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On unique determination of partially coated polyhedral scatterers with far field measurements

Hongyu Liu and Jun Zou[1]

Department of Mathematics, The Chinese University of Hong Kong, Shatin, NT, Hong Kong

E-mail: hyliu@math.cuhk.edu.hk and zou@math.cuhk.edu.hk

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Abstract
This work is a continuation of our early study in Liu and Zou (2006 Uniqueness in an inverse acoustic obstacle scattering problem for both sound-hard and sound-soft polyhedral scatterers *Inverse Problems* **22** 515–24; 2006 Uniqueness in determining multiple polygonal or polyhedral scatterers of mixed type *Technical Report* 2006-03(337) The Chinese University of Hong Kong) and addresses the unique determination of partially coated polyhedral scatterers in $\mathbb{R}^N$ $(N \geqslant 2)$ along with their surface impedance from far field data. Two global uniqueness results are established for this inverse problem with a scatterer consisting of multiple solid polyhedra: the first one is to determine such a scatterer of mixed sound-soft and impedance type by a single incident plane wave and the other is to determine such a scatterer of mixed sound-soft, sound-hard and impedance type by N different incident waves in the N-dimensional case with $N \geqslant 3$ and by only one incident wave for the two-dimensional case. Then we present some examples to show that as long as a scatterer admits the presence of (sound-hard) crack-type obstacles, then one cannot determine the scatterer uniquely by any less than N different incident plane waves. These examples also reveal that the uniqueness results achieved earlier in [15, 16] for polyhedral scatterers are optimal. Finally, the uniqueness results that have been solved or are still unsolved for the polyhedral-type scatterers with both solid and crack components are summarized in the conclusion.

1. Introduction

The purpose of the present paper is to continue our study on global uniqueness in inverse obstacle scattering with very general polyhedral scatterers in $\mathbb{R}^N$ $(N \geqslant 2)$. Let $\mathbf{D}$ be an impenetrable obstacle which is a compact set in $\mathbb{R}^N$ $(N \geqslant 2)$ and assumed to be the closure

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of the open complement of some unbounded connected Lipschitz domain $\mathbf{G}$ in $\mathbb{R}^N$. In the following, without loss of generality, we will assume that $\mathbf{G} = \mathbb{R}^N \backslash \mathbf{D}$. Let $u^i(x) = \exp\{ikx \cdot d\}$ be a plane wave, where $k > 0$ and $d \in \mathbb{S}^{N-1}$ are, respectively, the wave number and the incident direction. We define the scattering solution $u(x) := u(x; \mathbf{D}, k, d)$, the sum of the scattered field $u^s(x)$ and the incident field $u^i(x)$, as the solution to the Helmholtz system (see, e.g., [9, 17])

$$\begin{cases} \Delta u + k^2 u = 0 & \text{in } \mathbf{G}, \\ \lim_{r\to\infty} r^{(N-1)/2} \left(\dfrac{\partial u^s}{\partial r} - iku^s \right) = 0, \end{cases} \tag{1.1}$$

where $r = |x|$ for any $x \in \mathbf{G}$. The following three different boundary conditions will be considered for system (1.1):

$$u = 0 \qquad \text{on } \partial\mathbf{G} \quad \text{(sound-soft scatterer)}, \tag{1.2}$$

$$\frac{\partial u}{\partial \nu} = 0 \qquad \text{on } \partial\mathbf{G} \quad \text{(sound-hard scatterer)}, \tag{1.3}$$

$$\frac{\partial u}{\partial \nu} + i\lambda u = 0 \qquad \text{on } \partial\mathbf{G} \quad \text{(impedance-type scatterer)}, \tag{1.4}$$

where ν is the unit normal to $\partial\mathbf{G}$ directed into the interior of $\mathbf{G}$ and the surface impedance $\lambda(x) \in C(\partial\mathbf{G})$ satisfies $\lambda(x) \geqslant 0$ for $x \in \partial\mathbf{G}$.

It is known that there exists a unique solution $u \in H^1_{\mathrm{loc}}(\mathbf{G})$ to the Helmholtz system (1.1) associated with either of the boundary conditions (1.2), (1.3) and (1.4), see [17]. From the interior regularity for solutions of (1.1), we know that u is analytic in any compact set of $\mathbf{G}$ (see [9, 17]). Moreover, the asymptotic behaviour of the scattered wave u^s at infinity is governed by

$$u^s(x; \mathbf{D}, k, d) = \frac{e^{ik|x|}}{|x|^{(N-1)/2}} \left\{ u_\infty(\hat{x}; \mathbf{D}, k, d) + \mathcal{O}\left(\frac{1}{|x|}\right) \right\} \qquad \text{as} \quad |x| \to \infty, \tag{1.5}$$

uniformly for all directions $\hat{x} = x/|x| \in \mathbb{S}^{N-1}$. The analytic function $u_\infty(\hat{x}; \mathbf{D}, k, d)$ is defined on the unit sphere $\mathbb{S}^{N-1}$, known as the far field pattern (see [9]). Now, the inverse problem is to determine the obstacle $\mathbf{D}$ from knowledge of $u_\infty(\hat{x}; \mathbf{D}, k, d)$. It is seen that for fixed $k > 0$ and $d \in \mathbb{S}^{N-1}$, and all $\hat{x} \in \mathbb{S}^{N-1}$, this is a formally determined problem, since the data depend on the same number of variables, $N - 1$, as does the object we want to recover. There is a widespread belief that a sound-soft obstacle is uniquely determined by a single incident plane wave at arbitrarily fixed $k > 0$ and $d \in \mathbb{S}^{N-1}$, whereas a sound-hard obstacle is uniquely determined by a single incoming wave at some (probably small) fixed $k > 0$ and $d \in \mathbb{S}^{N-1}$ (see [10, 12]). However, this remains a challenging open problem. Recent progress on the uniqueness investigation has validated principally those beliefs in the case of polyhedral-type scatterers (see [1, 11, 15, 16]). In fact, we have stronger results, which will be summarized and extended to general scatterers in subsequent sections. It is noted that all of the literature mentioned so far on polyhedral scatterers has not included impedance-type obstacles in their investigations. One of the main reasons might be that the impedance hyperplane (i.e., an open subset of a hyperplane in $\mathbb{R}^N$ on which u assumes the impedance condition (1.4)) does not have those fine properties of Dirichlet and Neumann hyperplanes (see [15] for the definitions of Dirichlet and Neumann hyperplanes), which can be analytically extended in $\mathbf{G}$ and can be used for the reflection arguments of solution u to the Helmholtz system (1.1). But as one can see, the impedance boundary condition is more realistic in real applications. In particular, a scattering problem is proposed in [6] for a time-harmonic electromagnetic plane wave with the obstacle being a perfectly conducting infinite cylinder in $\mathbb{R}^3$, that is partially coated by a thin dielectric material. In the TM mode, the polarized electric field leads to the Helmholtz equation (1.1) in $\mathbb{R}^2$ associated with the mixed Dirichlet and impedance boundary conditions. Recently, the inverse problem of determining the shape of an obstacle together with its surface impedance from the measurement of the far field pattern has been extensively investigated, where the linear sampling method was shown to be an effective tool for the reconstruction, see, e.g., [4–7]. On the theoretical side, there is still little work on the unique determination of such partially coated obstacles other than the known result due to Kirsch and Kress by infinitely many incident waves (see [13]; see also [10] and remark 2.5 in section 2). This problem will be the focus in sections 2 and 3 where we will prove that a partially coated polyhedral scatterer can be uniquely determined by finitely many incident plane waves.

In section 2 and the concluding section 6, we will also summarize the existing uniqueness results for polyhedral scatterers and extend in several aspects. Finally, in section 5, we shall present some examples to demonstrate that as long as a scatterer admits the presence of (sound-hard) crack-type obstacles (see section 5 for the description of a crack-type scatterer), then one cannot uniquely determine the scatterer by any fewer than N different incident plane waves. These examples also reveal that the uniqueness results established earlier in [15, 16] for polyhedral scatterers are optimal. In certain circumstances, we will point out that our analysis works also for general scatterers other than polyhedral type.

2. Uniqueness with partially coated obstacles

We consider the scattering system (1.1) defined in the exterior to the compact scatterer $\mathbf{D}$ in $\mathbb{R}^N$ $(N \geqslant 2)$. We will assume that $\mathbf{D}$ consists of a finite number of compact polyhedra in $\mathbb{R}^N$, namely $\mathbf{D} = \cup_{l=1}^{n} D_l$, with each D_l being a compact polyhedron in $\mathbb{R}^N$, and $\mathbf{G} := \mathbb{R}^N \backslash \mathbf{D}$ is connected.

We are interested in the case that the boundary $\partial\mathbf{D}$ of scatterer $\mathbf{D}$ is partially coated such that $\partial\mathbf{D}$ is divided into two different parts, on which the boundary conditions of Dirichlet and impedance type are respectively specified. That is, we assume that $\partial\mathbf{D}$ has a Lipschitz dissection (see [17]) $\partial\mathbf{D} = \partial\mathbf{D}_D \cup \mathbf{C} \cup \partial\mathbf{D}_I$, where $\partial\mathbf{D}_D$ and $\partial\mathbf{D}_I$ are disjoint and relatively open subsets of $\partial\mathbf{D}$ having $\mathbf{C}$ as their common boundary on $\partial\mathbf{D}$, and the total field u to system (1.1) satisfies

$$u = 0 \quad \text{on} \quad \partial\mathbf{D}_D, \qquad \frac{\partial u}{\partial \nu} + \mathrm{i}\lambda u = 0 \quad \text{on} \quad \partial\mathbf{D}_I, \tag{2.1}$$

where $\lambda(x) \in C(\overline{\partial\mathbf{D}_I})$ and $\lambda(x) \geqslant \lambda_0 > 0$ for $x \in \partial\mathbf{D}_I$. The first major result in this paper is to establish the following global uniqueness with such partially coated polyhedral scatterers.

Theorem 2.1. *For any fixed $k_0 > 0$ and $d_0 \in \mathbb{S}^{N-1}$, let $\mathbf{D}$ and $\tilde{\mathbf{D}}$ be two polyhedral scatterers on which the condition (2.1) is enforced with respective surface impedance $\lambda(x)$ and $\tilde{\lambda}(x)$. Then we have $\mathbf{D} = \tilde{\mathbf{D}}$ and $\lambda = \tilde{\lambda}$, as long as $u_\infty(\hat{x}; \mathbf{D}, \lambda, k_0, d_0) = u_\infty(\hat{x}; \tilde{\mathbf{D}}, \tilde{\lambda}, k_0, d_0)$ for $\hat{x} \in \mathbb{S}^{N-1}$.*

We remark that if $\partial\mathbf{D}_D = \emptyset$ or $\partial\mathbf{D}_I = \emptyset$ in (2.1), we will have a pure Dirichlet or impedance boundary condition problem, for which the uniqueness result in theorem 2.1 still holds. But for the pure Dirichlet case, the result holds for much more general polyhedral scatterers which can include the crack-type scatterers simultaneously, see [1, 15].

Next, we consider the more general case than (2.1) by allowing $\lambda(x)$ to vanish on part of $\partial\mathbf{D}$; more specifically, we shall consider the following boundary conditions:

$$u = 0 \quad \text{on } \partial\mathbf{D}_D, \qquad \frac{\partial u}{\partial \nu} = 0 \quad \text{on } \partial\mathbf{D}_N, \qquad \frac{\partial u}{\partial \nu} + \mathrm{i}\lambda u = 0 \quad \text{on } \partial\mathbf{D}_I, \tag{2.2}$$

where $\partial\mathbf{D}_D$, $\partial\mathbf{D}_N$ and $\partial\mathbf{D}_I$ are disjoint, relative open subsets of $\partial\mathbf{D}$ and form a Lipschitz dissection of $\partial\mathbf{D}$. We will assume that $\lambda(x) \in C(\overline{\partial\mathbf{D}_I})$ and $\lambda(x) \geqslant \lambda_0 > 0$. Then we have the following uniqueness results:

Theorem 2.2. *For any fixed* $\mathrm{k}_0 > 0$ *and* N *linearly independent directions* $d_1, \ldots, d_N \in \mathbb{S}^{N-1}$, *let* $u_1(x), \ldots, u_N(x)$ *be the total fields of the scattering problem associated with the boundary condition (2.2) corresponding to the incident waves* $\exp\{\mathrm{i}k_0 x \cdot d_1\}, \ldots, \exp\{\mathrm{i}k_0 x \cdot d_N\}$, *respectively. Let* $\mathbf{D}$ *and* $\tilde{\mathbf{D}}$ *be two polyhedral scatterers on which the condition (2.2) is enforced with respective surface impedance* $\lambda(x)$ *and* $\tilde{\lambda}(x)$. *Then we have* $\mathbf{D} = \tilde{\mathbf{D}}$ *and* $\lambda = \tilde{\lambda}$, *as long as* $u_\infty(\hat{x}; \mathbf{D}, \lambda, k_0, d_j) = u_\infty(\hat{x}; \tilde{\mathbf{D}}, \tilde{\lambda}, k_0, d_j)$ *for* $j = 1, \ldots, N$ *and* $\hat{x} \in \mathbb{S}^{N-1}$.

It is noted that in (2.2), if $\partial\mathbf{D}_I = \emptyset$, this will lead to a mixed Dirichlet and Neumann problem, and in such a situation we have a much stronger uniqueness result for much more general type of polyhedral scatterers which can include crack-type obstacles simultaneously (cf [16]). Starting from now and throughout the rest of the paper, *by all possible physical properties of an underlying scatterer we shall always mean the sound-soft, sound-hard and impedance boundary conditions, or their mixed type with partial coatings, together with the surface impedance*. In this sense, theorem 2.2 can be reformulated as follows.

Theorem 2.3. *For any fixed* $k_0 > 0$ *and* N *linearly independent directions* $d_j \in \mathbb{S}^{N-1}$, $j = 1, 2, \ldots, N$, *let* $\mathbf{D}$ *and* $\tilde{\mathbf{D}}$ *be two polyhedral scatterers with unknown physical properties and unknown number of component polyhedral obstacles. Then we have* $\mathbf{D} = \tilde{\mathbf{D}}$, *and both scatterers have the same physical properties and the same number of component polyhedral obstacles, as long as* $u_\infty(\hat{x}; \mathbf{D}, k_0, d_j) = u_\infty(\hat{x}; \tilde{\mathbf{D}}, k_0, d_j)$ *for* $j = 1, \ldots, N$ *and* $\hat{x} \in \mathbb{S}^{N-1}$.

If we do not allow the scatterers with impedance surface, i.e., $\partial\mathbf{D}_I = \emptyset$, and only consider polygonal scatterers in $\mathbb{R}^2$ consisting of finitely many compact sound-soft or sound-hard polygons, there is a very strong uniqueness in [16] by means of only a single incoming wave. However, following the spirit in [16], together with the help of lemma 4.3 given in this paper to deal with the impedance boundary condition (namely $\partial\mathbf{D}_I \neq \emptyset$), we can show the following more general results allowing all possible physical conditions.

Theorem 2.4. *For any fixed* $d_0 \in \mathbb{S}^1$ *and* $k_0 > 0$, *let* $\mathbf{D}$ *and* $\tilde{\mathbf{D}}$ *be two polygonal scatterers in* $\mathbb{R}^2$ *with unknown physical properties. Then we have* $\mathbf{D} = \tilde{\mathbf{D}}$, *and both scatterers have the same physical properties, as long as* $u_\infty(\hat{x}; \mathbf{D}, k_0, d_0) = u_\infty(\hat{x}; \tilde{\mathbf{D}}, k_0, d_0)$ *for* $\hat{x} \in \mathbb{S}^1$.

Proof. The theorem can be proved in a straightforward way following the proof of theorem 2.2 in [16] together with the help of lemma 4.3 given in section 4 of the present paper. □

Remark 2.5. A similar uniqueness result to theorem 2.1 for the inverse scattering problem with general smooth obstacles associated with mixed Dirichlet, Neumann and impedance boundary conditions was essentially established in [13] by means of infinitely many incident plane waves (see also [10]); that is, under the assumptions that $u_\infty(\hat{x}; \mathbf{D}, \lambda, k_0, d) = u_\infty(\hat{x}; \tilde{\mathbf{D}}, \tilde{\lambda}, k_0, d)$ for fixed $k_0 > 0$ and all possible $\hat{x}, d \in \mathbb{S}^{N-1}$, we have $\mathbf{D} = \tilde{\mathbf{D}}$ and $\lambda = \tilde{\lambda}$.

3. Proof of theorem 2.1

We first introduce some notation and definitions for subsequent use. $B_r(x)$ shall denote an open ball in $\mathbb{R}^N$ centred at x with radius r, and $\bar{B}_r(x)$ and $S_r(x)$ denote, respectively, the closure and the boundary of $B_r(x)$. Unless specified otherwise, ν shall always denote the outward normal to a concerned domain or the normal to a line or a hyperplane. Also we may often write $u(E) = 0$ for any subset $E \subset \mathbb{R}^N$ if $u(x) = 0$ for $x \in E$. The distance between two sets $\mathcal{A}$ and $\mathcal{B}$ is defined by $\mathbf{d}(\mathcal{A}, \mathcal{B}) = \inf_{x\in\mathcal{A}, y\in\mathcal{B}} |x - y|$. Finally, a curve $\gamma = \gamma(t)$ $(t \geqslant 0)$ is said to be regular if it is C^1-smooth and $\frac{\mathrm{d}}{\mathrm{d}t}\gamma(t) \neq 0$. In what follows, $u(x)$ shall denote the total field to the system (1.1) and (2.1) with fixed k and d, namely $u(x) = \exp\{\mathrm{i}kx \cdot d\} + u^s(x)$. The following definition is introduced in [15]:

Definition 3.1. *$\mathcal{D}_u$ is called a Dirichlet set of u in $\mathbf{G}$, if*

$\mathcal{D}_u = \{x \in \mathbf{G};\ u\ |_{\Pi\cap B_r(x)\cap\mathbf{G}} = 0$ *for some* $r > 0$ *and hyperplane* Π *passing through* $x\}$.

Since u is analytic in any compact subset of $\mathbf{G}$, by using the analytic continuation and the asymptotic expression (1.5), one can show (see lemmas 4 and 5, [15])

Lemma 3.2.

(i) For any $x \in \mathcal{D}_u$, let $\widetilde{\Pi}$ be the corresponding open-connected component of $\Pi\backslash\mathbf{D}$ containing x, then $u(\widetilde{\Pi}) = 0$. $\widetilde{\Pi}$ must be an open-connected subset of a hyperplane with its boundary on $\partial\mathbf{G}$ and will be referred to as a Dirichlet hyperplane;
(ii) The Dirichlet set $\mathcal{D}_u$ and all Dirichlet hyperplanes are bounded.

The following lemma is crucial for our subsequent investigation.

Lemma 3.3. *Let $E \subset \mathbf{G}$ be a bounded polyhedral domain and $\partial E = \overline{\partial E_1} \cup \overline{\partial E_2}$ with ∂E_1 and ∂E_2 being two disjoint relative open subsets of ∂E. If $u = 0$ on ∂E_1 and $\partial u/\partial\nu + \mathrm{i}\xi u = 0$ (or $\partial u/\partial\nu - \mathrm{i}\xi u = 0$) on ∂E_2, where ν is the unit normal to ∂E directed to the interior of E and $\xi(x) \in C(\partial E_2)$ is nonnegative, then $\xi = 0$ on ∂E_2.*

Proof. We consider only the case that $\partial u/\partial\nu + \mathrm{i}\xi u = 0$ on ∂E_2, whereas the case with $\partial u/\partial\nu - \mathrm{i}\xi u = 0$ on ∂E_2 can be proved in a similar manner. It suffices to show that there is no open portion of ∂E_2 on which $\xi(x) > 0$, then the lemma is proved by the continuity of $\xi(x)$. Assume contrarily that $\Sigma \subset \partial E_2$ is a relative open subset of ∂E_2 such that $\xi(x) > 0$ on Σ. Without loss of generality, we may assume that Σ is contained in the interior of one of the faces which form the entire boundary ∂E. Noting that u solves the Helmholtz equation in E, we have by Green's formula that

$$-k^2 \int_E |u|^2\,\mathrm{d}x + \int_E |\nabla u|^2\,\mathrm{d}x = \int_{\partial E} \frac{\partial u}{\partial \nu}\bar{u}\,\mathrm{d}s, \tag{3.1}$$

which further implies by the homogeneous boundary condition

$$-k^2 \int_E |u|^2\,\mathrm{d}x + \int_E |\nabla u|^2\,\mathrm{d}x = -\mathrm{i}\int_{\{x\in\partial E_2:\xi(x)>0\}} \xi(x)|u|^2\,\mathrm{d}s. \tag{3.2}$$

By taking the imaginary part of (3.2), we see that

$$\int_{\{x\in\partial E_2:\xi(x)>0\}} \xi(x)|u|^2\,\mathrm{d}s = 0,$$

hence we know $\int_\Sigma \xi(x)|u|^2\,\mathrm{d}s = 0$. Since $\xi \in C(\partial E_2)$ and $\xi > 0$ on Σ, we easily deduce that $u(\Sigma) = 0$, so $\partial u/\partial\nu(\Sigma) = 0$ by the impedance boundary condition on ∂E_2. Finally, by

Holmgren's theorem (see theorem 6.12 in [8]), we see that $u = 0$ in $\mathbf{G}$, which contradicts lemma 3.2. The proof is completed. □

With the help of lemma 3.3, we are now ready to prove theorem 2.1, which is done in the remainder of this section.

We first show that $\lambda = \tilde{\lambda}$ if it is already known that $\mathbf{D} = \tilde{\mathbf{D}}$. We write $u = u(\mathbf{D})$ and $\tilde{u} = u(\tilde{\mathbf{D}})$ to denote, respectively, the total fields corresponding to the obstacles $\mathbf{D}$ and $\tilde{\mathbf{D}}$ associated with the fixed incident wave $u^i = \exp\{ik_0 x \cdot d_0\}$. Since $u_\infty(\mathbf{D}) = u_\infty(\tilde{\mathbf{D}})$, we have $u = \tilde{u}$ in $\mathbf{G} = \mathbb{R}^N\backslash\mathbf{D} = \mathbb{R}^N\backslash\tilde{\mathbf{D}}$ by Rellich's theorem (see theorem 2.13, [9]), hence $u = \tilde{u}$ and $\partial u/\partial\nu = \partial\tilde{u}/\partial\nu$ on $\partial\mathbf{D}$. We remark that these equalities hold on $\partial\mathbf{D}$ only in the weak sense. On the other hand, one can see that if $x \in \partial\mathbf{D}$ is an interior point of one of the faces forming $\partial\mathbf{D}$, then it is a regular point for the forward problem, hence those relations hold in the classical sense. By the boundary conditions, we know

$$u = 0 \qquad \text{on } \partial\mathbf{D}_D, \qquad \frac{\partial u}{\partial\nu} + i\lambda u = 0 \qquad \text{on } \partial\mathbf{D}_I,$$

and

$$\tilde{u} = 0 \qquad \text{on } \partial\tilde{\mathbf{D}}_D, \qquad \frac{\partial\tilde{u}}{\partial\nu} + i\tilde{\lambda}\tilde{u} = 0 \qquad \text{on } \partial\tilde{\mathbf{D}}_I.$$

First, it must have $\partial\mathbf{D}_D = \partial\tilde{\mathbf{D}}_D$, otherwise we can easily deduce that $u = \partial u/\partial\nu = 0$ on an relative open portion $\Sigma \subset \partial\mathbf{D}$, which gives $u = 0$ in $\mathbf{G}$ by Holmgren's theorem, contradicting lemma 3.2. Hence, we know $\partial\mathbf{D}_D = \partial\tilde{\mathbf{D}}_D$, which then leads to $\partial\mathbf{D}_I = \partial\tilde{\mathbf{D}}_I$. Next, by observing $(\lambda - \tilde{\lambda})u = 0$ on $\partial\mathbf{D}_I$, we show that there is no open portion of $\partial\mathbf{D}_I$ over which $\lambda \neq \tilde{\lambda}$. In fact, if there is an open portion $\tilde{\Sigma} \subset \partial\mathbf{D}_I$ such that $\lambda \neq \tilde{\lambda}$, then it is easily deduced that $u = 0$ and $\frac{\partial u}{\partial\nu} = 0$ on $\tilde{\Sigma}$, which gives a contradiction by Holmgren's theorem. Therefore, $\lambda = \tilde{\lambda}$ by continuity.

Now we proceed to the major part of the proof and show $\mathbf{D} = \tilde{\mathbf{D}}$ by contradiction. Assume that $\mathbf{D} \neq \tilde{\mathbf{D}}$. Let Ω be the unbounded connected component of $\mathbb{R}^N\backslash(\mathbf{D} \cup \tilde{\mathbf{D}})$. Since both $\mathbf{D}$ and $\tilde{\mathbf{D}}$ are bounded, we know that the unbounded component Ω is unique and $\partial\Omega$ forms the boundary of a polyhedral domain. Using $u_\infty(\mathbf{D}) = u_\infty(\tilde{\mathbf{D}})$, we see $u = \tilde{u}$ over Ω by Rellich's theorem, that implies

$$u = \tilde{u} \quad \text{and} \quad \partial u/\partial\nu = \partial\tilde{u}/\partial\nu \qquad \text{on} \quad \partial\Omega. \tag{3.3}$$

Without loss of generality, we can assume that $D^* := (\mathbb{R}^N\backslash\overline{\Omega})\backslash\mathbf{D}$ is nonempty. It is easy to see that $D^* \subset \tilde{\mathbf{D}}$, so D^* is bounded. Moreover, by choosing a sub-connected component if necessary, we may assume D^* is connected. Clearly, D^* forms a polyhedral domain in $\mathbf{G} = \mathbb{R}^N\backslash\mathbf{D}$ and u is defined in D^*. Furthermore, by noting $\partial D^* \subset (\partial\Omega \cup \partial\mathbf{D})$, from (3.3) and the homogeneous boundary condition, we see that for any regular point $x \in \partial D^*$, either $u(x) = 0$ or $(\partial u/\partial\nu + i\lambda u)(x) = 0$, or $(\partial u/\partial\nu + i\tilde{\lambda}u)(x) = 0$. But the latter two cases cannot happen by using lemma 3.3, otherwise both λ and $\tilde{\lambda}$ would vanish on ∂D^*, this contradicts our positiveness assumption on λ and $\tilde{\lambda}$. Therefore, we must have $u = 0$ on ∂D^*. Since $\partial D^* \subset \partial\Omega \cup \partial\mathbf{D}$ and $\mathbb{R}^N\backslash\mathbf{D}$ is connected, it is easy to see that $\partial D^*\backslash\partial\mathbf{D} \neq \emptyset$. Hence, there exists a point $\tilde{x}' \in \partial D^*\backslash\partial\mathbf{D}$. We can also assume that $\tilde{x}'$ belongs to the interior of one of the faces of ∂D^*, and so there exists a hyperplane Π_1 and $r > 0$ such that $\tilde{x}' \in \Pi_1 \cap B_r(\tilde{x}') \subset \partial D^*\backslash\partial\mathbf{D}$. Obviously, we have $u = 0$ on $\Pi_1 \cap B_r(\tilde{x}')$, and hence $\tilde{x}' \in \mathcal{D}_u$ and $\Pi_1 \cap B_r(\tilde{x}')$ is contained in a Dirichlet hyperplane of u in $\mathbf{G}$, which we denote by $\widetilde{\Pi}_1$.

In the following, the symbol Π_l for an integer l shall always represent a hyperplane in $\mathbb{R}^N$, which contains a Dirichlet hyperplane $\widetilde{\Pi}_l$. We also use R_l to denote the reflection in $\mathbb{R}^N$ with respect to Π_l. Since $\mathbf{G}$ is unbounded and connected, the open set $\mathbf{G}\backslash\widetilde{\Pi}_1$ must contain an unbounded open-connected component, which we denote by $\mathbf{G}'$. By lemma 3.2, $\widetilde{\Pi}_1$ is bounded, which together with the boundedness of $\partial\mathbf{G}$ shows that the unbounded component $\mathbf{G}'$ is unique. Noting that every point on $\widetilde{\Pi}_1$ is in $\mathbf{G}$ and thus can be connected to infinity in $\mathbf{G}$, we have $\widetilde{\Pi}_1 \subset \partial\mathbf{G}'$. Next, we set $x_1 := \tilde{x}' \in \widetilde{\Pi}_1$ and let $\gamma = \gamma(t)$ $(t \geqslant 0)$ be a regular curve such that $\gamma(0) = x_1$, $\gamma(t)$ $(t > 0)$ lies entirely in $\mathbf{G}'$ and $\lim_{t\to\infty} |\gamma(t)| = +\infty$. Through our above construction, we see that $\gamma(t) \in \widetilde{\Pi}_1$ iff $t = 0$. Then we set $t_1 = 0$ and $r_0 = \mathbf{d}(\gamma, \mathbf{D})/2$. Noting that γ is a closed set in $\mathbb{R}^N$ and $\mathbf{D}$ is compact, the distance $r_0 > 0$ is attainable. Obviously, we have $\bar{B}_{r_0}(x) \subset \mathbf{G}$ for any point $x \in \gamma(t)$.

Let $\tilde{x}_2^+ = \gamma(\tilde{t}_2) \in S_{r_0}(x_1) \cap \gamma$ and $\tilde{x}_2^- \in S_{r_0}(x_1)$ be the symmetric point of $\tilde{x}_2^+$ with respect to Π_1. By the fundamental property of the connected set we know that γ must intersect $S_{r_0}(x_1)$, but the intersection need not necessarily be a unique point. For definiteness, we take $\tilde{t}_2 = \max\{t > 0; \gamma(t) \in S_{r_0}(x_1)\}$. Now, let G_1^+ be the connected component of $\mathbf{G}\backslash\widetilde{\Pi}_1$ containing $\tilde{x}_2^+$ and G_1^- be the connected component of $\mathbf{G}\backslash\widetilde{\Pi}_1$ containing $\tilde{x}_2^-$. It is remarked that it may happen that $G_1^+ = G_1^-$. Then let E_1^+ be the connected component of $G_1^+ \cap R_1(G_1^-)$ containing $\tilde{x}_2^+$ and E_1^- be the connected component of $G_1^- \cap R_1(G_1^+)$ containing $\tilde{x}_2^-$. Observe that $E_1^+ = R_1(E_1^-)$, and if we set $E_1 = E_1^+ \cup \widetilde{\Pi}_1 \cup E_1^-$, then E_1 contains the closed ball $\bar{B}_{r_0}(x_1)$ and is symmetric with respect to Π_1. Moreover, E_1 is a connected open set with the boundary composed of subsets of $\partial\mathbf{D}$ and $R_1(\partial\mathbf{D})$. For any $x \in \mathbf{G}$, define $R_1u(x) = u(R_1(x))$, then one can easily verify that $u(x) + R_1u(x)$ is a solution to the Helmholtz equation in E_1 with zero Dirichlet and Neumann data on $\widetilde{\Pi}_1 \cap \bar{B}_{r_0}(x_1)$, therefore $u(x) = -R_1u(x)$ in E_1 by Holmgren's theorem, i.e., u is odd symmetric in E_1 with respect to the hyperplane Π_1. This indicates $u\,|_{E_1\cap\Pi_1} = 0$. Next, we show that E_1 is bounded. Clearly, we first see ∂E_1, $\partial G_1^{\pm}$ and $R_1\big(\partial G_1^{\pm}\big)$ are bounded by our construction. If E_1 is unbounded, then E_1 would contain $\mathbb{R}^N\backslash B_r(x_1)$ for some sufficiently large $r > 0$. Then using $u\,|_{E_1\cap\Pi_1} = 0$ and analytic continuation, $\Pi_1\backslash B_r(x_1)$ are parts of some Dirichlet hyperplanes. But $\Pi_1\backslash B_r(x_1)$ is unbounded, so it contradicts with lemma 3.2. Hence, E_1 is bounded and it is obviously a polyhedral domain in $\mathbf{G}$. Now by the unboundedness of γ, there must exist a $t_2 > \tilde{t}_2$ such that $x_2 = \gamma(t_2) \in \partial E_1$. Noting ∂E_1 is composed of subsets of $\partial\mathbf{D}$ and $R_1(\partial\mathbf{D})$, for any $x \in \partial E_1 \cap \partial\mathbf{D}$, we clearly have from the specified boundary condition that either $u(x) = 0$ or $\frac{\partial u}{\partial \nu}(x) + \mathrm{i}\lambda u(x) = 0$, where ν is the unit normal to ∂E_1 at x directed to the interior of E_1. Whereas for any $x \in \partial E_1 \cap R_1(\partial\mathbf{D})$, by making use of the fact that $u(x) = -R_1u(x)$ in E_1 and that $\frac{\partial u}{\partial \nu}(x) = -\frac{\partial u}{\partial \nu_R}(R_1(x))$, we can easily deduce that either $u(x) = 0$ or $u(x) + \lambda(R_1(x))\frac{\partial u}{\partial \nu}(x) = -u(R_1(x)) - \lambda(R_1(x))\frac{\partial u}{\partial \nu_R}(R_1(x)) = 0$, where ν and ν_R are, respectively, the unit normals to ∂E_1 at x and $R_1(x)$ both directed to the interior of E_1. Then by lemma 3.3, we must have that $u = 0$ on ∂E_1, otherwise, we would get $\lambda = 0$ and this is a contradiction to our assumption that λ is positive. Thus by analytic continuation, there exists a Dirichlet hyperplane passing through x_2, which we denote by $\widetilde{\Pi}_2$, and we know $x_2 = \gamma(t_2) \in \mathcal{D}_u$. Furthermore, we may assume that $\gamma(t_2)$ is the 'last' point on γ to intersect $\widetilde{\Pi}_2$, that is,

$$t_2 = \max\{t > 0; \gamma(t) \in \widetilde{\Pi}_2\} < \infty.$$

From above, we see the following two facts, which shall be crucial: $\widetilde{\Pi}_2$ is different from $\widetilde{\Pi}_1$, since $\widetilde{\Pi}_1$ intersects γ only at x_1; the length of $\gamma(t)$ from t_1 to t_2 is larger than r_0, i.e.,

$$|\gamma(t_1 \leqslant t \leqslant t_2)| \geqslant |\gamma(t_1 \leqslant t \leqslant \tilde{t}_2)| \geqslant r_0.$$

Next, let $\tilde{x}_3^+ = \gamma(\tilde{t}_3) \in S_{r_0}(x_2) \cap \gamma$, and $\tilde{x}_3^- \in S_{r_0}(x_2)$ be the symmetric point of $\tilde{x}_3^+$ with respect to Π_2, then let G_2^+ be the connected component of $\mathbf{G}\backslash\widetilde{\Pi}_2$ containing $\tilde{x}_3^+$, and G_2^- be the connected component of $\mathbf{G}\backslash\widetilde{\Pi}_2$ containing $\tilde{x}_3^-$. Let E_2^+ be the connected component of $G_2^+ \cap R_2(G_2^-)$ containing $\tilde{x}_3^+$ and E_2^- be the connected component of $G_2^- \cap R_2(G_2^+)$ containing $\tilde{x}_3^-$. Set $E_2 = E_2^+ \cup \widetilde{\Pi}_2 \cup E_2^-$, then we see E_2 contains the closed ball $\bar{B}_{r_0}(x_2)$, and its boundary is composed of subsets of $\partial\mathbf{D}$ and $R_2(\partial\mathbf{D})$. By a similar argument as used earlier for deriving $x_2 = \gamma(t_2)$ and $\widetilde{\Pi}_2$, there exists a point $x_3 = \gamma(t_3)$ $(t_3 > t_2)$ and a Dirichlet hyperplane $\widetilde{\Pi}_3$ passing through x_3. Again, we may assume that x_3 is the 'last' point to pass through Π_3. We see that $\widetilde{\Pi}_3$ is different from $\widetilde{\Pi}_1$ and $\widetilde{\Pi}_2$, since $x_1 = \gamma(t_1)$ and $x_2 = \gamma(t_2)$ are, respectively, the last point to pass through $\widetilde{\Pi}_1$ and $\widetilde{\Pi}_2$, and the length of $\gamma(t)$ from t_2 to t_3 is larger than r_0, i.e.,

$$|\gamma(t_2 \leqslant t \leqslant t_3)| \geqslant r_0.$$

Continuing with the above procedure, we can construct a strictly increasing sequence $\{t_n\}_{n=1}^{\infty}$ such that for any $n, x_n = \gamma(t_n) \in \mathcal{D}_u$, and $\widetilde{\Pi}_n$ is a Dirichlet hyperplane passing through x_n. Moreover, those Dirichlet hyperplanes are different from each other, and the length of $\gamma(t)$ from t_n to t_{n+1} is not less than r_0, i.e.,

$$|\gamma(t_n \leqslant t \leqslant t_{n+1})| \geqslant r_0. \tag{3.4}$$

Since $\mathcal{D}_u$ is bounded and $\lim_{t\to\infty} |\gamma(t)| = +\infty$, so we must have $\lim_{n\to\infty} t_n = t_0$ for some finite t_0. Otherwise, we would have $\lim_{n\to\infty} t_n = +\infty$ due to the fact that t_n is strictly increasing and this further implies $\lim_{n\to\infty} |\gamma(t_n)| = +\infty$, contradicting the boundedness of $\mathcal{D}_u$ and the fact that $\gamma(t_n) = x_n \in \mathcal{D}_u$ for each n. Finally, because $\gamma(t)$ is a C^1-smooth curve, we must have that

$$\lim_{n\to\infty} |\gamma(t_n \leqslant t \leqslant t_{n+1})| = \lim_{n\to\infty} \int_{t_n}^{t_{n+1}} |\gamma'(t)|\, \mathrm{d}t = 0, \tag{3.5}$$

which contradicts the inequality (3.4), thus completing the proof of theorem 2.1.

4. Proof of theorem 2.2

For convenience, we first introduce some notation and definitions. Corresponding to the incident waves $\exp\{\mathrm{i}kx \cdot d_j\}$, $j = 1, \ldots, N$, with each $d_j \in \mathbb{S}^{N-1}$, we denote the total fields of the direct scattering problem associated with the boundary condition (2.2), respectively, by $u_1(x), \ldots, u_N(x)$. We will write $\mathcal{U} = \{u_1, u_2, \ldots, u_N\}$, and any operation on $\mathcal{U}$ is always meant elementwise. In correspondence to the Dirichlet set of section 3, we introduce the notion of a Neumann set (cf [15]).

Definition 4.1. *$\mathcal{Z}_{\mathcal{U}}$ is called a Neumann set of $\mathcal{U}$ in* **G** *if*

$$\mathcal{Z}_{\mathcal{U}} = \left\{ x \in \mathbf{G};\ \frac{\partial \mathcal{U}}{\partial \nu}\bigg|_{\Pi\cap B_r(x)\cap G} = 0 \text{ for some } r > 0 \text{ and hyperplane } \Pi \text{ passing through } x \right\},$$

where ν is the unit normal to the hyperplane Π.

The following properties on the Neumann set $\mathcal{Z}_{\mathcal{U}}$ were proved in [15] (lemmas 1 and 2):

Lemma 4.2.

(i) For any $x \in \mathcal{Z}_{\mathcal{U}}$, let $\widetilde{\Pi}$ be the corresponding open-connected component of $\Pi\backslash\mathbf{D}$ containing x, then $\frac{\partial\mathcal{U}}{\partial\nu}(\widetilde{\Pi}) = 0$. $\widetilde{\Pi}$ must be an open-connected subset of a hyperplane with its boundary on $\partial\mathbf{G}$ and will be referred to as a Neumann hyperplane.

(ii) The Neumann set $\mathcal{Z}_u$ and all Neumann hyperplanes are bounded.

For our subsequence purpose, we redefine the *Dirichlet set* as

$$\mathcal{D}_{\mathcal{U}} = \{x \in \mathbf{G}; \mathcal{U}|_{\Pi\cap B_r(x)\cap G} = 0 \text{ for some } r > 0 \text{ and hyperplane } \Pi \text{ passing through } x\}.$$

It is easy to verify that the Dirichlet set $\mathcal{D}_{\mathcal{U}}$ has the same properties as those given in lemma 3.2 with u replaced by $\mathcal{U}$. Furthermore, the definition for a Dirichlet hyperplane is modified accordingly. Next, similarly to lemma 3.3, we can show that

Lemma 4.3. *Let $E \subset \mathbf{G}$ be a bounded polyhedral domain and $\partial E = \overline{\partial E_1} \cup \overline{\partial E_2} \cup \overline{\partial E_3}$ with $\partial E_l, l = 1, 2, 3,$ being disjoint relative open subsets of ∂E. If $u = 0$ on ∂E_1, $\frac{\partial u}{\partial \nu} = 0$ on ∂E_2 and $\partial u/\partial\nu + \mathrm{i}\xi u = 0$ (or $\partial u/\partial\nu - \mathrm{i}\xi u = 0$) on ∂E_3, where ν is the unit normal to ∂E directed to the interior of E and $\xi \in C(\partial E_3)$ is nonnegative. Then $\xi = 0$ on ∂E_3.*

Now, with the above preparations, we are ready to prove theorem 2.2. The proof follows in a very similar manner to that of theorem 2.1, so we only outline the major modifications.

We first prove $\mathbf{D} = \tilde{\mathbf{D}}$ by contradiction. If $\mathbf{D} \neq \tilde{\mathbf{D}}$, then with the help of lemma 4.3, using the same argument as that for theorem 2.1, we can assume that there exists a Neumann or a Dirichlet hyperplane $\widetilde{\Pi}_1 \subset \mathbf{G} := \mathbb{R}^N \backslash \mathbf{D}$. Next, we fix an arbitrary point $x_1 \in \widetilde{\Pi}_1$. Let $\gamma(t)$ ($t \geqslant 0$) be a regular curve such that $\gamma(0) = x_1, \gamma(t)$ ($t > 0$) lies entirely in $\mathbf{G}'$, the unique unbounded connected component of $\mathbf{G}\backslash\widetilde{\Pi}_1$, and $\lim_{t\to\infty} |\gamma(t)| = +\infty$. Then, if $\widetilde{\Pi}_1$ is a Neumann hyperplane, we can make an even symmetric reflection argument (see the proof of theorem 1, [15]), while if $\widetilde{\Pi}_1$ is a Dirichlet hyperplane, we can make an odd symmetric reflection argument (see the proof of theorem 2.1 of the present paper). And in both cases, with the help of lemma 4.3, we can find another Dirichlet or Neumann hyperplane $\widetilde{\Pi}_2$ which is different to $\widetilde{\Pi}_1$ such that $\gamma(t) \cap \widetilde{\Pi}_2 = x_2 := \gamma(t_1)$, and by the same trick to that in the proof of theorem 2.1 we have

$$|\gamma(0 \leqslant t \leqslant t_1)| \geqslant r_0 > 0 \tag{4.1}$$

where $r_0 > 0$ is a constant defined to be half the distance between γ and $\mathbf{D}$. Continuing with the above procedure, we will be led to the same contradiction as that in the proof of theorem 2.1, which completes the proof of $\mathbf{D} = \widetilde{\mathbf{D}}$.

Finally, using the Holmgren's theorem and a similar argument to the first part of the proof of theorem 2.1, we can show that $\partial\mathbf{D}_D = \partial\tilde{\mathbf{D}}_D$, $\partial\mathbf{D}_N = \partial\tilde{\mathbf{D}}_N$, $\partial\mathbf{D}_I = \partial\tilde{\mathbf{D}}_I$ and $\lambda = \tilde{\lambda}$.

5. Non-unique determination of scatterers by a single plane wave

In the previous sections, we have only considered scatterers of solid type but not of crack type. We refer to a scatterer as a solid-type obstacle if it coincides with the closure of its interior, or a crack-type obstacle if it has an empty interior, for instance, a line segment in $\mathbb{R}^2$ or a surface in $\mathbb{R}^3$. For definiteness, we define a solid body to be the closure of some bounded Lipschitz domain in $\mathbb{R}^N$, and a crack to be a Lipschitz patch of the boundary of some bounded Lipschitz domain in $\mathbb{R}^N$. Furthermore, we assume that a crack is bounded, simply connected, oriented, and if it is in $\mathbb{R}^3$, then it is bounded by a Jordan curve. Whenever it concerns a polyhedral scatterer, a solid polyhedral body is meant to be a solid polyhedra in $\mathbb{R}^N$, while a crack polyhedron is defined to be the closure of an open subset of a hyperplane in $\mathbb{R}^N$. Now, a solid-type obstacle is considered to be composed of a finite number of solid bodies, while a crack-type obstacle is composed of a finite number of cracks.

Next, we consider the inverse scattering problem with scatterers consisting of both solid- and crack-type obstacles.

We still use $\mathbf{D}$ to denote an impenetrable scatterer which might consist of finitely many both solid- and crack-type obstacles, which is a compact set in $\mathbb{R}^N$ ($N \geqslant 2$) with a connected complement $\mathbf{G} := \mathbb{R}^N\backslash\mathbf{D}$. We assume that the boundary $\partial\mathbf{G}$ is Lipschitz. We stress that $\mathbf{D}$ is not necessarily of polyhedral type in this section. One can show that there exists a unique solution $u \in H^1_{\mathrm{loc}}(\mathbf{G})$ to the Helmholtz system (1.1) associated with either of the boundary conditions in (1.2), (1.3) and (1.4). For the existence and regularity, we refer to [19, 21], where the derivations were carried out without the use of the popular boundary integral equation approach. On the other hand, following the spirit of [9, 17] for solid-type obstacles, and [2, 3, 14, 18] for sole crack-type obstacles, one can also show the unique existence of solutions for the above Helmholtz system by using boundary integral equations.

When polyhedral-type scatterers are concerned, the following theorem collects the existing global uniqueness results from [1, 15, 16] and the current work.

Theorem 5.1. *For any fixed $k_0 > 0$, and N linearly independent directions $d_1, \ldots, d_N \in \mathbb{S}^{N-1}$, let $\mathbf{D}$ and $\tilde{\mathbf{D}}$ be two polyhedral scatterers in $\mathbb{R}^N$ which admit the simultaneous presence of both solid- and crack-type obstacles. We assume that the physical properties of $\mathbf{D}$ and $\tilde{\mathbf{D}}$ are unknown a priori, but the crack-type obstacles must be either sound-soft or sound-hard. Then $\mathbf{D} = \tilde{\mathbf{D}}$, and both obstacles have the same physical properties as long as $u_\infty(\hat{x}; \mathbf{D}, k_0, d_j) = u_\infty(\hat{x}; \tilde{\mathbf{D}}, k_0, d_j)$ for $\hat{x} \in \mathbb{S}^{N-1}$ and $j = 1, 2, \ldots, N$.*

Proof. The proof is a combination of the proofs of theorem 3.1 in [16] and theorem 2.2 in the current work. □

If one restricts the crack-type obstacles to be of sound-soft type only, then we can show the following results.

Theorem 5.2. *For any fixed $k_0 > 0$ and $d_0 \in \mathbb{S}^{N-1}$, let $\mathbf{D}$ and $\tilde{\mathbf{D}}$ be two polyhedral scatterers in $\mathbb{R}^N$ which admit the simultaneous presence of both solid- and crack-type obstacles. We assume that the crack-type obstacles must be sound-soft, and on the solid-type obstacles boundary conditions similar to (2.1) are enforced with respective surface impedance $\lambda(x)$ and $\tilde{\lambda}(x)$. Then we have $\mathbf{D} = \tilde{\mathbf{D}}$ and $\lambda = \tilde{\lambda}$, as long as $u_\infty(\hat{x}; \mathbf{D}, \lambda, k_0, d_0) = u_\infty(\hat{x}; \tilde{\mathbf{D}}, \tilde{\lambda}, k_0, d_0)$ for $\hat{x} \in \mathbb{S}^{N-1}$.*

Proof. The theorem can be proved by a natural modification of the proof of theorem 2.1 in section 2, along with the proof of theorem 2 in [15]. □

We wish to mention that there is also a uniqueness result for a more general scatterer of purely sound-soft type in $\mathbb{R}^3$ consisting of both solid- and crack-type obstacles by a single incident plane wave at fixed $d \in \mathbb{S}^2$ and a sufficiently small $k > 0$ (see [20]).

Theorem 5.3. *For any fixed $k_0 > 0$ and direction $d_0 \in \mathbb{S}^1$, let $\mathbf{D}$ and $\tilde{\mathbf{D}}$ be two polygonal scatterers in $\mathbb{R}^2$ which admit the simultaneous presence of both solid- and crack-type obstacles. We assume that the physical properties of $\mathbf{D}$ and $\tilde{\mathbf{D}}$ are unknown a priori, but the crack-type obstacles must be sound-soft. Then $\mathbf{D} = \tilde{\mathbf{D}}$, and both obstacles have the same physical properties, as long as $u_\infty(\hat{x}; \mathbf{D}, k_0, d_0) = u_\infty(\hat{x}; \tilde{\mathbf{D}}, k_0, d_0)$ for $\hat{x} \in \mathbb{S}^1$.*

Proof. With the help of lemma 4.3 in the present paper, the theorem can be proved by an appropriate modification of the proof of theorem 2.2 in [16]. □

Whether theorem 5.3 holds in $\mathbb{R}^N$ for $N \geqslant 3$ is still unclear.

Next, we are going to give several examples to demonstrate that for a general scatterer in $\mathbb{R}^N$, not necessarily confined to polyhedral scatterers, if there are any sound-hard crack-type components present, then one cannot determine the scatterer uniquely by any less than N incident waves.

We start with a relative simple example. Let $\mathbf{D} \subset \mathbb{R}^N$ be a simply connected compact set with a boundary $\partial\mathbf{D}$, which can be smooth or only Lipschitz continuous, and Π be a hyperplane in $\mathbb{R}^N$ such that $\mathbf{D}$ is symmetric with respect to Π (see figure 1, left). Since the Laplacian

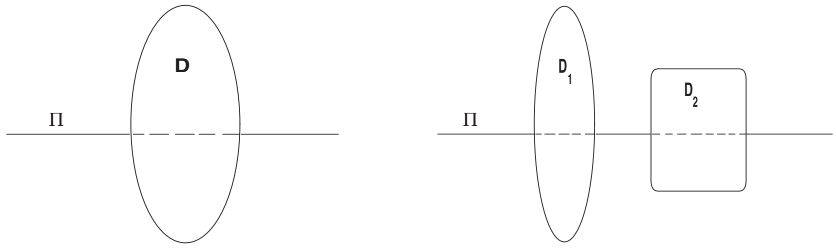

Figure 1. Illustration of the example.

is invariant with respect to rigid motions, we may assume that the origin lies in $\mathbf{D}$ and Π is the $\mathbb{R}_{x'}$ hyperplane composed of all possible points of the form $x' = (\mathrm{x}_1, \mathrm{x}_2, \ldots, \mathrm{x}_{N-1}, 0)$. Let $\Gamma = \{x = (\mathrm{x}_1, \ldots, \mathrm{x}_N) \in \partial\mathbf{D};\, \mathrm{x}_N \geqslant 0\}$ and $\Gamma^\rho = \left\{x^\rho = \left(\mathrm{x}_1^\rho, \ldots, \mathrm{x}_N^\rho\right) \in \partial\mathbf{D};\, \mathrm{x}_N^\rho \leqslant 0\right\}$. Clearly, Γ^ρ is the reflection of Γ about $\mathbb{R}_{x'}$. Let $d \in \mathbb{S}^{N-1} \cap \mathbb{R}_{x'}$ be an incident direction. We consider the scattering problem by the sound-hard obstacle $\mathbf{D}$. It is easily seen that $u^s(x)$ satisfies the Helmholtz equation, i.e., $\Delta u^s + k^2 u^s = 0$, with the boundary condition $(\partial u^s/\partial\nu)(x) = -(\partial u^i/\partial\nu)(x) = -k(d\cdot\nu)\exp\{\mathrm{i}kx\cdot d\}$ on $\partial\mathbf{D}$, where ν is the unit outward normal to $\partial\mathbf{D}$ at x. Now for an arbitrary point x on Γ, let $x^\rho \in \Gamma^\rho$ be the reflection of x with respect to Π, and ν and ν^ρ be the unit normals to $\partial\mathbf{D}$ at x and x^ρ, respectively. Then one can easily see that $d\cdot x = d\cdot x^\rho$ and $d\cdot\nu = d\cdot\nu^\rho$. Hence, the boundary data of $(\partial u^s/\partial\nu)$ on $\partial\mathbf{D}$ are symmetric with respect to Π. By the symmetry of $\mathbf{D}$, $u^s(x)$ must also be symmetric with respect to Π, i.e., $u^s(x) = u^s(x^\rho)$. To see this, let $R_{x_N^+} = \{x \in \mathbb{R}^N;\, x_N \geqslant 0\}$, and $R_{x_N^-} = \{x \in \mathbb{R}^N;\, x_N \leqslant 0\}$, and set $w(x) := u^s(x)$ for $x \in R_{x_N^+} \cap \mathbf{G}$ and $w(x) := u^s(R_\Pi(x))$ for $x \in R_{x_N^-} \cap \mathbf{G}$, where R_Π is the reflection with respect to Π. Then $w(x)$ is a $H^1_{\mathrm{loc}}(\mathbf{G})$ solution to the Helmholtz equation in $\mathbf{G}$. By the uniqueness of the solution, we conclude that $u^s(x) = w(x)$ for $x \in \mathbf{G}$ or $u^s(x)$ is symmetric about Π. Therefore, we must have $(\partial u^s/\partial\nu_\Pi)(x) = 0$ on $\Pi\cap\mathbf{G}$, where ν_Π is the normal to Π. But noting that $d\cdot\nu_\Pi = 0$ and $u = u^s + u^i$, we then have $(\partial u/\partial\nu_\Pi)(x) = 0$ on $\Pi\cap\mathbf{G}$. Now consider two arbitrary sound-hard crack-type obstacles $\widehat{C}$ and $\widetilde{C}$ lying on $\Pi\cap(\mathbb{R}^N\setminus\mathbf{D})$, with which we can form two sound-hard scatterers $\widehat{\mathbf{D}} := \mathbf{D}\cup\widehat{C}$ and $\widetilde{\mathbf{D}} := \mathbf{D}\cup\widetilde{C}$. Let $d_1, \ldots, d_{N-1} \in \mathbb{S}^{N-1}\cap\Pi$ be any $N-1$ different incident directions, possibly linearly dependent or linearly independent. Now using the symmetry of $u^s(x)$ with respect to Π, we know that the scattered fields $u^s(x)$ corresponding to two scatterers $\widehat{\mathbf{D}}$ and $\widetilde{\mathbf{D}}$ must be equal, which implies that $u_\infty(\hat{x}; \widehat{\mathbf{D}}, k, d_i) = u_\infty(\hat{x}; \widetilde{\mathbf{D}}, k, d_i)$ for $i = 1, \ldots, N-1$ and any fixed $k > 0$. On the other hand, it is not difficult to find that $u_\infty(\hat{x}; \widehat{\mathbf{D}}, d_0, k) = u_\infty(\hat{x}; \widetilde{\mathbf{D}}, d_0, k)$ for any fixed arbitrary $d_0 \in \mathbb{S}^{N-1}\cap\Pi$ and arbitrary $k > 0$. From the above, we can conclude that one cannot determine the scatterers $\widehat{\mathbf{D}}$ and $\widetilde{\mathbf{D}}$ uniquely by any less than N different incident plane waves.

Next, we consider the same compact set $\mathbf{D}$ as mentioned above, but assume it to be a sound-soft obstacle or an obstacle on which the impedance boundary condition (1.4) is enforced with a constant impedance. As done earlier, one can show that the boundary data of the corresponding scattered field $u^s(x)$ are still symmetric with respect to Π, so the same conclusions as the case when $\mathbf{D}$ is a sound-hard obstacle follow provided the concerned scatterer admits any sound-hard crack-type components.

In a similar manner, we can show that we cannot uniquely determine a scatterer of mixed type, see, e.g., figure 1 (right), where $\mathbf{D}_1$ is a sound-hard obstacle and $\mathbf{D}_2$ is a sound-soft obstacle or an obstacle satisfying an impedance boundary condition, using any less than N incident waves if the scatterer admits any sound-hard crack-type components. In this sense, the uniqueness results established and summarized in this section are optimal.

6. Concluding remarks

This work studies the unique determination of a scatterer associated with the inverse obstacle scattering problem by time-harmonic incident plane waves. When the scatterer consists of finitely many solid polyhedral obstacles, which may be either sound-soft, sound-hard or of impedance type, or of general mixed type with partially coated obstacles, and it may also contain some crack-type obstacles but only sound-soft ones, then it is believed that one can uniquely determine the scatterer by a single incident plane wave at some fixed $k_0 > 0$ and $d_0 \in \mathbb{S}^{N-1}$. This is affirmatively verified in any dimensions whenever there is no sound-hard obstacle present; when there is any sound-hard obstacle, the uniqueness is validated in the $\mathbb{R}^2$ case, but still incomplete in the $\mathbb{R}^N$ case with $N \geqslant 3$, which is proved to be true only with N different incident plane waves. Whenever the scatterer contains some sound-hard crack-type obstacles, one cannot uniquely determine the scatterer by any fewer than N incident waves. So in the case with sound-hard crack-type obstacles, the result that one can uniquely determine a scatterer by N incident waves at any fixed wave number and arbitrary N linearly independent incident directions is optimal.

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