A nonlinear vacillating dynamo induced by an electrically heterogeneous mantle

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Abstract. This paper reports the first spherical numerical dynamo based on a three-dimensional finite element method. We investigate a nonlinear dynamo in a turbulent electrically conducting fluid spherical shell of constant electric conductivity surrounded by an electrically heterogeneous mantle. Magnetic fields in the form of a three-dimensional azimuthally traveling dynamo wave are generated by a prescribed time-dependent α in the fluid shell. In the inner sphere, we assume that there is a solid electrical conductor with the same conductivity as that of the fluid shell. Equilibration of the generated magnetic fields is achieved by the nonlinear process of α-quenching. We show for the first time that finite element methods can be effectively and efficiently employed to simulate three-dimensional dynamos in spherical systems. We also show that an electrically heterogeneous mantle can modulate the core dynamo, leading to a vacillating dynamo whose amplitude depends upon the relative phases between the generated magnetic field and the heterogeneous mantle.

Introduction

There is evidence indicating that the Earth's lower mantle adjacent to the core-mantle interface is thermally and chemically heterogeneous (for example, Weber, 1993; Lay et al., 1998). Thermal variations at the base of the mantle impose a nonuniform thermal boundary condition that affects convection in the outer fluid core (Bloxham and Gubbins, 1987; Zhang and Gubbins, 1993; Sumita and Olson, 1999). Chemical variations may lead to spatially varying magnetic diffusivity in the lowermost mantle (Jeanloz, 1990). The magnetic interaction between the core and lower mantle would certainly influence the behavior of the geodynamo. In fact, it was shown by Busse and Wicht (1992) that magnetic fields can be generated by a uniform flow over an electrically heterogeneous wall like the mantle. Buffett (1996) showed that conductivity variations in the lowermost mantle can significantly change the velocity and magnetic field at the top of the core. Both these studies used a Cartesian approximation.

Nearly all existing geodynamo models employ spectral methods based on spherical harmonic expansions (Zhang and Busse, 1989; Glatzmaier and Roberts, 1995; Sarson and Jones, 1999; Kuang and Bloxham, 1997; Olson et al., 1999; Christensen et al., 1999). Dormy et al. (2000) recently provide a comprehensive summary and comparison of various geodynamo models. When a geodynamo model assumes that the whole mantle is a perfect insulator, the magnetic field $\mathbf{B}$ in the mantle satisfies

$$\nabla \times \mathbf{B} = 0, \tag{1}$$

$$\nabla \cdot \mathbf{B} = 0. \tag{2}$$

It follows that the magnetic field $\mathbf{B}$ in the perfectly insulating mantle can be simply written as

$$\mathbf{B} = -\nabla \left[\sum_l \sum_m \frac{C_{lm}}{r^{l+1}} Y_l^m(\theta,\phi) \right], \tag{3}$$

where (r,θ,ϕ) are spherical polar coordinates, Y_l^m are spherical harmonics and C_{lm} are coefficients of the expansion. If the magnetic field in the fluid outer core is also expanded in terms of spherical harmonics similar to (3), the values of C_{lm} can be determined by matching conditions at the interface between the electrically conducting fluid core and the electrically insulating mantle. As a result, we do not need to solve (1) and (2) explicitly for the mantle. This is perhaps the most significant advantage in using the spectral method for simulating a spherical dynamo. However, spatially varying electric conductivity in the lower mantle, which does not permit a solution given by (3), would complicate the corresponding analysis substantially.

There are a number of important reasons why we benefit from a numerical geodynamo that uses a finite element method. The main one is that the Legendre transform which is computationally inefficient severely limits the efficiency of the spectral methods in large-scale parallel simulations. The global nature of the spectral methods causes difficulties in

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an efficient implementation on massively parallel computers with a large number of processors. A new generation of geodynamo model using fundamentally different numerical methods such as finite element methods is highly desirable. Other reason is that the spatial local variation of conductivity can be readily incorporated into a finite element geodynamo model. However, there is a major numerical difficulty in using finite element methods for a spherical dynamo problem: the conflict between the local nature of finite element methods and the global nature of magnetic field boundary conditions. In this paper, we use a simple nonlinear α^2 dynamo model to show that the conflict can be resolved by employing an asymptotic magnetic boundary condition. We also use the finite element model to examine how the behavior of nonlinear time-dependent dynamos in the fluid core can be influenced by an electrically heterogeneous mantle.

Model, Method and Results

We consider the Earth's outer core as a turbulent fluid spherical shell of inner radius r_i and outer radius r_o with constant magnetic diffusivity λ. We assume that magnetic fields are generated by an α-effect (Roberts, 1972, Moffatt, 1978) in the fluid shell. The inner sphere $0 \le r < r_i$ is assumed to be a solid electrical conductor that has the same magnetic diffusivity λ as the shell. The fluid shell is surrounded by a mantle, $r_o \le r \le r_m$, that has a spatially varying magnetic diffusivity, λ_m, of the form

$$\lambda_m = \lambda e^{qQ(r)} M(\theta, \phi), \tag{4}$$

where q is a positive constant and $M(\theta, \phi)$ and $Q(r)$ are positive functions. With this simple choice, the lowermost mantle has moderate magnetic diffusivity while the rest of the mantle behaves like an insulator.

In the solid inner core and mantle, the magnetic field $\mathbf{B}$ is governed by the magnetic diffusion equation. In the fluid shell, the magnetic field is generated by a nonlinear spherical α^2 dynamo governed by the equations

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times \left[\left(\frac{\alpha_0 \alpha(r,\theta,\phi,t)}{(1+s|\mathbf{B}|^2)} \right) \mathbf{B} \right] - \lambda \nabla \times \nabla \times \mathbf{B}, \tag{5}$$

$$\nabla \cdot \mathbf{B} = 0, \tag{6}$$

where α_0 is a parameter and the factor $(1+s|\mathbf{B}|^2)^{-1}$ involves the nonlinear process of alpha quenching which saturates the growing dynamo (Roberts and Soward, 1992). We take $s = 1$ in our model. We nondimensionalize length by the thickness of the shell $(r_o - r_i)$ and time by the magnetic diffusion time $(r_o - r_i)^2/\lambda$ of the fluid shell. In consequence, the key nondimensional parameters characterizing the dynamo problem are the radii ratio η and the magnetic Reynolds number R_α,

$$\eta = \frac{r_i}{r_o}, \quad R_\alpha = \frac{(r_o - r_i)|\alpha_0|}{\lambda}. \tag{7}$$

At the two interfaces of the fluid shell, $r = r_i$ and $r = r_o$, we impose the conditions that all components of the magnetic field and the tangential component of the electric field are continuous. Furthermore, we require that there are no sources at infinity

$$\mathbf{B} = O(r^{-3}), \text{ as } r \to \infty. \tag{8}$$

We assume a three-dimensional, time-dependent α of the form

$$\alpha(r,\theta,\phi,t) = \cos\theta \left[1 + \epsilon_\phi \sin\pi(r - r_i) \sin(m\phi - \omega t) \right], \tag{9}$$

in the region $r_i \le r \le r_o$, where m is an azimuthal wavenumber (we choose $m = 2$) and $\epsilon_\phi \le O(1)$. Our choice of α is suggested by the recent laboratory experiment that simulates convective motions in the Earth's fluid core in the presence of a thermally heterogeneous lowermost mantle (Sumita and Olson, 1999). It shows that there exists two widely different scales of convection: a small-scale turbulent flow associated with the asymptotic law $l = O(E^{1/3})$, where l is the scale of the flow and E is a small Ekman number (Gubbins and Roberts, 1987); and a large-scale non-axisymmetric flow in connection with the large-scale spatial thermal structure of the lowermost mantle. In other words, our choice (9) attempts to reflect the two-scale feature of convective motions in the Earth's fluid core that are strongly affected by both rapid rotation and large-scale non-uniformity of the lower mantle (Bloxham and Gubbins, 1987; Olson and Glatzmaier, 1996, Sumita and Olson, 1999).

The major difficulty in employing a finite element method for a spherical dynamo problem involves the satisfaction of the magnetic field boundary conditions. We have resolved this difficulty by making the following asymptotic approximation. We replace an infinitely extended outer exterior of the fluid core by a thick spherical layer in the region $r_o \le r \le r_m$ such that

$$\left(\frac{r_m}{r_o} \right)^3 \gg 1. \tag{10}$$

For a sufficiently large $(r_m/r_o)^3$, we can replace condition (8) by a simple local boundary condition

$$\mathbf{B} = 0 \text{ at } r = r_m, \tag{11}$$

which can be readily implemented by finite element methods. We have performed various tests comparing our finite element solutions using condition (11) with analytical or semi-analytical solutions using the asymptotic condition (8). It is found that a finite element solution with $(r_m/r_o) \approx 4$ is usually sufficiently large to yield the asymptotic result within about 1% accuracy. For example, taking $r_o = 5/3 (\eta = 0.4)$ for an electrically uniform mantle, we obtain analytically the slowest decay rate, $\sigma = 3.55$, for the largest scale of poloidal magnetic fields in the limit $r_m/r_o \to \infty$. When we use the asymptotic boundary condition (11) with $(r_m/r_o) = 6$, our time-dependent finite element model yields $\sigma = 3.57$.

We now briefly describe the three-dimensional finite element discretization of our model. We first triangulate the outer spherical surface. In our calculation, meshes on a spherical surface comprise 320 or 1280 triangles. The next level will have $1280 \times 4 = 5120$ triangles on the surface. For our dynamo problem, the resolution with 1280 triangles on a surface is sufficient. To avoid a large number of nodes in the neighborhood of the center, we use two different schemes to divide the whole spherical system into smaller tetrahedra: One is for the spherical shell and the other is for the inner sphere. In short, the three-dimensional tetrahedralization produces a uniform mesh distribution on a spherical surface without the pole problem and nearly uniform nodes in the inner sphere without the origin problem. As a typical

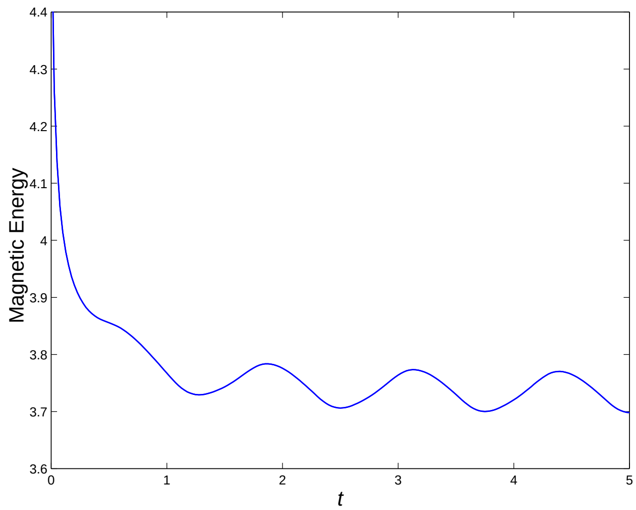

Figure 1. Magnetic energy as a function of time for $R_\alpha = 15$ at $\eta = 0.4$.

feature of a finite element method, the meshes in a particular region such as the core-mantle boundary can be refined locally if a higher resolution is required there. In each tetrahedron, a linear or quadratic interpolation function is used as a basis function. Application of the standard finite element procedure yields a system of nonlinear equations. We then use the Crank-Nicholson method for time integration while the nonlinear term is treated explicitly by a second-order extrapolation scheme. Additional technical details of our finite element method will be discussed in a separate paper (Chan et al., 2001).

Any spatial variation in $\lambda_m(r,\theta,\phi)$ can be readily implemented in our finite-element model. However, we do not know the actual spatial variation of $\lambda_m(r,\theta,\phi)$ in the lower mantle. As a result, we assume a simple azimuthal periodic variation of the magnetic diffusivity in the lower mantle similar to that used by Buffet (1996),

$$\lambda_m = \lambda\,[1 + \epsilon_g \sin(m\phi)\ \sin\theta]\, e^{qQ(r)}, \quad r_o \le r \le r_m. \quad (12)$$

We have used two different profiles, $Q(r) = (r - r_o)$ and $Q(r) = r/r_o$, in our calculations and they give rise to similar solutions. We choose $q = 0.6$, $\eta = r_i/r_o = 0.4$, and $m = 2$ with either $r_m/r_o = 3$ or $r_m/r_o = 6$. We have simulated many nonlinear dynamos at various values of the magnetic Reynolds number. When the magnetic Reynolds number R_α is less than about 10, there are no growing dynamo solutions. When ϵ_g is zero, we always obtain a three-dimensional nonlinear dynamo wave with constant amplitude as a result of a uniform core-mantle interface. When ϵ_g is non-zero, *i.e.*, when the core-mantle interface is electrically nonuniform, the resulting nonlinear dynamo is in the form of a vacillating solution with a periodic variation of its amplitude. This is because the amplitude of the generated magnetic field is determined by the relative phases between the dynamo wave and the heterogeneous mantle. Figure 1 shows the time dependence of the magnetic energy of a nonlinear dynamo obtained at $R_\alpha = 15$, $\omega = 5$ and $\epsilon_g = 0.9$. The period of vacillation, τ, is, of course, associated with the value of the speed of the wave, ω/m. In this case with $m = 2$ and $\omega = 5$, we have $\tau = 4\pi/5 = 2.51$. But the period of the magnetic energy $|\mathbf{B}|^2$ shown in Figure 1 is $\tau = 2.51/2$. In Figure 2, we show the structure of the radial component of the generated magnetic fields at the core-mantle interface at a particular instant. The non-axisymmetric magnetic fields of the vacillating dynamo drift azimuthally in a direction dependent upon the sign of ω. The radial component of the magnetic field is dominated by two magnetic flux tubes and the vacillating dynamo solution has dipolar symmetry.

The nonlinear dynamo shown in Figures 1 and 2 is calculated with $r_m/r_o \approx 6$. Figure 3 shows the numerical solution in the whole domain of our calculation $0 \le r \le r_m$. It clearly indicates that the penetration of the toroidal field

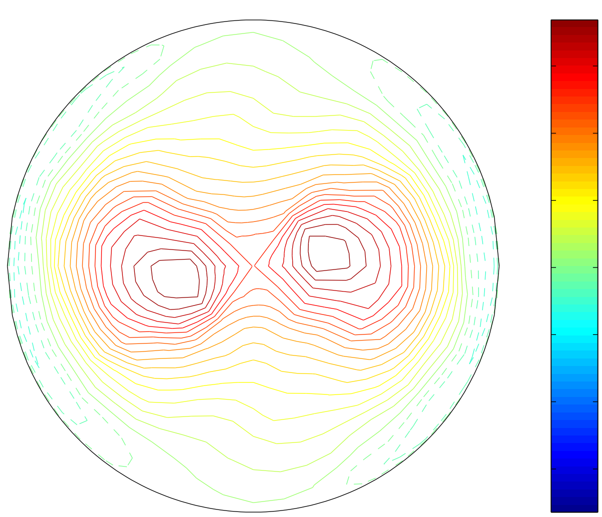

Figure 2. Contours of the radial magnetic field B_r at the core-mantle interface plotted at $t = 4.5$. and viewed at the north pole. Solid (red) contours indicate that the field lines point out of the interface and dashed (blue) contours correspond to field lines that point into the fluid core. The parameters are $R_\alpha = 15.0$ and $\eta = 0.4$.

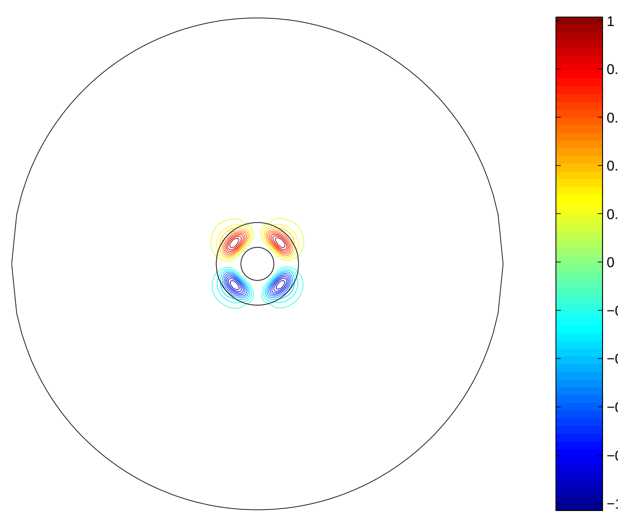

Figure 3. Contours of B_ϕ in a meridional plane showing the three domains of our dynamo calculation, the inner sphere, the fluid shell and the outer exterior. Dashed (blue) contours indicate that the toroidal field lines point out of the plane of the figure and solid (red) contours correspond to the fields that point into the plane. The parameters are the same as those in Figure 2.

is confined to a thin layer adjacent to the interface $r = r_o$. At the same time, the poloidal field with $(r_m/r_o)^3 \approx 200$ is so weak that the local condition (11) represents a fairly accurate boundary condition. In fact, we have also performed the same simulation using $r_m/r_o = 3$, which produces nearly the same dynamo solution as that with $r_m/r_o \approx 6$.

Some Remarks

Finite element methods have been widely used in many areas of scientific computing including fluid dynamics. This paper represents the first attempt to employ finite-element techniques for solving the problem of spherical planetary dynamos. We have resolved the two most important problems in finite-element geodynamo modeling: (i) the conflict between the local method and the global boundary condition of the generated magnetic field, and (ii) the three-dimensional finite-element discretization for the whole spherical system. Although this paper is not concerned with convection-driven dynamos, an extension of our model to include thermal convection is a straightforward matter. This is because the temperature and velocity have local boundary conditions which can be easily treated by the finite element method.

A key feature of our finite-element geodynamo model is that the region exterior to the fluid core (the mantle and surrounding space) is a part of the system whose governing equations must be solved numerically together with those for the fluid core. It follows that the three-dimensional structure of a thermally or an electrically heterogeneous mantle can be readily incorporated into our finite-element model without extra effort. In this sense, this is a convenient and powerful method when one constructs a whole-earth geodynamo model that includes both the fluid core and a heterogeneous mantle.

This paper represents the first and most important step in our effort to construct a convection-driven, whole-earth geodynamo model using the finite element method, which is also particularly suitable for a massively parallel computer. There are two related investigations that are currently underway. The first is to extend our present model to include the equation of motion which couples with the dynamo equation (5) by the Lorenz force. The second is to design a suitable parallel algorithm for the fully convection-driven dynamo problem.

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