(1 - zg(z))\varphi(z)| \|y\|$$

$$:= (\mathbf{I})_{1, \alpha} + (\mathbf{I})_{2, \alpha},$$

where we apply Theorem 2.1 (c) and assumption (**A**). Now using the arguments similar to that leading to (3.20), one can bound $(\mathbf{I})_{1, \alpha}$ and $(\mathbf{I})_{2, \alpha}$ by $C_h R_1 c' \varphi(\alpha)\|y\|$ and $C_h R_1 c_\varphi \varphi(\alpha)\|y\|$, respectively. This yields our desired result readily. $\square$

4.5. Convergence rates of the regularized solutions

Now, we are in a position to present our main results. To this end, we first establish a logarithmic convexity result under the assumption (**A**), which follows from the composition rule (Theorem 4.20) and the moment inequality for sectorial operators, which goes back to the 1960s and can be found in several references (see [17,32], for instance).

Lemma 4.32. *Let $(e^{-tA})_{t \geq 0}$ be the analytic semigroup generated by $-A$. Then, it holds for each $t \in (0, T)$ that*

$$\|e^{-tA}x\| \leq C_{co}\|x\|^{1-t/T}\|e^{-TA}x\|^{t/T},$$

where C_{co} depends only on t, and is uniformly bounded for all $t \in (0, T)$.

Theorem 4.33. *Let $\{f_\alpha\}_{\alpha \in I_s}$ be a family of functions in $\mathcal{M}_\mathcal{A}$ of the form:*

$$f_\alpha(\cdot) = \frac{1}{T} \ln(g_\alpha(\cdot)), \quad \alpha \in I,$$

where $\{g_\alpha\}_{\alpha \in I}$ satisfies the following conditions:
(i) *$\{g_\alpha\}_{\alpha \in I_s}$ is a regularization associated with a qualification pair (ρ_c, ρ_r), which covers a proper index function φ, and satisfies for a constant $C_\varphi^* > 0$,*

$$\sup_{z \in \Sigma_{\omega_T, b_T}} \left| \frac{z\varphi(z)}{\rho_c(z)} \right| \leq C_\varphi^*. \tag{4.45}$$

(ii) *The following estimate holds for some $\widetilde{c} > 0$ and $q > 0$,*

$$\sup_{z \in \Sigma_{\omega_T, b_T}} |g_\alpha(z)| \leq \frac{\widetilde{c}}{\sqrt[q]{\alpha}}, \quad \alpha \in I.$$

Then for $x^\dagger = u(0) \in X_\varphi$ with $\|x^\dagger\|_\varphi \leq Q$, we have the error estimate

$$\|M_{\alpha, t} f^\delta - u(t)\| \leq C_H \left(\frac{\delta}{\sqrt[q]{\alpha}} + Q\varphi(\alpha) \right)^{1-t} (\delta + Q\rho_r(\alpha))^t \quad \forall t \in [0, T], \quad \alpha \in I, \tag{4.46}$$

for some $C_H > 0$. Furthermore, under a priori parameter choice

$$\alpha(\delta) = \delta^\kappa \quad \text{with} \quad 0 < \kappa/q < 1, \tag{4.47}$$

we have the following estimate of the convergence rate

$$\|M_{\alpha, t} f^\delta - u(t)\| = \mathcal{O}\left((\delta^\eta + Q\varphi(\delta^\kappa))^{1-t} (\delta + Q\rho_r(\delta^\kappa))^t \right) \quad \text{as} \quad \delta \to +0,$$

for all $0 \leq t \leq T$, with $\eta := 1 - \kappa/q$.

Proof. For any fixed $\alpha \in I$, an interplay of Lemma 4.22 and condition **(ii)** implies that

$$x_{\alpha, \delta} := e^{Tf_\alpha(\mathcal{A})} f^\delta = g_\alpha(\mathcal{A}) f^\delta \tag{4.48}$$

is well-defined, so is the forward evolution equation (4.37). Then we have

$$\|M_{\alpha, 0} f^\delta - u(0)\| \leq \|g_\alpha(\mathcal{A})(f^\delta - f)\| + \|(g_\alpha(\mathcal{A})\mathcal{A} - I)x^\dagger\|$$
$$:= (\mathbf{J})_{1, \alpha} + (\mathbf{J})_{2, \alpha}.$$

By Lemma 4.22 and condition **(ii)**, $(\mathbf{J})_{1, \alpha}$ can be bounded by $\frac{C_h \widetilde{c}\delta}{\sqrt[q]{\alpha}}$. Applying Proposition 4.31, we have $(\mathbf{J})_{2, \alpha} \leq C_{\varphi, h}\varphi(\alpha)\|x^\dagger\|_\varphi$ for some $C_{\varphi, h} > 0$. Then the combination of these two estimates yields

$$\|M_{\alpha,0} f^\delta - u(0)\| \leq \frac{C_h \widetilde{c} \delta}{\sqrt[q]{\alpha}} + C_{\varphi,h} Q\varphi(\alpha). \tag{4.49}$$

On the other hand, we can compute at $t = T$:

$$\|M_{\alpha,T} f^\delta - e^{-AT} u(0)\| \leq \|Ag_\alpha(\mathcal{A})(f^\delta - f)\| + \|A(g(\mathcal{A})\mathcal{A} - I)x^\dagger\|$$
$$:= (\mathbf{J}')_{1,\alpha} + (\mathbf{J}')_{2,\alpha}.$$

By Definition 4.27 and Proposition 4.31, we can estimate $(\mathbf{J}')_{1,\alpha}$ by $C_h(1 + R_1)\delta$. Moreover, we can see from Definition 4.27 that

$$(\mathbf{J}')_{2,\alpha} \leq C_h \sup_{z \in \Sigma_{\omega_T, b_T}} |(1 - zg_\alpha(z))\rho_c(z) \frac{z\varphi(z)}{\rho_c(z)}| \|x^\dagger\|_\varphi \leq C_h C_\varphi^* R_1 \|x^\dagger\|_\varphi \rho_r(\alpha).$$

Then an application of Lemma 4.32 implies that

$$\|M_{\alpha,t} f^\delta - u(t)\| \leq C_H (\frac{\delta}{\sqrt[q]{\alpha}} + Q\varphi(\alpha))^{1-t}(\delta + Q\rho_r(\alpha))^t$$

for all $t \in [0, T]$, where $C_H := \max\{C_h\widetilde{c} + C_{\varphi,h}, C_h(1 + R_{1,}), C_h C_\varphi^* R_1\}\max\{C_{co}, 1\}$. Now the application of the parameter choice (4.47) completes the proof. $\square$

Remark 4.34. It is worth mentioning that the condition (4.45) is trivial if $\rho_c(z)$ decays reasonably slowly at origin, i.e., $|z|^\alpha \lesssim |\rho_c(z)|$ over the sector Σ_{ω_T, b_T} for some $0 < \alpha \leq 1$. In this case, any bounded index function will satisfy (4.45). This condition is reasonable in many applications, which shall be seen in subsection 4.6. On the other hand, the function $g(z) := z\varphi(z)$ is always an index function. If ρ_c is an index function, then an equivalent interpretation of the condition (4.45) is that X_g is embedded into X_{ρ_c} continuously.

To complete the picture of regularization, we now discuss some adaptive strategy. More precisely, we discuss a strategy of choosing the regularization parameter α without precise knowledge of index functions. Fix $q_* > 1$ and $\alpha_0 > \sqrt[q]{\delta}$, and set $\alpha_k := \alpha_0 q_*^k$, $k = 1, 2, \cdots, N$ such that $N = \max\{k \geq 1 : \alpha_{k-1} \leq a_s\}$, and

$$\Delta_{q_*} := \{\alpha_k; \quad k = 0, 1, \cdots, N\}.$$

Now we propose the following adaptive strategy:

Compute $x_{\alpha_k, \delta}$ successively for $\alpha_0, \alpha_1, \cdots$ as long as the following estimate holds:

$$\|x_{\alpha_k, \delta} - x_{\alpha_{k-1}, \delta}\| \leq 4C_K \frac{\delta}{\sqrt[q]{\alpha_{k-1}}} \quad \text{with } C_K := C_H \max\{Q, 1\}.$$

Then we choose the regularization parameter after the termination:

$$\hat{\alpha} := \max\{\alpha_k : \|x_{\alpha_k, \delta} - x_{\alpha_{k-1}, \delta}\| \leq 4C_K \frac{\delta}{\sqrt[q]{\alpha_{k-1}}}\}.$$

Let us define a class of index functions by setting

$$\mathcal{F}_{q_*,D} := \{\varphi;\ \varphi(q_*t) \le D\varphi(t)\ \forall t \in (0, b_T/q_*]\ \text{with constants}\ D > 0\ \text{and}\ q_* > 1\}. \quad (4.50)$$

If $\varphi|_{(0,b_T]}$ is also a concave function, it is not difficult to verify that $\varphi \in \mathcal{F}_{c,c}$ for any $c > 1$.

Theorem 4.35. *Under the same assumptions as in Theorem 4.33, the above adaptive strategy yields the following error estimate for any $\varphi \in \mathcal{F}_{q_*,D}$:*

$$\sup_{t\in[0,T]} \|M_{\hat{\alpha},t}f^\delta - u(t)\| \le C_{K,q} D \sqrt[q]{q_*}\varphi(\Theta_H^{-1}(\delta)),$$

where $C_{K,q_*} = 2C_h C_K(\frac{2\sqrt[q]{q_*}}{\sqrt[q]{q_*}-1} + 1)$ and $\Theta_H(t) := \sqrt[q]{t}\varphi(t),\ t \in I.$

Proof. Let

$$\alpha_* = \max\{\alpha \in \Delta_{q_*},\ \sqrt[q]{\alpha}\varphi(\alpha) \le \delta\},$$

and $\alpha_* = \alpha_l$ and $\hat{\alpha} = \alpha_m$ for some $1 \le l, m \le N$. In addition, we have $l \le m$. We first notice from the assumptions that φ is a proper index function, then we can use the monotonicity of $\varphi|_{(0,b_T]}$ and the estimate (4.49) to obtain

$$\|x_{\alpha_l,\delta} - x_{\alpha_{l-1},\delta}\| \le \|x^\dagger - x_{\alpha_l,\delta}\| + \|x^\dagger - x_{\alpha_{l-1},\delta}\|$$

$$\le C_K(\varphi(\alpha_l) + \frac{\delta}{\sqrt[q]{\alpha_l}} + \varphi(\alpha_{l-1}) + \frac{\delta}{\sqrt[q]{\alpha_{l-1}}})$$

$$\le 4C_K \frac{\delta}{\sqrt[q]{\alpha_{l-1}}}.$$

In view of this and the triangle's inequality, we derive

$$\|x_{\hat{\alpha},\delta} - x^\dagger\| \le \|x^\dagger - x_{\alpha_l,\delta}\| + \sum_{k=0}^{m-l-1} \|x_{\alpha_{m-k},\delta} - x_{\alpha_{m-k-1},\delta}\|$$

$$\le C_K(\varphi(\alpha_l) + \frac{\delta}{\sqrt[q]{\alpha_l}}) + \sum_{k=0}^{m-l-1} \frac{4C_K\delta}{\sqrt[q]{\alpha_{m-k-1}}}$$

$$\le \frac{2C_K\delta}{\sqrt[q]{\alpha_*}} + \frac{4C_K\delta}{\sqrt[q]{\alpha_*}} \sum_{k=1}^{m-l} \frac{1}{\sqrt[q]{q_*^{m-l-k}}}$$

$$\le \frac{C_{K,q_*}}{C_h} \frac{\delta}{\sqrt[q]{\alpha_*}}.$$

As $\sqrt[q]{q_*\alpha_*}\varphi(q_*\alpha_*) > \delta$, we can bound $\delta/\sqrt[q]{q_*\alpha_*}$ by $\varphi(q_*\alpha_*)$. Then we infer by noting that $\varphi \in \mathcal{F}_{q_*,D}$, the boundedness of semigroup $(e^{-tA})_{t\ge0}$, and the monotonicity of Θ_H:

$$\|M_{\hat{\alpha},t}f^\delta - u(t)\| \le C_h \|x_{\hat{\alpha},\delta} - x^\dagger\| \le C_{K,q_*} D \sqrt[q]{q_*}\varphi(\alpha_*)$$

$$\le C_{K,q_*} D \sqrt[q]{q_*}\varphi(\Theta_H^{-1}(\delta))\ \forall t \in [0, T]. \quad \square$$

4.6. Application to specific regularizations

We recall that operator A in (1.1) satisfies the condition **(A)** throughout this subsection. Then we consider the regularized backward evolution equation

$$\begin{cases} v'_{\alpha,\delta}(t) + A_\alpha v_{\alpha,\delta}(t) = 0, & 0 < t < T, \\ v_{\alpha,\delta}(T) = f^\delta, \end{cases} \tag{4.51}$$

where A_α is given by

$$A_\alpha := -\frac{1}{pT} \ln(\alpha^p + e^{-pTA}), \quad \alpha > 0, \tag{4.52}$$

with the parameter $p \geq 1$ satisfying $pT\omega_H \in [0, \frac{\pi}{2})$. In the sequel we shall use the parameters ω_T and b_T defined by (4.34).

Now we will approximate the backward evolution equation (1.1) with the family of regularized solutions $M_{\alpha,t} f^\delta$ defined by (4.38). Writing $\mathcal{A} := e^{-TA}$, A_α can be rewritten as

$$A_\alpha := \frac{1}{T} \ln g_\alpha(\mathcal{A}) \quad \alpha \in (0, b_T],$$

where $g_\alpha(z) := \sqrt[p]{\frac{1}{\alpha^p + z^p}}$ for $\alpha \in (0, b_T]$.

Lemma 4.36. *We have the following assertions:*
(a) There exists some positive constant γ_0, depending only on ω_T and b_T, such that

$$|z - \sqrt[p]{\alpha^p + z^p}| \leq \gamma_0 \alpha \quad \forall z \in \Sigma_{\omega_T, b_T}, \ \alpha \in (0, b_T].$$

(b) One can find a constant $\gamma > 0$, depending only on ω_T and b_T, such that

$$\left| \frac{z}{\sqrt[p]{(\alpha^p + z^p)}} - 1 \right| \leq \gamma \frac{\alpha}{\alpha + |z|} \quad \forall z \in \Sigma_{\omega_T, b_T}, \ \alpha \in (0, b_T].$$

Proof. From the binomial series for $(1+z)^{1/p}$ it follows immediately that

$$|\sqrt[p]{(1 + z^p)} - 1| \leq C_{ineq} |z| \quad \forall z \in \Sigma_{\omega_T, 1} \tag{4.53}$$

for some $C_{ineq} > 0$. Thus one has

$$|\lambda - \sqrt[p]{1 + \lambda^p}| = |\lambda| |1 - \sqrt[p]{\lambda^{-p} + 1}| \leq C_{ineq}$$

for $|\lambda| \geq 1$ and $\lambda \in \Sigma_{\omega_T}$. We then conclude that the function $f(\lambda) := |\lambda - \sqrt[p]{1 + \lambda^p}|$ is uniformly bounded on Σ_{ω_T}, which implies the first assertion.

Now let us deal with the assertion (*b*). By assertion (*a*), it suffices to show that there exists some constant $\gamma_1 > 0$ such that

$$\left| \frac{\alpha + |z|}{\sqrt[p]{(\alpha^p + z^p)}} \right| \leq \gamma_1 \quad \forall z \in \Sigma_{\omega_T, b_T}, \ \alpha \in (0, b_T]. \tag{4.54}$$

To this end, we only need to consider the asymptotic behavior of the function $|\frac{1+|z|}{\sqrt[p]{(1+z^p)}}|$ as $z \to \infty$. Setting $\lambda = z^p$, we can easily see that $\frac{(1+|\lambda|^{1/p})}{(1+\lambda)^{1/p}}$ is uniformly bounded on $\Sigma_{p\omega_T}$, using the fact that $p\omega_T < \frac{\pi}{2}$. This completes the proof. $\quad\square$

From Lemma 4.36 it follows immediately that $\{g_\alpha\}_{\alpha \in (0, b_T]}$ is a regularization associated with the qualification pair (z, α). On the other hand, using the following inequality:

$$\left| \frac{\alpha}{\sqrt[p]{\alpha^p + z^p}} \right| \leq \frac{1}{\cos \omega_T} \quad \forall z \in \Sigma_{\omega_T},$$

we know that condition (**ii**) in Theorem 4.33 holds with $q = 1$. By Remark 4.34, condition (4.45) in Theorem 4.33 is always true for any bounded index function. The following proposition gives a criterion for a proper index function φ to be covered by the qualification pair (z, α).

Proposition 4.37. *Let φ be a proper index function, then φ is covered by the qualification pair (z, α) with a constant $K > 0$ if one of the following conditions holds, where $K = c_\varphi$ for the cases (a) and (b), and $K = c_\varphi \varphi(b_T)/\varphi(\hat{t})$ for the cases (c) and (d), and c_φ is from Definition 4.28:*

(a) $\varphi(t)/t$ *is monotonically decreasing on* $(0, b_T]$.
(b) $\varphi(t)$ *is concave on* $(0, b_T]$.
(c) $\varphi(t)/t$ *is monotonically decreasing on* $(0, \hat{t}] \subset (0, b_T]$.
(d) $\varphi(t)$ *is concave on* $(0, \hat{t}] \subset (0, b_T]$.

Let us consider the function $\varphi(z) := \log^{-\beta}(1/z)$ with $\beta > 0$, which is a holomorphic function on Σ_{ω_T, b_T} due to the fact that $0 < b_T < 1$ and $0 < \omega_T < \frac{\pi}{2}$. In addition, we can easily check

$$|\varphi(z)| \leq \log^{-\beta}(1/|z|) \quad \forall z \in \Sigma_{\omega_T, b_T},$$

and the function $\varphi|_{[0, b_T]}$ is strictly increasing on $[0, b_T]$, and concave on $(0, e^{-(\beta+1)}]$. Obviously, the identity (4.45) holds trivially. In view of Proposition 4.37, the index function φ is covered by the qualification pair (z, α). By the reasoning above we know that Theorem 4.33 is applicable, and hence the following result follows immediately.

Corollary 4.38. *Let $x^\dagger = u(0) \in \mathcal{D}(A^\beta)$ with some $\beta > 0$, and $M_{\alpha,t}$ be the family of regularizing operators associated with (4.52). Then we have the following convergence rate*

$$\|M_{\alpha,t} f^\delta - u(t)\| = \mathcal{O}(\log^{-\beta(1-t/T)}(1/\sqrt{\delta})\sqrt{\delta}^{t/T}) \quad as \ \delta \to +0, \tag{4.55}$$

for any $t \in [0, T]$ with the parameter choice $\alpha = \sqrt{\delta}$.

Remark 4.39. By Lemma 3.6, A is also a sectorial operator. For each $\beta \in (0, 1)$, let us denote by $(B, D(B))$ the β power of A as a sectorial operator. Then it raises a natural question that whether B is identical to A^β. The answer is affirmative. Indeed, using [17, corollary 3.3.6] and Fubini's theorem, we have

$$B^{-1}x = \frac{1}{\Gamma(\beta)} \int_0^\infty t^{\beta-1} e^{-tA} x \, dt = A^{-\beta}x \quad \forall x \in \mathcal{D}(A^2).$$

As $D(A^2)$ is dense in X, and both $A^{-\beta}$ and B belong to $\mathcal{L}(X)$, we can extend the identity above for all $x \in X$. This proves our claim.

Compared with the existing results, e.g., [3, Theorem 4.4], which were for self-adjoint operators and Hilbert spaces, our results apply to much more general operators and spaces. Even for self-adjoint operators A, our smoothness requirement is also much weaker. Indeed, Corollary 4.38 requires that $x^\dagger \in D(A^\beta)$ for some $\beta > 0$, while it needs $x^\dagger \in \bigcap_{n=1}^\infty D(A^n)$ in [3, Theorem 4.4].

Example 4.40. Suppose that Ω is a bounded domain in $\mathbb{R}^n$ with smooth boundary $\partial\Omega$. Consider the heat equation with Dirichlet boundary condition of the form:

$$\begin{cases} \frac{\partial u(t,\xi)}{\partial t} - \sum_{i,j=1}^n D_j(\alpha(\xi)D_i u(t,\xi)) + \beta(\xi)u(t,\xi) = 0, & (t,\xi) \in (0, T) \times \Omega, \\ u(x,t) = 0, & (t,x) \in (0, T) \times \partial\Omega, \\ u(0,x) = u_0, \end{cases} \quad (4.56)$$

where $\alpha \in C^1(\overline{\Omega})$, and $\beta \in C(\overline{\Omega})$. We further assume that $\alpha(\xi)$ is real-valued and strictly positive, and $\mathfrak{Re}(\beta)(\xi) \geq 0$ for all $\xi \in \Omega$. Set $a := \min_{\xi \in \Omega}(a(\xi))$ and $\omega = \max_{\xi \in \Omega} \mathfrak{Im}(\beta(\xi))$. Let $\mathbb{H} = L^2(\Omega)$ and $A : \mathcal{D}(A) \subset \mathbb{H} \to \mathbb{H}$ be defined by

$$Au = -\sum_{i,j=1}^n D_j(\alpha(\xi)D_i u(\xi)) + \beta(\xi)u(\xi) \quad \forall u \in \mathcal{D}(A),$$

with the domain $\mathcal{D}(A) := H^2(\Omega) \cap H_0^1(\Omega)$. If we set $Cu = \mathfrak{Im}(\beta)(\xi)u(\xi)$ for any $u \in X$, and

$$Bu = -\sum_{i,j=1}^n D_j(\alpha(\xi)D_i u(\xi)) + \mathfrak{Re}(\beta)(\xi)u(\xi) \quad \forall u \in \mathcal{D}(A),$$

then it is not difficult to see that $A = B + iC$, and that both B and C are self-adjoint operator with $B \geq \frac{a}{C_p}$ and $-\omega \leq B \leq \omega$, where C_p is the constant in Poincaré inequality for the domain Ω, i.e., the minimal eigenvalue of the Laplacian in the domain Ω. Propositions 4.24 and 4.25 imply that $A \in H(a', \omega)$ and A has a bounded H^∞-calculus over $H_{a',\omega}$ for any $0 < a' < a/C_p$. So assumption (**A**) is fulfilled if both ω and T are small enough. Therefore, both Theorem 4.33 and Corollary 4.38 are applicable. In addition, A is also a sectorial operator, and its fractional power spaces have a deep connection with Sobolev spaces. To be more precise, $\mathcal{D}(A^\beta)$ is equivalent to $H^{2\beta}(\Omega)$ if $\beta \in [0, 3/4)$ and $\beta \neq \frac{1}{4}$ [38]. Thus Corollary 4.38 implies that

$$\|u_{\alpha,\delta}(t) - u(t)\| = \mathcal{O}(\log^{-\beta(1-t/T)}(1/\sqrt{\delta})\delta^{\frac{t}{2T}}) \quad \text{as} \quad \delta \to +0, \quad \forall t \in [0, T], \qquad (4.57)$$

for the parameter choice $\alpha = \sqrt{\delta}$ if $x^\dagger = u(0) \in H^{2\beta}(\Omega)$ with $\beta \in [0, 1/4]$.

Acknowledgments

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