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Research Article

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Fully Discrete Finite Element Approximation of the MHD Flow

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Abstract: In this work, we consider a fully discrete finite element approximation of the 3D incompressible magnetohydrodynamic system. The velocity and magnetic field are approximated both by piecewise quadratic finite elements, while the pressure is approximated by piecewise linear finite elements. The time discretization is based on the Crank–Nicolson scheme for the linear terms in the model and the explicit Adams–Bashforth for the nonlinear terms. We establish the optimal error estimates of both the approximate velocity and magnetic field in $\mathbf{H}^1$-norm and of the approximate pressure in L^2-norm. In order to achieve the optimal L^2-norm error estimates of both the approximate velocity and magnetic field, we shall make use of a special negative norm technique.

Keywords: MHD Flow, Finite Element Approximations, Error Estimates, Negative-Norm Technique

MSC 2010: 65N30, 35Q35, 65N12, 76M10, 76W05

1 Introduction

This work is concerned with the following 3D incompressible magnetohydrodynamic (MHD) equations under the influence of body forces [3, 11, 21, 25, 28]:

$$\begin{cases} \dfrac{\partial \mathbf{u}}{\partial t} - \nu\Delta\mathbf{u} + \nabla p + (\mathbf{u}\cdot\nabla)\mathbf{u} + s\mathbf{B}\times(\nabla\times\mathbf{B}) = \mathbf{f} & \text{in } \Omega\times(0,T], \\ \dfrac{\partial \mathbf{B}}{\partial t} + \mu\nabla\times(\nabla\times\mathbf{B}) - \nabla\times(\mathbf{u}\times\mathbf{B}) = 0 & \text{in } \Omega\times(0,T], \\ \nabla\cdot\mathbf{u} = 0, \quad \nabla\cdot\mathbf{B} = 0 & \text{in } \Omega\times(0,T], \end{cases} \tag{1.1}$$

where $\Omega \subset R^3$ is an open bounded domain with a smooth boundary, $\mathbf{u}$, $\mathbf{B}$ and p denote the velocity field, the magnetic field and the pressure, respectively. The three parameters ν, μ and s are given by $\nu = Re^{-1}$, $\mu = Re_m^{-1}$, $s = M^2/(Re\,Re_m)$, respectively, with Re and Re_m being the Reynolds number and the magnetic Reynolds number, and $M > 0$ being the Hartman number. For convenience, we shall often write $\mathbf{r} = (x, y, z) \in \Omega$, and the magnetic field $\mathbf{B}$ as $\mathbf{B}(t)$ or $\mathbf{B}(\mathbf{r}, t)$; the same for the pressure p and velocity $\mathbf{u}$.

System (1.1) couples the incompressible Navier–Stokes equations with Maxwell's equations, and is considered in conjunction with the following initial boundary conditions [3, 11, 21, 25, 28]:

$$\begin{aligned} &\mathbf{u}(0) = \mathbf{u}_0(\mathbf{r}), \quad \mathbf{B}(0) = \mathbf{B}_0(\mathbf{r}) && \text{in } \Omega, \\ &\mathbf{u} = 0, \quad \mathbf{B}\cdot\mathbf{n} = 0, \quad \mathbf{n}\times\nabla\times\mathbf{B} = 0 && \text{on } \partial\Omega\times[0,T], \end{aligned} \tag{1.2}$$

where $\nabla\cdot\mathbf{u}_0(\mathbf{r}) = 0$ and $\nabla\cdot\mathbf{B}_0(\mathbf{r}) = 0$, with $\mathbf{n}$ being the unit outward normal vector of $\partial\Omega$.

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We will frequently use the following spaces in our subsequent analysis:

$$M = L_0^2(\Omega), \quad \mathbf{X} = H_0^1(\Omega)^3, \quad \mathbf{W} = \{\mathbf{C} \in \mathbf{H}^1(\Omega);\ \mathbf{C}\cdot\mathbf{n}|_{\partial\Omega} = 0\},$$
$$\mathbf{H} = \{\mathbf{v} \in L^2(\Omega)^3, \operatorname{div}\mathbf{v} = 0,\ \mathbf{v}\cdot\mathbf{n}|_{\partial\Omega} = 0\},$$
$$\mathbf{V} = \mathbf{X}\cap\mathbf{H}, \quad \mathbf{W}_0 = \mathbf{W}\cap\mathbf{H}, \quad \mathbf{H}^k(\Omega) = H^k(\Omega)^3, \quad k = 0, 1, 2, 3,$$

as well as the following trilinear forms:

$$\begin{aligned} b(\mathbf{w},\mathbf{u},\mathbf{v}) &= \Big((\mathbf{w}\cdot\nabla)\mathbf{u} + \frac{1}{2}(\nabla\cdot\mathbf{w})\mathbf{u}, \mathbf{v}\Big)_\Omega \\ &= \frac{1}{2}((\mathbf{w}\cdot\nabla)\mathbf{u},\mathbf{v})_\Omega - \frac{1}{2}((\mathbf{w}\cdot\nabla)\mathbf{v},\mathbf{u})_\Omega \quad \text{for all } \mathbf{w},\mathbf{u},\mathbf{v}\in\mathbf{X}, \\ d(\mathbf{v},\mathbf{B},\mathbf{C}) &= (\mathbf{v}\times\mathbf{B}, \nabla\times\mathbf{C})_\Omega \quad \text{for all } \mathbf{v}\in\mathbf{X},\ \mathbf{B},\mathbf{C}\in\mathbf{W}. \end{aligned} \tag{1.3}$$

For the coupled MHD flow system (1.1)–(1.2), we are interested in its variational formulation: find $(\mathbf{u}(t), p(t), \mathbf{B}(t)) \in \mathbf{X}\times M\times\mathbf{W}$ such that it satisfies for all $(\mathbf{v}, q, \mathbf{C}) \in \mathbf{X}\times M\times\mathbf{W}$ the system

$$(\mathbf{u}_t,\mathbf{v})_\Omega + \nu(\nabla\mathbf{u},\nabla\mathbf{v})_\Omega - (p,\nabla\cdot\mathbf{v})_\Omega + (\nabla\cdot\mathbf{u}, q)_\Omega + b(\mathbf{u},\mathbf{u},\mathbf{v}) + sd(\mathbf{v},\mathbf{B},\mathbf{B}) = (\mathbf{f},\mathbf{v})_\Omega, \tag{1.4}$$
$$(\mathbf{B}_t,\mathbf{C})_\Omega + \mu(\nabla\times\mathbf{B},\nabla\times\mathbf{C})_\Omega - d(\mathbf{u},\mathbf{B},\mathbf{C}) = 0. \tag{1.5}$$

There are wide studies of the stability and convergence of the fully discrete second-order schemes based on the finite element spatial discretization for solving the time-dependent Navier–Stokes equations. For further exposition, we let $0 < h < 1$ be the spatial mesh size and $0 < \tau = \frac{T}{N} < 1$ a time step size, and we let $\mathbf{u}_h^m$, $\mathbf{B}_h^m$ and p_h^m be the finite element approximate solutions of $\mathbf{u}$, $\mathbf{B}$ and p at $t = t_m$, respectively. We know that, for second-order or higher-order schemes for solving the time-dependent Navier–Stokes equations when the nonlinear term is treated explicitly, the stability or convergence is often achieved under the conditions of the form $\tau h^{-\alpha} \le C_0$ for some $\alpha > 0$; see, for example, [2, 20, 23, 29]. These conditions can be improved, e.g., in [13], where the Adams–Bashforth and Crank–Nicolson schemes were considered in combination with a finite element spatial discretization for solving the 2D time-dependent Navier–Stokes equations. It was proved in [13] that the stability and convergence are guaranteed under the condition $\tau \le C_0$ and with the following error estimates:

$$\|\mathbf{u}(t_m) - \mathbf{u}_h^m\|_0 \le \kappa\sigma^{-1}(t_m)\tau^2,$$
$$\|\nabla(\mathbf{u}(t_m) - \mathbf{u}_h^m)\|_0 \le \kappa\sigma^{-\frac{1}{2}}(t_m)\tau,$$
$$\|p(t_m) - p_h^m\|_0 \le \kappa\sigma^{-1}(t_m)\tau$$

for all $t_m \in (0, T]$. Here $\sigma(t) = \min\{1, t\}$, and κ is a generic positive constant depending on the data ν, Ω, T, u_0 and f. Hereafter, we use κ to denote a generic positive constant depending on the data ν, μ, s, Ω, T, u_0, B_0 and f. It was further shown in [15] that the convergence and stability of the Euler semi-implicit scheme for the 3D MHD equations can be also ensured under the condition $\tau \le C_0$.

In this work, we consider the Adams–Bashforth and Crank–Nicolson schemes for the time discretizations of the nonlinear and linear terms, respectively, and the finite element method for the spatial discretization for solving the time-dependent coupled MHD flow system (1.1)–(1.2) in three dimensions. Due to the explicit treatment of the nonlinear terms in the model, we cannot achieve the desired stability estimates of the fully discrete finite element solutions; therefore, it is hard for us to derive the optimal error estimates by means of existing approaches for both the Navier–Stokes equations and the coupled MHD flow system. We shall overcome this technical difficulty by a delicate induction method. Moreover, we are able to establish the optimal second-order convergence of the numerical velocity and magnetic field in the L^2-norm by using a special negative-norm technique, while the standard discrete duality strategy [12–14, 19, 30] does not work.

The rest of the work is organized as follows. In § 2, some basic assumptions and inequalities related to the MHD flow and the basic properties of the finite element solution for the 3D MHD flow are presented. In § 3, we provide some estimates and smoothing properties of the finite element solution $(\mathbf{u}_h, p_h, \mathbf{B}_h)$. In § 4, we develop a fully discrete finite element approximation based on the Crank–Nicolson/Adams–Bashforth scheme for the 3D MHD equations and then derive its truncation error estimates in § 5. In § 6, we establish the optimal error estimates for the numerical solution $(\mathbf{u}_h^n, \mathbf{B}_h^n, p_h^n)$ and present some numerical experiments in § 7 to verify the convergence orders of the numerical scheme.

2 Finite Element Spatial Discretization for the MHD Flow

In this section, we discuss the finite element spatial discretization for the MHD equations (1.1)–(1.2). To do so, we first introduce a triangulation of the domain Ω. For the sake of technical treatments, we assume that the boundary of domain Ω is a closed convex polyhedron; the actual curved boundary case can be treated using some well-developed technicalities for the curved boundary in combination with the finite element error estimates established in the current work. Let $\mathcal{T}_h$ be a triangulation of the polyhedral domain Ω, and let $\mathbf{X}_h \subset \mathbf{X}$, $M_h \subset M = L_0^2(\Omega)$ and $\mathbf{W}_h \subset \mathbf{W}$ be some continuous piecewise finite element spaces defined on $\mathcal{T}_h$. We shall also need the following subspace of $\mathbf{X}_h$:

$$\mathbf{V}_h = \{\mathbf{v}_h \in \mathbf{X}_h; (\nabla \cdot \mathbf{v}_h, q_h)_\Omega = 0 \text{ for all } q_h \in M_h\}.$$

The finite element spaces $\mathbf{X}_h \subset \mathbf{X}$, $M_h \subset M = L_0^2(\Omega)$ and $\mathbf{W}_h \subset \mathbf{W}$ will be made clearer later.

Throughout this paper, we make the following two assumptions on the prescribed data and the solutions to system (1.4)–(1.5), which specify the regularities of the data and solutions needed for our main results. Hereafter, c, κ and c_i, κ_i for $i \geq 0$ are some generic positive constants depending only on some of the given data ν, μ, s, Ω, T, $\mathbf{u}_0$, $\mathbf{B}_0$ and $\mathbf{f}$.

Assumption (A0). The initial data $\mathbf{u}_0 \in \mathbf{H}^2(\Omega) \cap \mathbf{V}$, $\mathbf{B}_0 \in \mathbf{H}^2(\Omega) \cap \mathbf{W}_0$ and the force $\mathbf{f}$ satisfy

$$\sup_{0 \leq t \leq T} \{\|\mathbf{f}(t)\|_{1,\Omega}^2 + \|\mathbf{f}_t(t)\|_{0,\Omega}^2 + \|\mathbf{f}_{tt}(t)\|_{0,\Omega}^2\} + \|\mathbf{u}_0\|_{2,\Omega}^2 + \|\mathbf{B}_0\|_{2,\Omega}^2 \leq \kappa_0.$$

Assumption (A1). Problem (1.4)–(1.5) has a unique solution $(\mathbf{u}(t), p(t), \mathbf{B}(t)) \in \mathbf{X} \times M \times \mathbf{W}$, with the a priori estimates

$$\int_0^T (\|\nabla \mathbf{u}(t)\|_{0,\Omega}^4 + \|\nabla \times \mathbf{B}(t)\|_{0,\Omega}^4)\, dt \leq \kappa.$$

Also, we assume that the domain Ω is smooth to ensure the following estimates [4, 6, 7, 17, 26].

Assumption (A2). The unique solution $(\mathbf{v}, q)$ of the steady Stokes problem

$$-\Delta \mathbf{v} + \nabla q = \mathbf{g}, \quad \nabla \cdot \mathbf{v} = 0 \text{ in } \Omega; \quad \mathbf{v}|_{\partial\Omega} = 0,$$

for prescribed $\mathbf{g} \in H^{k-2}(\Omega)^3$ with $k = 2, 3$ satisfies

$$\|\mathbf{v}\|_{k,\Omega} + \|q\|_{k-1,\Omega} \leq c\|\mathbf{g}\|_{k-2,\Omega},$$

and the steady Maxwell equations

$$\nabla \times \nabla \times \mathbf{C} = \mathbf{h}, \quad \nabla \cdot \mathbf{C} = 0 \quad \text{in } \Omega, \qquad \mathbf{n} \times \nabla \times \mathbf{C} = 0, \quad \mathbf{C} \cdot \mathbf{n} = 0 \quad \text{on } \partial\Omega,$$

for the prescribed $\mathbf{h} \in \mathbf{H}^{k-2}(\Omega)$ with $k = 2, 3$ admit a unique solution $\mathbf{C} \in \mathbf{W}_0$ which satisfies

$$\|\mathbf{C}\|_{k,\Omega} \leq c\|\mathbf{h}\|_{k-2,\Omega}.$$

We shall often use the Stokes operator $A_1 = -P\Delta$ and the Maxwell operator $A_2 = P\nabla \times \nabla\times$, with

$$D(A_1) = H^2(\Omega)^3 \cap \mathbf{V}, \quad D(A_2) = \mathbf{H}^2(\Omega) \cap \mathbf{W}_0.$$

Here P is an L^2-projection from $L^2(\Omega)^3$ into $\mathbf{H}$.

Now we make a standard approximation assumption on the finite element space system $(\mathbf{X}_h, M_h, \mathbf{W}_h)$ [1, 5, 9, 10, 17, 18, 24, 27].

Assumption (A3). For each $\mathbf{v} \in \mathbf{H}^i(\Omega) \cap \mathbf{V}$, $q \in H^{i-1}(\Omega) \cap M$ and $\mathbf{C} \in \mathbf{H}^i(\Omega)^3 \cap \mathbf{W}_0$ for $i = 2, 3$, there exist the approximations $\pi_h v \in \mathbf{V}_h$, $\rho_h q \in M_h$ and $J_h C \in \mathbf{W}_h$ such that

$$\|\nabla(\mathbf{v} - \pi_h \mathbf{v})\|_{0,\Omega} \leq c_1 h^{i-1}\|\mathbf{v}\|_{i,\Omega}, \quad \|q - \rho_h q\|_{0,\Omega} \leq c_1 h^{i-1}\|q\|_{i-1,\Omega},$$
$$\|\mathbf{C} - J_h\mathbf{C}\|_{0,\Omega} + \|\nabla \times (\mathbf{C} - J_h\mathbf{C})\|_{0,\Omega} \leq c_1 h^{i-1}(\|\mathbf{C}\|_{i-1,\Omega} + \|\nabla \times \mathbf{C}\|_{i-1,\Omega}),$$

along with the inverse inequalities (with $q \geq p$)

$$\|\nabla \mathbf{v}_h\|_{0,\Omega} \leq c_1 h^{-1}\|\mathbf{v}_h\|_{0,\Omega} \quad \text{for all } \mathbf{v}_h \in \mathbf{X}_h,$$
$$\|\mathbf{C}_h\|_{L^q} \leq c h^{3(\frac{1}{q}-\frac{1}{p})}\|\mathbf{C}_h\|_{L^p}, \quad \|\nabla \mathbf{C}_h\|_{0,\Omega} \leq c_1 h^{-1}\|\mathbf{C}_h\|_{0,\Omega} \quad \text{for all } \mathbf{C}_h \in \mathbf{W}_h,$$

and the so-called inf-sup inequality

$$\sup_{\mathbf{v}_h \in \mathbf{X}_h} \frac{(\nabla \cdot \mathbf{v}_h, q_h)_\Omega}{\|\nabla \mathbf{v}_h\|_{0,\Omega}} \geq \beta_1 \|q_h\|_{0,\Omega} \quad \text{for all } q_h \in M_h.$$

Here is an example of the finite element spaces $(\mathbf{X}_h, M_h, \mathbf{W}_h)$ that satisfy Assumption (A3) above (cf. [3, 25]):

$$\begin{aligned} \mathbf{X}_h &= \{\mathbf{v}_h \in C(\bar{\Omega}) \cap \mathbf{X};\ \mathbf{v}_h|_K \in P_2(K)^3 \text{ for all } K \in \mathcal{T}_h\}, \\ M_h &= \{q_h \in C(\bar{\Omega}) \cap M;\ q_h|_K \in P_1(K) \text{ for all } K \in \mathcal{T}_h\}, \\ \mathbf{W}_h &= \{\mathbf{C}_h \in \mathbf{C}(\bar{\Omega}) \cap \mathbf{W};\ \mathbf{C}_h|_K \in P_2(K)^3 \text{ for all } K \in \mathcal{T}_h\}. \end{aligned}$$

Now we formulate the semi-discrete finite element approximation of system (1.4)–(1.5), whose properties will be crucial to our error estimates of the finite element solution to the fully discrete scheme: find $(\mathbf{u}_h(t), p_h(t), \mathbf{B}_h(t)) \in \mathbf{X}_h \times M_h \times \mathbf{W}_h$ such that it satisfies, for all $(\mathbf{v}_h, q_h, \mathbf{C}_h) \in \mathbf{X}_h \times M_h \times \mathbf{W}_h$,

$$(\mathbf{u}_{ht}, \mathbf{v}_h)_\Omega + \nu(\nabla \mathbf{u}_h, \nabla \mathbf{v}_h)_\Omega - (\nabla \cdot \mathbf{v}_h, p_h)_\Omega + (\nabla \cdot \mathbf{u}_h, q_h)_\Omega + b(\mathbf{u}_h, \mathbf{u}_h, \mathbf{v}_h) + sd(\mathbf{v}_h, \mathbf{B}_h, \mathbf{B}_h) = (\mathbf{f}, \mathbf{v}_h)_\Omega, \tag{2.1}$$
$$(\mathbf{B}_{ht}, \mathbf{C}_h)_\Omega + \mu(\nabla \times \mathbf{B}_h, \nabla \times \mathbf{C}_h)_\Omega + \mu(\nabla \cdot \mathbf{B}_h, \nabla \cdot \mathbf{C}_h)_\Omega - d(\mathbf{u}_h, \mathbf{B}_h, \mathbf{C}_h) = 0, \tag{2.2}$$

with $\mathbf{u}_h(0) = P_h \mathbf{u}_0$ and $\mathbf{B}_h(0) = R_{0h}\mathbf{B}_0$, where P_h and R_{0h} are the L^2-projections from $L^2(\Omega)^3$ to $\mathbf{V}_h$ and $\mathbf{W}_h$. It follows from Assumption (A3) that, for any $\mathbf{v} \in H^i(\Omega)^3 \cap \mathbf{V}$ and $\mathbf{C} \in \mathbf{H}^i(\Omega) \cap \mathbf{W}_0$ for $i = 1, 2, 3$,

$$\begin{aligned} \|\mathbf{v} - P_h \mathbf{v}\|_{0,\Omega} + h\|\nabla(\mathbf{v} - P_h \mathbf{v})\|_{0,\Omega} &\leq c_2 h^i \|\mathbf{v}\|_{i,\Omega}, \\ \|\mathbf{C} - R_{0h}\mathbf{C}\|_{0,\Omega} + h\|\nabla(\mathbf{C} - R_{0h}\mathbf{C})\|_{0,\Omega} &\leq c_2 \|\mathbf{C}\|_{i,\Omega}, \end{aligned} \tag{2.3}$$

We shall often use the discrete Stokes operator $A_h = -P_h \Delta_h$, where $-\Delta_h$ is defined by

$$(-\Delta_h \mathbf{u}_h, \mathbf{v}_h) = (\nabla \mathbf{u}_h, \nabla \mathbf{v}_h) \quad \text{for all } \mathbf{u}_h, \mathbf{v}_h \in \mathbf{X}_h,$$

and the A_h-induced discrete norm $\|\mathbf{v}_h\|_\alpha = \|A_h^{\frac{\alpha}{2}} \mathbf{v}_h\|_{0,\Omega}$ for $\alpha \in R$. In particular, we have

$$\|\mathbf{v}_h\|_1 = \|\nabla \mathbf{v}_h\|_{0,\Omega}, \quad \|\mathbf{v}_h\|_2 = \|A_{1h}\mathbf{v}_h\|_{0,\Omega}, \quad \|\mathbf{v}_h\|_{-1} = \|A_{1h}^{-\frac{1}{2}} \mathbf{v}_h\|_{0,\Omega} \quad \text{for all } \mathbf{v}_h \in \mathbf{V}_h.$$

Similarly to the discrete Stokes operator A_h, we define the discrete Maxwell operator A_{2h} as follows: for any $\mathbf{B}_h \in \mathbf{W}_h$, $A_{2h}\mathbf{B}_h = R_{0h}(\nabla_h \times \nabla \times \mathbf{B}_h + \nabla_h \nabla \cdot \mathbf{B}_h) \in \mathbf{W}_h$, satisfying

$$(A_{2h}\mathbf{B}_h, \mathbf{C}_h)_\Omega = (A_{2h}^{\frac{1}{2}}\mathbf{B}_h, A_{2h}^{\frac{1}{2}}\mathbf{C}_h)_\Omega = (\nabla \times \mathbf{B}_h, \nabla \times \mathbf{C}_h)_\Omega + (\nabla \cdot \mathbf{B}_h, \nabla \cdot \mathbf{C}_h)_\Omega \quad \text{for all } \mathbf{C}_h \in \mathbf{W}_h.$$

We will also use the A_{2h}-induced discrete norm $\|\mathbf{B}_h\|_\alpha = \|A_{2h}^{\frac{\alpha}{2}}\mathbf{B}_h\|_{0,\Omega}$ for any $\alpha \in R$. Then we have

$$\begin{aligned} \|\mathbf{B}_h\|_0^2 &= \|\mathbf{B}_h\|_{0,\Omega}^2, \quad \|\mathbf{B}_h\|_1^2 = \|A_{2h}^{\frac{1}{2}}\mathbf{B}_h\|_{0,\Omega}^2 = \|\nabla \cdot \mathbf{B}_h\|_{0,\Omega}^2 + \|\nabla \times \mathbf{B}_h\|_{0,\Omega}^2, \\ \|\mathbf{B}_h\|_2^2 &= \|A_{2h}\mathbf{B}_h\|_{0,\Omega}^2 = \|R_{0h}(\nabla_h \times \nabla \times \mathbf{B}_h + \nabla_h \nabla \cdot \mathbf{B}_h)\|_{0,\Omega}^2, \\ \|\mathbf{B}_h\|_{-1} &= \|A_{2h}^{-\frac{1}{2}}\mathbf{B}_h\|_{0,\Omega} = \sup_{\mathbf{C}_h \in \mathbf{W}_h} \frac{(\mathbf{B}_h, \mathbf{C}_h)_\Omega}{\|\mathbf{C}_h\|_1}. \end{aligned}$$

We end this section by recalling some stability and error estimates of the solution $(\mathbf{u}(t), p(t), \mathbf{B}(t))$ to system (1.4)–(1.5) and the numerical solution $(\mathbf{u}_h(t), p_h(t), \mathbf{B}_h(t))$ to the semi-discrete system (2.1)–(2.2) [16].

Lemma 2.1. *Under Assumptions* (A0)–(A3)*, it holds that*

$$\|\mathbf{u}(t)\|_{2,\Omega}^2 + \|p(t)\|_{1,\Omega}^2 + \|\mathbf{B}(t)\|_{2,\Omega}^2 + \int_0^T [\|\mathbf{u}(t)\|_{3,\Omega}^2 + \|p(t)\|_{2,\Omega}^2 + \|\mathbf{B}(t)\|_{3,\Omega}^2]\, dt \leq \kappa,$$
$$\|\mathbf{u}_h(t)\|_{0,\Omega}^2 + \|\mathbf{B}_h(t)\|_{0,\Omega}^2 + \int_0^T [\|\nabla \mathbf{u}_h(t)\|_{0,\Omega}^2 + \|\nabla \mathbf{B}_h(t)\|_{0,\Omega}^2]\, dt \leq \kappa.$$

Theorem 2.1. *Under Assumptions* (A0)–(A3), *the following error estimates hold for all* $t \in (0, T]$*:*

$$\|\mathbf{u}(t) - \mathbf{u}_h(t)\|_{0,\Omega}^2 + \|\mathbf{B}(t) - \mathbf{B}_h(t)\|_{0,\Omega}^2 + \int_0^T [\|\nabla(\mathbf{u}(t) - \mathbf{u}_h(t))\|_{0,\Omega}^2 + \|\nabla(\mathbf{B}(t) - \mathbf{B}_h(t))\|_{0,\Omega}^2]\,dt \le \kappa h^4,$$
$$\int_0^T [\|\mathbf{u}(t) - \mathbf{u}_h(t)\|_{0,\Omega}^2 + \|\mathbf{B}(t) - \mathbf{B}_h(t)\|_{0,\Omega}^2]\,dt \le \kappa h^6,$$
$$\sigma(t)[\|\mathbf{u}(t) - \mathbf{u}_h(t)\|_{0,\Omega}^2 + \|\mathbf{B}(t) - \mathbf{B}_h(t)\|_{0,\Omega}] + h^2\sigma^2(t)\|p(t) - p_h(t)\|_{0,\Omega}$$
$$+\sigma(t)h^2[\|\nabla(\mathbf{u}(t) - \mathbf{u}_h(t))\|_{0,\Omega}^2 + \|\nabla(\mathbf{B}(t) - \mathbf{B}_h(t))\|_{0,\Omega}^2] \le \kappa h^6.$$

Lemma 2.2. *Under Assumptions* (A0)–(A3), *there hold*

$$\|A_h^{-1}P_hA_1\mathbf{v} - \mathbf{v}\|_{0,\Omega} + h\|\nabla(A_h^{-1}P_hA_1\mathbf{v} - \mathbf{v})\|_{0,\Omega} \le ch^i\|\mathbf{v}\|_{i,\Omega},$$
$$\|A_1^{-1}PA_h\mathbf{v}_h - \mathbf{v}_h\|_{0,\Omega} + h\|\nabla(A_1^{-1}PA_h\mathbf{v}_h - \mathbf{v}_h)\|_{0,\Omega} \le ch^i\|\mathbf{v}_h\|_i$$

for all $\mathbf{v} \in \mathbf{H}^i(\Omega) \cap \mathbf{V}$ *and* $\mathbf{v}_h \in V_h$ *with* $i = 2, 3$, *and*

$$\|A_{2h}^{-1}R_{0h}A_2\mathbf{C} - \mathbf{C}\|_{0,\Omega} + h\|\nabla(A_{2h}^{-1}R_{0h}A_2\mathbf{C} - \mathbf{C})\|_{0,\Omega} \le ch^i\|\mathbf{C}\|_{i,\Omega},$$
$$\|A_2^{-1}PA_{2h}\mathbf{C}_h - \mathbf{C}_h\|_{0,\Omega} + h\|\nabla(A_2^{-1}PA_{2h}\mathbf{C}_h - \mathbf{C}_h)\|_{0,\Omega} \le ch^i\|\mathbf{C}_h\|_i$$

for all $\mathbf{C} \in \mathbf{H}^i(\Omega) \cap \mathbf{W}_0$ *and* $\mathbf{C}_h \in \mathbf{W}_h$ *with* $i = 2, 3$.

3 Basic Estimates and Smoothing Properties of Discrete Solutions

In this section, we present some discrete inequalities of finite element functions and some smoothing properties of the finite element solution $(\mathbf{u}_h, p_h, \mathbf{B}_h)$ to the semi-discrete system (2.1)–(2.2). First, we recall some discrete Gadliardo–Nirenberg estimates [17, 18]:

$$\|\nabla\mathbf{v}_h\|_{L^3} + \|\mathbf{v}_h\|_{L^\infty} \le c\|\nabla\mathbf{v}_h\|_{0,\Omega}^{\frac{1}{2}}\|A_{1h}\mathbf{v}_h\|_{0,\Omega}^{\frac{1}{2}}, \quad \|\nabla\mathbf{v}_h\|_{L^6} \le c\|A_h\mathbf{v}_h\|_0 \quad \text{for all } \mathbf{v}_h \in \mathbf{V}_h,$$
$$\|A_hP_h\mathbf{v}\|_{0,\Omega} \le c\|A_1\mathbf{v}\|_{0,\Omega} \quad \text{for all } \mathbf{v} \in D(A_1). \tag{3.1}$$

By using Assumptions (A2)–(A3) and some similar arguments to the ones used by Heywood and Rannacher in [17], we can prove the following discrete Gadliardo–Nirenberg estimates.

Lemma 3.1. *Under Assumptions* (A2)–(A3), *there hold, for any* $\mathbf{C}_h \in \mathbf{W}_h$,

$$\|\mathbf{C}_h\|_{L^6} \le c\|\mathbf{C}_h\|_{1,\Omega}, \quad \|\mathbf{C}_h\|_{1,\Omega} \le c\|A_{2h}^{\frac{1}{2}}\mathbf{C}_h\|_{0,\Omega}, \tag{3.2}$$
$$\|\mathbf{C}_h\|_{L^\infty} + \|\nabla\mathbf{C}_h\|_{L^3} \le c\|A_{2h}^{\frac{1}{2}}\mathbf{C}_h\|_{0,\Omega}^{\frac{1}{2}}\|A_{2h}\mathbf{C}_h\|_{0,\Omega}^{\frac{1}{2}},$$
$$\|\nabla\mathbf{C}_h\|_{L^6} \le c\|A_{2h}\mathbf{C}_h\|_0. \tag{3.3}$$

Proof. First, we can easily get (3.2) using the fact that $\mathbf{W}_h \subset \mathbf{W}$ and the following inequalities from [7]:

$$\|\mathbf{C}\|_{L^6} \le c\|\mathbf{C}\|_{1,\Omega}, \quad \|\mathbf{C}\|_{1,\Omega} \le c\|\nabla \times \mathbf{C}\|_{0,\Omega} + c\|\nabla \cdot \mathbf{C}\|_{0,\Omega} \quad \text{for all } \mathbf{C} \in \mathbf{W}.$$

To derive (3.3), we define the discrete Laplace operator Δ_h on $\mathbf{W}_h$ by

$$(-\Delta_h\mathbf{C}_h, \phi_h)_\Omega = (\nabla\mathbf{C}_h, \nabla\phi_h)_\Omega \quad \text{for all } \mathbf{C}_h, \phi_h \in \mathbf{W}_h,$$

and recall the following inequalities [17]:

$$\|\nabla\mathbf{C}_h\|_{L^6} \le c\|\Delta_h\mathbf{C}_h\|_{0,\Omega},$$
$$\|\mathbf{C}_h\|_{L^\infty} + \|\nabla\mathbf{C}_h\|_{L^3} \le c\|\nabla\mathbf{C}_h\|_{0,\Omega}^{\frac{1}{2}}\|\Delta_h\mathbf{C}_h\|_{0,\Omega}^{\frac{1}{2}} \quad \text{for all } \mathbf{C}_h \in \mathbf{W}_h. \tag{3.4}$$

Now let $\mathbf{C} \in \mathbf{W}_0$ be the solution to the system

$$\nabla \times \nabla \times \mathbf{C} = PA_{2h}\mathbf{C}_h, \quad \mathbf{n} \times \nabla \times \mathbf{C}|_{\partial\Omega} = 0.$$

Then we have from Lemma 2.2 and Assumption (A2) that

$$\|\nabla(\mathbf{C} - \mathbf{C}_h)\|_{0,\Omega} + h\|\mathbf{C}\|_{2,\Omega} \le ch\|A_{2h}\mathbf{C}_h\|_{0,\Omega}. \tag{3.5}$$

By using Assumption (A3) and (3.5), we can derive

$$\|\Delta_h \mathbf{C}_h\|_{0,\Omega} \le \sup_{\phi_h \in \mathbf{W}_h} \frac{(\nabla \mathbf{C}_h, \nabla\phi_h)_\Omega}{\|\phi_h\|_{0,\Omega}} \le ch^{-1}\|\nabla(\mathbf{C}_h - \mathbf{C})\|_{0,\Omega} + c\|\mathbf{C}\|_{2,\Omega} \le c\|A_{2h}\mathbf{C}_h\|_{0,\Omega},$$

Now (3.3) follows from this and (3.4) and hence concludes the proof of Lemma 3.1. □

Lemma 3.2. *Under Assumptions* (A2)–(A3), *the following estimates hold for all* $\mathbf{u}_h, \mathbf{v}_h, \mathbf{w}_h \in \mathbf{V}_h$*:*

$$\begin{aligned} &b(\mathbf{u}_h, \mathbf{v}_h, \mathbf{w}_h) = -b(\mathbf{u}_h, \mathbf{w}_h, \mathbf{v}_h), \\ &|b(\mathbf{u}_h, \mathbf{v}_h, \mathbf{w}_h)| + |b(\mathbf{w}_h, \mathbf{u}_h, \mathbf{v}_h)| + |b(\mathbf{v}_h, \mathbf{u}_h, \mathbf{w}_h)| \\ &\qquad \le \frac{c_0}{2}\|A_h\mathbf{u}_h\|_{0,\Omega}\|\mathbf{v}_h\|_1^{\frac12}\|\mathbf{v}_h\|_2^{\frac12}\|\mathbf{w}_h\|_{-1} + \frac{c_0}{2}\|A_h\mathbf{v}_h\|_{0,\Omega}\|\mathbf{u}_h\|_1^{\frac12}\|\mathbf{u}_h\|_2^{\frac12}\|\mathbf{w}_h\|_{-1}. \end{aligned} \tag{3.6}$$

Proof. The first relation comes from a direct computing. To derive (3.6), we let $\phi_h = A_h^{-1}\mathbf{w}_h$ for each $\mathbf{w}_h \in \mathbf{V}_h$, then apply the inverse inequality to get

$$\|\mathbf{w}_h\|_0^2 = (\mathbf{w}_h, A_h\phi_h)_\Omega = (\nabla\mathbf{w}_h, \nabla\phi_h)_\Omega \le \|\mathbf{w}_h\|_1\|\mathbf{w}_h\|_{-1} \le ch^{-1}\|\mathbf{w}_h\|_0\|\mathbf{w}_h\|_{-1}.$$

Now we rewrite $b(\mathbf{u}_h, \mathbf{v}_h, \mathbf{w}_h)$ as

$$\begin{aligned} b(\mathbf{u}_h, \mathbf{v}_h, \mathbf{w}_h) &= ((\mathbf{u}_h \cdot \nabla)(\mathbf{v}_h - A_1^{-1}PA_h\mathbf{v}_h), \mathbf{w}_h) + (P_h(\mathbf{u}_h \cdot \nabla)A_1^{-1}PA_h\mathbf{v}_h, \mathbf{w}_h) \\ &\quad + \frac{1}{2}(\nabla \cdot (\mathbf{u}_h - A_1^{-1}PA_h\mathbf{u}_h)\mathbf{v}_h, \mathbf{w}_h). \end{aligned} \tag{3.7}$$

But, using Lemma 2.2, we can readily derive

$$\begin{aligned} |((\mathbf{u}_h \cdot \nabla)(\mathbf{v}_h - A_1^{-1}PA_h\mathbf{v}_h), \mathbf{w}_h)| &\le \|\mathbf{u}_h\|_{L^\infty}\|\nabla(\mathbf{v}_h - A_1^{-1}PA_h\mathbf{v}_h)\|_{0,\Omega}\|\mathbf{w}_h\|_0 \\ &\le ch\|\nabla\mathbf{u}_h\|_{0,\Omega}^{\frac12}\|A_h\mathbf{u}_h\|_{0,\Omega}^{\frac12}\|A_h\mathbf{v}_h\|_{0,\Omega}\|\mathbf{w}_h\|_0 \\ &\le c\|\mathbf{u}_h\|_1^{\frac12}\|\mathbf{u}_h\|_2^{\frac12}\|A_h\mathbf{v}_h\|_{0,\Omega}\|\mathbf{w}_h\|_{-1}, \\ |(P_h(\mathbf{u}_h \cdot \nabla)A_1^{-1}PA_h\mathbf{v}_h, \mathbf{w}_h)| &\le \|\nabla(P_h(\mathbf{u}_h \cdot \nabla)A^{-1}PA_h\mathbf{v}_h)\|_{0,\Omega}\|\mathbf{w}_h\|_{-1} \\ &\le c\|\nabla((\mathbf{u}_h \cdot \nabla)A_1^{-1}PA_h\mathbf{v}_h)\|_{0,\Omega}\|\mathbf{w}_h\|_{-1} \\ &\le c\|\nabla\mathbf{u}_h\|_{L^3}\|\nabla(A_1^{-1}PA_h\mathbf{v}_h)\|_{L^6}\|\mathbf{w}_h\|_{-1} \\ &\le c\|\nabla\mathbf{u}_h\|_{0,\Omega}^{\frac12}\|A_h\mathbf{u}_h\|_{0,\Omega}^{\frac12}\|A_h\mathbf{v}_h\|_{0,\Omega}\|\mathbf{w}_h\|_{-1}, \\ \frac{1}{2}|(\nabla \cdot (\mathbf{u}_h - A_1^{-1}PA_h\mathbf{u}_h)\mathbf{v}_h, \mathbf{w}_h)| &\le \|\nabla(\mathbf{u}_h - A^{-1}PA_h\mathbf{v}_h)\|_{0,\Omega}\|\mathbf{v}_h\|_{L^\infty}\|\mathbf{w}_h\|_0 \\ &\le c\|A_h\mathbf{u}_h\|_{0,\Omega}\|\nabla\mathbf{v}_h\|_{0,\Omega}^{\frac12}\|A_h\mathbf{v}_h\|_{0,\Omega}^{\frac12}\|\mathbf{w}_h\|_{-1}. \end{aligned}$$

Combining the above inequalities with (3.7) yields

$$|b(\mathbf{u}_h, \mathbf{v}_h, \mathbf{w}_h)| \le \frac{1}{6}c_0\|A_h\mathbf{u}_h\|_{0,\Omega}\|\mathbf{v}_h\|_2^{\frac12}\|\mathbf{v}_h\|_1^{\frac12}\|\mathbf{w}_h\|_{-1} + \frac{1}{6}c_0\|A_h\mathbf{v}_h\|_{0,\Omega}\|\mathbf{u}_h\|_2^{\frac12}\|\mathbf{u}_h\|_1^{\frac12}\|\mathbf{w}_h\|_{-1}.$$

Similarly, we can estimate $b(\mathbf{v}_h, \mathbf{u}_h, \mathbf{w}_h)$, $b(\mathbf{w}_h, \mathbf{u}_h, \mathbf{v}_h)$; then (3.6) is a consequence of these estimates. □

Lemma 3.3. *For all* $\mathbf{u}_h \in \mathbf{V}_h$, $\mathbf{B}_h$ *and* $\mathbf{C}_h \in \mathbf{W}_{0h}$, *the trilinear form* d *in* (1.3) *satisfies the estimates*

$$\begin{aligned} |d(\mathbf{u}_h, \mathbf{B}_h, \mathbf{C}_h)| + |d(\mathbf{u}_h, \mathbf{C}_h, \mathbf{B}_h)| &\le \frac{c_3}{2}\|A_h\mathbf{u}_h\|_{0,\Omega}\|\mathbf{B}_h\|_2^{\frac12}\|\mathbf{B}_h\|_1^{\frac12}\|\mathbf{C}_h\|_{-1} \\ &\quad + \frac{c_3}{2}\|\mathbf{u}_h\|_1^{\frac12}\|\mathbf{u}_h\|_2^{\frac12}\|A_{2h}\mathbf{B}_h\|_{0,\Omega}\|\mathbf{C}_h\|_{-1}, \end{aligned} \tag{3.8}$$

$$\begin{aligned} |d(\mathbf{u}_h, \mathbf{B}_h, \mathbf{C}_h)| + |d(\mathbf{u}_h, \mathbf{C}_h, \mathbf{B}_h)| &\le \frac{c_3}{2}\|\mathbf{u}_h\|_{-1}\|\mathbf{B}_h\|_1^{\frac12}\|\mathbf{B}_h\|_2^{\frac12}\|A_{2h}\mathbf{C}_h\|_{0,\Omega} \\ &\quad + \frac{c_3}{2}\|\mathbf{u}_h\|_{-1}\|A_{2h}\mathbf{B}_h\|_{0,\Omega}\|\mathbf{C}_h\|_1^{\frac12}\|\mathbf{C}_h\|_2^{\frac12}. \end{aligned} \tag{3.9}$$

Proof. We write $\mathbf{u} = A_1^{-1}PA_h\mathbf{u}_h$, $\mathbf{B} = A_2^{-1}PA_{2h}\mathbf{B}_h$, and rewrite $d(\mathbf{u}_h, \mathbf{B}_h, \mathbf{C}_h)$ as

$$\begin{aligned} d(\mathbf{u}_h, \mathbf{B}_h, \mathbf{C}_h) &= (\mathbf{u}_h \times \mathbf{B}_h, \nabla \times \mathbf{C}_h)_\Omega \\ &= (\mathbf{u}_h \times (\mathbf{B}_h - \mathbf{B}), \nabla \times \mathbf{C}_h)_\Omega + (\mathbf{u}_h - \mathbf{u}) \times \mathbf{B}, \nabla \times \mathbf{C}_h)_\Omega + (R_{0h}[(\mathbf{B}\cdot\nabla)\mathbf{u} - (\mathbf{u}\cdot\nabla)\mathbf{B}], \mathbf{C}_h)_\Omega \\ &=: I_1 + I_2 + I_3. \end{aligned} \tag{3.10}$$

Then we can derive by using (3.1)–(3.3), (2.3), Lemma 2.2 and Assumptions (A2)–(A3) that

$$\begin{aligned} \|\mathbf{u}\|_{1,\Omega} &\le c\|\mathbf{u}_h\|_1, & \|\mathbf{B}\|_{1,\Omega} &\le c\|\mathbf{B}_h\|_1, \\ \|\mathbf{u}\|_{2,\Omega} &\le c\|A_h\mathbf{u}_h\|_{0,\Omega}, & \|\mathbf{B}\|_{2,\Omega} &\le c\|A_{2h}\mathbf{B}_h\|_{0,\Omega}, \\ \|\nabla \times \mathbf{C}_h\|_{0,\Omega} &\le ch^{-1}\|\mathbf{C}_h\|_{0,\Omega}, & \|\mathbf{C}_h\|_0^2 &\le \|\mathbf{C}_h\|_1\|\mathbf{C}_h\|_{-1} \le ch^{-1}\|\mathbf{C}_h\|_0\|\mathbf{C}_h\|_{-1}. \end{aligned}$$

Using these estimates, along with (3.1)–(3.3), we can bound I_1, I_2 and I_3 as follows:

$$\begin{aligned} I_1 &\le \sqrt{2}\|\mathbf{u}_h\|_{L^\infty}\|\mathbf{B}_h - A_2^{-1}PA_{2h}\mathbf{B}_h\|_{0,\Omega}\|\nabla \times \mathbf{C}_h\|_{0,\Omega} \le c\|\mathbf{u}_h\|_1^{\frac12}\|\mathbf{u}_h\|_2^{\frac12}\|A_{2h}\mathbf{B}_h\|_{0,\Omega}\|\mathbf{C}_h\|_{-1}, \\ I_2 &\le \sqrt{2}\|\mathbf{B}_h\|_{L^\infty}\|\mathbf{u}_h - A_1^{-1}PA_h\mathbf{u}_h\|_{0,\Omega}\|\nabla \times \mathbf{C}_h\|_{0,\Omega} \le c\|\mathbf{B}_h\|_1^{\frac12}\|\mathbf{B}_h\|_2^{\frac12}\|A_h\mathbf{u}_h\|_{0,\Omega}\|\mathbf{C}_h\|_{-1}, \\ I_3 &\le \|A_{2h}^{\frac12}R_{0h}[(\mathbf{B}\cdot\nabla)\mathbf{u} - (\mathbf{u}\cdot\nabla)\mathbf{B}]\|_{0,\Omega}\|\mathbf{C}_h\|_{-1} \le c\|A_2^{\frac12}[(\mathbf{B}\cdot\nabla)\mathbf{u} - (\mathbf{u}\cdot\nabla)\mathbf{B}]\|_{0,\Omega}\|\mathbf{C}_h\|_{-1} \\ &\le c\|\mathbf{B}\|_{L^\infty}(\|\nabla\mathbf{u}\|_{L^2} + \|\nabla\nabla\mathbf{u}\|_{L^2})\|\mathbf{C}_h\|_{-1} + c[\|\nabla\mathbf{B}\|_{L^3}(\|\mathbf{u}\|_{L^6} + \|\nabla\mathbf{u}\|_{L^6}) + c\|\nabla\nabla\mathbf{B}\|_{L^2}\|\mathbf{u}\|_{L^\infty}]\|\mathbf{C}_h\|_{-1} \\ &\le c[\|\mathbf{B}_h\|_1^{\frac12}\|\mathbf{B}_h\|_2^{\frac12}\|A_h\mathbf{u}_h\|_{0,\Omega} + c\|A_{2h}\mathbf{B}_h\|_{0,\Omega}\|\mathbf{u}_h\|_1^{\frac12}\|\mathbf{u}_h\|_2^{\frac12}]\|\mathbf{C}_h\|_{-1}. \end{aligned}$$

Now it follows readily from the above inequalities and (3.10) that

$$|d(\mathbf{u}_h, \mathbf{B}_h, \mathbf{C}_h)| \le c\|\mathbf{B}_h\|_1^{\frac12}\|\mathbf{B}_h\|_2^{\frac12}\|A_h\mathbf{u}_h\|_{0,\Omega}\|\mathbf{C}_h\|_{-1} + c\|A_{2h}\mathbf{B}_h\|_{0,\Omega}\|\mathbf{u}_h\|_1^{\frac12}\|\mathbf{u}_h\|_2^{\frac12}\|C_h\|_{-1}. \tag{3.11}$$

Similarly, we can rewrite $d(\mathbf{u}_h, \mathbf{C}_h, \mathbf{B}_h)$ as

$$\begin{aligned} d(\mathbf{u}_h, \mathbf{C}_h, \mathbf{B}_h) &= ((\mathbf{u}_h - \mathbf{u}) \times \mathbf{C}_h, \nabla \times \mathbf{B}_h)_\Omega - (\mathbf{u} \times (\nabla \times (\mathbf{B}_h - \mathbf{B})), \mathbf{C}_h)_\Omega - (R_{0h}[\mathbf{u} \times (\nabla \times \mathbf{B})], \mathbf{C}_h)_\Omega \\ &=: J_1 + J_2 + J_3. \end{aligned} \tag{3.12}$$

Then we can deduce from (3.1)–(3.3), (2.3), Lemma 2.2 and Assumption (A3) that

$$\begin{aligned} J_1 &\le \sqrt{2}\|\mathbf{u}_h - \mathbf{u}\|_{L^6}\|\mathbf{C}_h\|_{L^2}\|\nabla \times \mathbf{B}_h\|_{L^3} \le c\|A_h\mathbf{u}_h\|_{0,\Omega}\|\mathbf{B}_h\|_1^{\frac12}\|\mathbf{B}_h\|_2^{\frac12}\|\mathbf{C}_h\|_{-1}, \\ J_2 &\le \sqrt{2}\|\mathbf{u}\|_{L^\infty}\|\nabla \times (\mathbf{B}_h - \mathbf{B})\|_{0,\Omega}\|\mathbf{C}_h\|_{0,\Omega} \le c\|\mathbf{u}_h\|_1^{\frac12}\|\mathbf{u}_h\|_2^{\frac12}\|A_{2h}\mathbf{B}_h\|_{0,\Omega}\|\mathbf{C}_h\|_{-1}, \\ J_3 &\le \|A_{2h}^{\frac12}R_{0h}[\mathbf{u} \times (\nabla \times \mathbf{B})]\|_{0,\Omega}\|\mathbf{C}_h\|_{-1} \le \|A_2^{\frac12}[\mathbf{u} \times (\nabla \times \mathbf{B})]\|_{0,\Omega}\|\mathbf{C}_h\|_{-1} \\ &\le c\|\mathbf{u}\|_{L^\infty}(\|\nabla \times \mathbf{B}\|_{L^2} + \|\nabla \times \nabla \times \mathbf{B}\|_{L^2})\|\mathbf{C}_h\|_{-1} + c\|\nabla\mathbf{u}\|_{L^3}\|\nabla \times \mathbf{B}\|_{L^6}\|\mathbf{C}_h\|_{-1} \\ &\le c\|\mathbf{u}_h\|_1^{\frac12}\|\mathbf{u}_h\|_2^{\frac12}\|A_{2h}\mathbf{B}_h\|_{0,\Omega}\|\mathbf{C}_h\|_{-1}. \end{aligned}$$

A direct application of these estimates to (3.12) yields

$$|d(\mathbf{u}_h, \mathbf{C}_h, \mathbf{B}_h)| \le c\|\mathbf{B}_h\|_1^{\frac12}\|\mathbf{B}_h\|_2^{\frac12}\|A_h\mathbf{u}_h\|_{0,\Omega}\|\mathbf{C}_h\|_{-1} + c\|A_{2h}\mathbf{B}_h\|_{0,\Omega}\|\mathbf{u}_h\|_1^{\frac12}\|\mathbf{u}_h\|_2^{\frac12}\|\mathbf{C}_h\|_{-1},$$

which, along with (3.11), gives (3.8). Similarly, we can prove (3.9). □

Next, we derive some smoothing properties of the semi-discrete solution $(\mathbf{u}_h(t), p_h(t), \mathbf{B}_h(t))$ to system (2.1)–(2.2).

Lemma 3.4. *Under Assumptions* (A0)–(A3), *it holds, for all* $t \in [0, T]$,

$$\|\mathbf{u}_h(t)\|_1^2 + s\|\mathbf{B}_h(t)\|_1^2 + \int_0^t [\nu\|\mathbf{u}_h\|_2^2 + s\mu\|\mathbf{B}_h\|_2^2 + \|\mathbf{u}_{ht}\|_0^2 + s\|\mathbf{B}_{ht}\|_0^2]\,dt \le \kappa.$$

Proof. Summing up equation (2.1) with $(\mathbf{v}_h, q_h) = (A_h\mathbf{u}_h, 0)$ and equation (2.2) with $\mathbf{C}_h = sA_{2h}\mathbf{B}_h$, and using Young's inequality, we obtain the energy inequality

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|A_h^{\frac{1}{2}}\mathbf{u}_h\|_1^2 + \nu\|A_h\mathbf{u}_h\|_{0,\Omega}^2 + \frac{s}{2}\frac{d}{dt}\|\mathbf{B}_h\|_1^2 + s\mu\|A_{2h}\mathbf{B}_h\|_{0,\Omega}^2 \\ + b(\mathbf{u}_h, \mathbf{u}_h, A_h\mathbf{u}_h) + sd(A_h\mathbf{u}_h, \mathbf{B}_h, \mathbf{B}_h) - sd(\mathbf{u}_h, \mathbf{B}_h, A_{2h}\mathbf{B}_h) \le \frac{\nu}{8}\|A_h\mathbf{u}_h\|_{0,\Omega}^2 + \frac{4}{\nu}\|\mathbf{f}\|_{0,\Omega}^2.\end{aligned} \tag{3.13}$$

Then, using (3.1)–(3.2) and Young's inequality, we further derive

$$\begin{aligned}|b(\mathbf{u}_h, \mathbf{u}_h, A_h\mathbf{u}_h)| &\le \|\mathbf{u}_h\|_{L^\infty}\|\nabla\mathbf{u}_h\|_{L^2}\|A_h\mathbf{u}_h\|_{L^2} \le c_0\|\nabla\mathbf{u}_h\|_{0,\Omega}^{\frac{3}{2}}\|A_h\mathbf{u}_h\|_{0,\Omega}^{\frac{3}{2}} \\ &\le \frac{\nu}{8}\|A_h\mathbf{u}_h\|_{0,\Omega}^2 + \Big(\frac{4}{\nu}\Big)^3 c_0^4\|\nabla\mathbf{u}_h\|_{0,\Omega}^4\|A_h^{\frac{1}{2}}\mathbf{u}_h\|_{0,\Omega}^2, \\ s|d(A_h\mathbf{u}_h, \mathbf{B}_h, \mathbf{B}_h)| &\le s\sqrt{2}\|A_h\mathbf{u}_h\|_{L^2}\|\mathbf{B}_h\|_{L^6}\|\nabla\times\mathbf{B}_h\|_{L^3} \\ &\le sc_0\|A_h\mathbf{u}_h\|_{0,\Omega}\|A_{2h}^{\frac{1}{2}}\mathbf{B}_h\|_{0,\Omega}\|A_{2h}\mathbf{B}_h\|_{0,\Omega}^{\frac{1}{2}}\|\nabla\times\mathbf{B}_h\|_{0,\Omega}^{\frac{1}{2}} \\ &\le \frac{\nu}{16}\|A_h\mathbf{u}_h\|_{0,\Omega}^2 + \frac{s\mu}{16}\|A_{2h}\mathbf{B}_h\|_{0,\Omega}^2 + 4^3\nu^{-2}\mu^{-1}s^3c_0^4\|\nabla\times\mathbf{B}_h\|_{0,\Omega}^2\|A_{2h}^{\frac{1}{2}}\mathbf{B}_h\|_{0,\Omega}^4, \\ s|d(\mathbf{u}_h, \mathbf{B}_h, A_{2h}\mathbf{B}_h)| &\le (\|\mathbf{B}_h\|_{L^6}\|\nabla\mathbf{u}_h\|_{L^3} + \|\mathbf{u}_h\|_{L^\infty}\|\nabla\mathbf{B}_h\|_{L^2})\|A_{2h}\mathbf{B}_h\|_{0,\Omega} \\ &\le sc_0\|\nabla\mathbf{u}_h\|_{0,\Omega}^{\frac{1}{2}}\|A_h\mathbf{u}_h\|_{0,\Omega}^{\frac{1}{2}}\|A_{2h}^{\frac{1}{2}}\mathbf{B}\|_{0,\Omega}\|A_{2h}\mathbf{B}_h\|_{0,\Omega} \\ &\le \frac{s\mu}{16}\|A_{2h}\mathbf{B}_h\|_{0,\Omega}^2 + \frac{\nu}{16}\|A_h\mathbf{u}_h\|_{0,\Omega}^2 + \frac{4}{\nu}\Big(\frac{4}{\mu}\Big)^2 s^2c_0^4\|\nabla\mathbf{u}_h\|_{0,\Omega}^2\|A_{2h}^{\frac{1}{2}}\mathbf{B}_h\|_{0,\Omega}^4.\end{aligned}$$

Combining the above inequalities with (3.13) yields

$$\begin{aligned}\frac{d}{dt}\|\mathbf{u}_h\|_1^2 + \nu\|A_h\mathbf{u}_h\|_{0,\Omega}^2 + s\frac{d}{dt}\|\mathbf{B}_h\|_1^2 + s\mu\|A_{2h}\mathbf{B}_h\|_{0,\Omega}^2 \\ \le \frac{8}{\nu}\|\mathbf{f}\|_{0,\Omega}^2 + c(\mu + \|\nabla\mathbf{u}_h\|_{0,\Omega}^4 + \|\mathbf{B}_h\|_{0,\Omega}^4 + \|\nabla\times\mathbf{B}_h\|_{0,\Omega}^4)(\|\mathbf{u}_h\|_1^2 + s\|\mathbf{B}_h\|_1^2).\end{aligned}$$

Integrating both sides from 0 to t and using Lemma 2.1 and Assumption (A0), we obtain

$$\begin{aligned}\|\mathbf{u}_h(t)\|_1^2 + s\|\mathbf{B}_h(t)\|_1^2 + \int_0^t (\nu\|A_h\mathbf{u}_h\|_{0,\Omega}^2 + s\mu\|A_{2h}\mathbf{B}_h\|_{0,\Omega}^2)\,ds \\ \le \kappa_1 + \int_0^t c(\|\nabla_h\mathbf{u}_h\|_{0,\Omega}^4 + \|\nabla\times\mathbf{B}_h\|_{0,\Omega}^4)(\|\mathbf{u}_h\|_1^2 + s\|\mathbf{B}_h\|_1^2)\,ds.\end{aligned} \tag{3.14}$$

Now, applying Gronwall's inequality to (3.14), we readily get

$$\|\mathbf{u}_h(t)\|_1^2 + s\|\mathbf{B}_h(t)\|_1^2 + \int_0^t (\nu\|A_h\mathbf{u}_h\|_{0,\Omega}^2 + s\mu\|A_{2h}\mathbf{B}_h\|_{0,\Omega}^2)\,ds \le \kappa\exp\Big\{c\int_0^T (\|\nabla_h\mathbf{u}_h\|_{0,\Omega}^4 + \|\nabla\times\mathbf{B}_h\|_{0,\Omega}^4)\,ds\Big\}. \tag{3.15}$$

But it follows from Assumption (A3) and Lemma 2.1 that

$$\begin{aligned}\int_0^T (\|\nabla_h\mathbf{u}_h\|_{0,\Omega}^4 + \|\nabla\times\mathbf{B}_h\|_{0,\Omega}^4)\,ds &\le 2\int_0^T (\|\nabla\mathbf{u}\|_{0,\Omega}^2\|\nabla\mathbf{u}_h\|_{0,\Omega}^2 + \|\nabla\times\mathbf{B}\|_{0,\Omega}^2\|\nabla\times\mathbf{B}_h\|_{0,\Omega}^2)\,ds \\ &\quad + 2\int_0^T (\|\nabla(\mathbf{u}-\mathbf{u}_h)\|_{0,\Omega}^2\|\nabla\mathbf{u}_h\|_{0,\Omega}^2 + \|\nabla\times(\mathbf{B}-\mathbf{B}_h)\|_{0,\Omega}^2\|\nabla\times\mathbf{B}_h\|_{0,\Omega}^2)\,ds \le \kappa.\end{aligned}$$

This, along with (3.15) and applying the Gronwall lemma to (3.14), yields

$$\|\mathbf{u}_h(t)\|_1^2 + s\|\mathbf{B}_h(t)\|_1^2 + \int_0^t (\nu\|A_h\mathbf{u}_h\|_{0,\Omega}^2 + s\mu\|A_{2h}\mathbf{B}_h\|_{0,\Omega}^2)\,ds \le \kappa. \tag{3.16}$$

Furthermore, we readily derive from (2.1)–(2.2), (3.1)–(3.2) and the Hölder inequality that

$$\begin{aligned}\|\mathbf{u}_{ht}\|_0 &\le \nu\|A_h\mathbf{u}_h\|_{0,\Omega} + c\|\mathbf{u}_h\|_{L^\infty}\|\nabla\mathbf{u}_h\|_{L^2} + cs\|\mathbf{B}_h\|_{L^6}\|\nabla\times\mathbf{B}_h\|_{L^3} + \|\mathbf{f}\|_{0,\Omega}\\ &\le \nu\|A_h\mathbf{u}_h\|_{0,\Omega} + c\|A_h\mathbf{u}_h\|_{0,\Omega}\|\mathbf{u}_h\|_1 + c\|A_{2h}\mathbf{B}_h\|_{0,\Omega}\|\mathbf{B}_h\|_1 + c\|\mathbf{f}\|_{0,\Omega},\\ \|\mathbf{B}_{ht}\|_0 &\le \mu\|A_{2h}\mathbf{B}_h\|_{0,\Omega} + c\|\nabla\mathbf{u}_h\|_{L^3}\|\mathbf{B}_h\|_{L^6} + c\|\nabla\mathbf{B}_h\|_{L^2}\|\mathbf{u}_h\|_{L^\infty}\\ &\le \mu\|A_{2h}\mathbf{B}_h\|_{0,\Omega} + c\|A_h\mathbf{u}_h\|_{0,\Omega}\|\mathbf{B}_h\|_1.\end{aligned}$$

Combining these inequalities with (3.16) concludes the proof of Lemma 3.4. □

Lemma 3.5. *Under Assumptions* (A0)–(A3)*, the solution* $(\mathbf{u}_h(t), p_h(t), \mathbf{B}_h(t))$ *to the semi-discrete system* (2.1)–(2.2) *satisfies*

$$\begin{aligned}\{\|\mathbf{u}_{ht}(t)\|_0^2 + s\|\mathbf{B}_{ht}(t)\|_0^2 + \|\mathbf{u}_h(t)\|_2^2 + \|-\nu\Delta_h\mathbf{u}_h(t) + \nabla_h p_h(t)\|_{0,\Omega}^2 + \|\mathbf{B}_h(t)\|_2\}\\ + \int_0^t (\nu\|\mathbf{u}_{ht}(s)\|_1^2 + s\mu\|\mathbf{B}_{ht}(s)\|_1^2 + \|u_{htt}\|_{-1}^2 + \|\mathbf{B}_{htt}\|_{-1}^2)\,ds \le \kappa \quad \textit{for all } t\in[0,T].\end{aligned}\tag{3.17}$$

Proof. Differentiating (2.1) and (2.2) with respect to t, we obtain, for all $(\mathbf{v}_h, q_h, \mathbf{C}_h) \in \mathbf{X}_h\times M_h\times\mathbf{W}_h$,

$$\begin{aligned}(\mathbf{u}_{htt}, \mathbf{v}_h)_\Omega + \nu(\nabla\mathbf{u}_{ht}, \nabla\mathbf{v}_h)_\Omega - (\nabla\cdot\mathbf{v}_h, p_{ht})_\Omega + (\nabla\cdot\mathbf{u}_{ht}, q_h)_\Omega\\ +b(\mathbf{u}_{ht}, \mathbf{u}_h, \mathbf{v}_h) + b(\mathbf{u}_h, \mathbf{u}_{ht}, \mathbf{v}_h) + sd(\mathbf{v}_h, \mathbf{B}_{ht}, \mathbf{B}_h) + sd(\mathbf{v}_h, \mathbf{B}_h, \mathbf{B}_{ht}) = (\mathbf{f}_t, \mathbf{v}_h)_\Omega,\end{aligned}\tag{3.18}$$

$$(\mathbf{B}_{htt}, \mathbf{C}_h)_\Omega + \mu(\nabla\times\mathbf{B}_{ht}, \nabla\times\mathbf{C}_h)_\Omega + \mu(\nabla\cdot\mathbf{B}_{ht}, \nabla\cdot\mathbf{C}_h)_\Omega - d(\mathbf{u}_{ht}, \mathbf{B}_h, \mathbf{C}_h) - d(\mathbf{u}_h, \mathbf{B}_{ht}, \mathbf{C}_h) = 0. \tag{3.19}$$

Summing up (3.18) with $(\mathbf{v}_h, q_h) = (\mathbf{u}_{ht}, p_{ht})$ and (3.19) with $\mathbf{C}_h = s\mathbf{B}_{ht}$ and then using (3.6), we derive

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\mathbf{u}_{ht}\|_0^2 + \nu\|\mathbf{u}_{ht}\|_1^2 + \frac{s}{2}\frac{d}{dt}\|\mathbf{B}_{ht}\|_0^2 + s\mu\|\mathbf{B}_{ht}\|_1^2\\ + b(\mathbf{u}_{ht}, \mathbf{u}_h, \mathbf{u}_{ht}) + sd(\mathbf{u}_{ht}, \mathbf{B}_{ht}, \mathbf{B}_h) - sd(\mathbf{u}_h, \mathbf{B}_{ht}, \mathbf{B}_{ht}) = \frac{\nu}{8}\|\mathbf{u}_{ht}\|_1^2 + \frac{4}{\nu}\gamma^2\|\mathbf{f}_t\|_{0,\Omega}^2,\end{aligned}\tag{3.20}$$

where γ is the Poincaré constant satisfying $\|\mathbf{u}\|_{0,\Omega} \le \gamma\|\nabla\mathbf{u}\|_{0,\Omega}$.

Using (3.1)–(3.2), we can estimate the three trilinear terms in (3.20) as follows:

$$\begin{aligned}|b(\mathbf{u}_{ht}, \mathbf{u}_h, \mathbf{u}_{ht})| &\le c_0\|\mathbf{u}_h\|_2\|\mathbf{u}_{ht}\|_0\|\mathbf{u}_{ht}\|_1 \le \frac{\nu}{16}\|\mathbf{u}_{ht}\|_1^2 + \frac{4}{\nu}c_0^2\|\mathbf{u}_h\|_2^2\|\mathbf{u}_{ht}\|_0^2,\\ s|d(\mathbf{u}_{ht}, \mathbf{B}_{ht}, \mathbf{B}_h)| &\le \sqrt{2}s\|\mathbf{u}_{ht}\|_{L^6}\|\mathbf{B}_{ht}\|_{L^2}\|\nabla\times\mathbf{B}_h\|_{L^3} \le \frac{\nu}{16}\|\mathbf{u}_{ht}\|_1^2 + s^2c_0^2\frac{4}{\nu}\|\mathbf{B}_h\|_2^2\|\mathbf{B}_{ht}\|_0^2,\\ s|d(\mathbf{u}_h, \mathbf{B}_{ht}, \mathbf{B}_{ht})| &\le sc_0\|\mathbf{B}_{ht}\|_1\|\mathbf{u}_h\|_2\|\mathbf{B}_{ht}\|_0 \le \frac{s\mu}{16}\|\mathbf{B}_{ht}\|_1^2 + \frac{4}{\mu}sc_0^2\|\mathbf{u}_h\|_2^2\|\mathbf{B}_{ht}\|_0^2.\end{aligned}$$

Combining these three estimates with (3.20) yields

$$\frac{d}{dt}(\|\mathbf{u}_{ht}\|_0^2 + s\|\mathbf{B}_{ht}\|_0^2) + (\nu\|\mathbf{u}_{ht}\|_1^2 + s\mu\|\mathbf{B}_{ht}\|_1^2) \le c(\mu + \nu\|\mathbf{u}_h\|_2^2 + s\mu\|\mathbf{B}_h\|_2^2)(\|\mathbf{u}_{ht}\|_0^2 + s\|\mathbf{B}_{ht}\|_0^2) + c\|\mathbf{f}_t\|_{0,\Omega}^2.$$

Integrating both sides from 0 to t and using Gronwall's lemma and Lemmas 2.1 and 3.4, we obtain

$$\|\mathbf{u}_{ht}(t)\|_0^2 + s\|\mathbf{B}_{ht}(t)\|_0^2 + \int_0^t (\nu\|\mathbf{u}_{ht}\|_1^2 + s\mu\|\mathbf{B}_{ht}\|_1^2)\,ds \le \kappa. \tag{3.21}$$

We continue estimating the remaining terms in (3.17). First, we can bound the two negative-norm terms by directly using equations (2.1)–(2.2),

$$\|\mathbf{u}_{htt}\|_{-1} \le \nu\|u_{ht}\|_1 + c\|\mathbf{u}_{ht}\|_1\|\mathbf{u}_h\|_1 + c\|\mathbf{B}_{ht}\|_1\|\mathbf{B}_h\|_1 + c\|\mathbf{f}_t\|_{0,\Omega}, \tag{3.22}$$

$$\|\mathbf{B}_{htt}\|_{-1} \le 2\mu\|\mathbf{B}_{ht}\|_1 + c\|\mathbf{u}_{ht}\|_1\|\mathbf{B}_h\|_1 + c\|\mathbf{u}_h\|_1\|\mathbf{B}_{ht}\|_1. \tag{3.23}$$

To bound the term $-\nu\Delta_h\mathbf{u}_h(t)+\nabla_h p_h(t)$, we can use (2.1)–(2.2), (3.1)–(3.2) and Young's inequality to deduce

$$\begin{aligned}
\nu\|A_h\mathbf{u}_h(t)\|_{0,\Omega} &\le \|\mathbf{u}_{ht}(t)\|_0+\|\mathbf{f}(t)\|_{0,\Omega}+\|\nabla\mathbf{u}_h\|_{L^2}\|\mathbf{u}_h\|_{L^\infty}+s\sqrt{2}\|\mathbf{B}_h\|_{L^6}\|\nabla\times\mathbf{B}_h\|_{L^3}\\
&\le \frac{\nu}{4}\|A_h\mathbf{u}_h(t)\|_{0,\Omega}+\frac{\mu}{4}\|A_{2h}\mathbf{B}_h\|_{0,\Omega}+c\|\mathbf{u}_h(t)\|_1^3+c\|\mathbf{B}_h\|_1^3+\|\mathbf{u}_{ht}(t)\|_{0,\Omega}+c\|\mathbf{f}(t)\|_{0,\Omega},\\
\mu\|A_{2h}\mathbf{B}_h(t)\|_{0,\Omega} &\le \|\mu\mathbf{B}_h(t)-\mathbf{B}_{ht}(t)\|_{0,\Omega}+c\|\nabla\mathbf{u}_h(t)\|_{L^3}\|\mathbf{B}_h(t)\|_{L^6}^2+c\|\mathbf{u}_h(t)\|_{L^\infty}\|\nabla\mathbf{B}_h(t)\|_{L^2}\\
&\le \frac{\nu}{4}\|A_h\mathbf{u}_h(t)\|_{0,\Omega}+c\|\mu\mathbf{B}_h(t)-\mathbf{B}_{ht}(t)\|_{0,\Omega}+c\|\mathbf{u}_h(t)\|_1\|\mathbf{B}_h(t)\|_1^2.
\end{aligned}\tag{3.24}$$

Then we can readily get

$$\begin{aligned}
\|-\nu\Delta_h\mathbf{u}_h+\nabla_h p_h\|_{0,\Omega} &\le \|\mathbf{u}_{ht}\|_0+c\|\mathbf{u}_h\|_{L^\infty}\|\nabla\mathbf{u}_h\|_{L^2}+cs\|\mathbf{B}_h\|_{L^6}\|\nabla\times\mathbf{B}_h\|_{L^3}+\|\mathbf{f}\|_{0,\Omega}\\
&\le \|\mathbf{u}_{ht}\|_0+c\|A_h\mathbf{u}_h\|_{0,\Omega}\|\mathbf{u}_h\|_1+c\|A_{2h}\mathbf{B}_h\|_{0,\Omega}\|\mathbf{B}_h\|_1+c\|\mathbf{f}\|_{0,\Omega},
\end{aligned}\tag{3.25}$$

Now, combining estimates (3.21)–(3.25) and using Lemma 3.4, we can see the desired estimate (3.17). □

Lemma 3.6. *Under Assumptions* (A0)–(A3), *the solution* $(\mathbf{u}_h(t),p_h(t),\mathbf{B}_h(t))$ *to the semi-discrete system* (2.1)–(2.2) *satisfies the estimate*

$$\begin{aligned}
&\sigma(t)[\|\mathbf{u}_{ht}(t)\|_1^2+s\|\mathbf{B}_{ht}(t)\|_1^2+\|\mathbf{u}_{htt}(t)\|_{-1}^2+\|\mathbf{B}_{htt}(t)\|_{-1}^2]\\
&\quad+\int_0^t\sigma(s)(\nu\|A_h\mathbf{u}_{ht}(s)\|_{0,\Omega}^2+\|-\nu\Delta_h\mathbf{u}_{ht}(s)+\nabla_h p_{ht}(s)\|_{0,\Omega}^2+s\mu\|A_{2h}\mathbf{B}_{ht}(s)\|_{0,\Omega}^2)\,ds\\
&\quad+\int_0^t\sigma(s)(\|\mathbf{u}_{htt}(s)\|_0^2+\|\mathbf{B}_{htt}(s)\|_{0,\Omega}^2)\,ds\le\kappa\quad\text{for all } t\in[0,T].
\end{aligned}$$

Proof. Taking the sum of (3.18) with $(\mathbf{v}_h,q_h)=(A_h\mathbf{u}_{ht},0)$ and (3.19) with $\mathbf{C}_h=sA_{2h}\mathbf{B}_{ht}$, we obtain

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|\mathbf{u}_{ht}\|_1^2+\nu\|A_h\mathbf{u}_{ht}\|_{0,\Omega}^2+\frac{s}{2}\frac{d}{dt}\|\mathbf{B}_{ht}\|_1^2+s\mu\|A_{2h}\mathbf{B}_{ht}\|_{0,\Omega}^2\\
&\quad+b(\mathbf{u}_h,\mathbf{u}_{ht},A_h\mathbf{u}_{ht})+b(\mathbf{u}_{ht},\mathbf{u}_h,A_h\mathbf{u}_{ht})\\
&\quad+sd(A_h\mathbf{u}_{ht},\mathbf{B}_{ht},\mathbf{B}_h)+sd(A_h\mathbf{u}_{ht},\mathbf{B}_h,\mathbf{B}_{ht})\\
&\quad-sd(\mathbf{u}_{ht},\mathbf{B}_h,A_{2h}\mathbf{B}_{ht})-sd(\mathbf{u}_h,\mathbf{B}_{ht},A_{2h}\mathbf{B}_{ht})=\frac{\nu}{8}\|A_h\mathbf{u}_{ht}\|_{0,\Omega}^2+\frac{4}{\nu}\|\mathbf{f}_t\|_{0,\Omega}^2.
\end{aligned}\tag{3.26}$$

Using (3.1)–(3.2), we can further bound all the trilinear terms above as follows:

$$\begin{aligned}
|b(\mathbf{u}_{ht},\mathbf{u}_h,A_h\mathbf{u}_{ht})|+|b(\mathbf{u}_h,\mathbf{u}_{ht},A_h\mathbf{u}_{ht})| &\le c_0\|\mathbf{u}_h\|_2\|\mathbf{u}_{ht}\|_1\|A_h\mathbf{u}_{ht}\|_{0,\Omega}\\
&\le \frac{\nu}{16}\|A_h\mathbf{u}_{ht}\|_{0,\Omega}^2+\frac{4}{\nu}c_0^2\|\mathbf{u}_h\|_2^2\|\mathbf{u}_{ht}\|_1^2,\\
s|d(A_h\mathbf{u}_{ht},\mathbf{B}_{ht},\mathbf{B}_h)|+s|d(A_h\mathbf{u}_{ht},\mathbf{B}_h,\mathbf{B}_{ht})| &\le sc_0\|A_h\mathbf{u}_{ht}\|_{0,\Omega}\|\mathbf{B}_h\|_2\|\mathbf{B}_{ht}\|_1\\
&\le \frac{\nu}{16}\|A_h\mathbf{u}_{ht}\|_{0,\Omega}^2+\frac{4}{\nu}s^2c_0^2\|\mathbf{B}_h\|_2^2\|\mathbf{B}_{ht}\|_1^2,\\
s|d(\mathbf{u}_{ht},\mathbf{B}_h,A_{2h}\mathbf{B}_{ht})|+s|d(\mathbf{u}_h,\mathbf{B}_{ht},A_{2h}\mathbf{B}_{ht})| &\le sc_0\|A_{2h}\mathbf{B}_{ht}\|_{0,\Omega}(\|\mathbf{u}_h\|_2\|\mathbf{B}_{ht}\|_1+\|\mathbf{B}_h\|_2\|\mathbf{u}_{ht}\|_1)\\
&\le \frac{s\mu}{16}\|A_{2h}\mathbf{B}_{ht}\|_{0,\Omega}^2+\mu^{-1}4^2sc_0^2(\|\mathbf{u}_h\|_2^2+\|\mathbf{B}_h\|_2^2)(\|\mathbf{u}_{ht}\|_1^2+\|\mathbf{B}_{ht}\|_1^2).
\end{aligned}$$

Combining the above inequalities with (3.26) yields

$$\begin{aligned}
&\frac{d}{dt}(\|\mathbf{u}_{ht}\|_1^2+s\|\mathbf{B}_{ht}\|_1^2)+(\nu\|A_h\mathbf{u}_{ht}\|_{0,\Omega}^2+s\mu\|A_{2h}\mathbf{B}_{ht}\|_{0,\Omega}^2)\\
&\quad\le c(\mu+\|\mathbf{u}_h\|_2^2+\|\mathbf{B}_h\|_2^2)(\|\mathbf{u}_{ht}\|_1^2+s\|\mathbf{B}_{ht}\|_1^2)+c\|\mathbf{f}_t\|_{0,\Omega}^2.
\end{aligned}\tag{3.27}$$

Multiplying (3.27) by $\sigma(t)$, integrating with respect to t, and then using Lemmas 3.4–3.5, we readily get

$$\sigma(t)(\|\mathbf{u}_{ht}(t)\|_1^2+s\|\mathbf{B}_{ht}(t)\|_1^2)+\int_0^t\sigma(t)(\nu\|A_h\mathbf{u}_{ht}\|_{0,\Omega}^2+s\mu\|A_{2h}\mathbf{B}_{ht}\|_{0,\Omega}^2)\,ds\le\kappa.\tag{3.28}$$

Furthermore, we can derive from (3.18)–(3.19) and (3.1)–(3.2) that

$$\|\mathbf{u}_{htt}\|_0 \le \|\mathbf{f}_t\|_0 + \nu\|A_h\mathbf{u}_{ht}\|_{0,\Omega} + c\|\mathbf{u}_{ht}\|_1\|\mathbf{u}_h\|_2 + sc\|\mathbf{B}_{ht}\|_1\|\mathbf{B}_h\|_2,$$

$$\|-\nu\Delta_h\mathbf{u}_{ht} + \nabla_h p_{ht}\|_{0,\Omega} \le \|\mathbf{u}_{htt}\|_0 + \|\mathbf{f}_t\|_0 + c\|\mathbf{u}_{ht}\|_1\|\mathbf{u}_h\|_2 + sc\|\mathbf{B}_{ht}\|_1\|\mathbf{B}_h\|_2, \tag{3.29}$$

$$\|\mathbf{B}_{htt}\|_0 \le \mu\|A_{2h}\mathbf{B}_{ht}\|_{0,\Omega} + c\|\mathbf{u}_{ht}\|_1\|\mathbf{B}_h\|_2 + c\|\mathbf{B}_{ht}\|_1\|\mathbf{u}_h\|_2. \tag{3.30}$$

Combining (3.29)–(3.30) with (3.28) and using Lemmas 3.4–3.5, we conclude the proof of Lemma 3.6. □

Lemma 3.7. *Under Assumptions* (A0)–(A3), *the solution* $(\mathbf{u}_h(t), p_h(t), \mathbf{B}_h(t))$ *to the semi-discrete system* (2.1)–(2.2) *satisfies the estimate*

$$\begin{aligned}\sigma^2(t)[\|\mathbf{u}_{htt}(t)\|_0^2 + s\|\mathbf{B}_{htt}(t)\|_0^2] &+ \int_0^t \sigma^2(t)[\nu\|\mathbf{u}_{htt}(s)\|_1^2 + s\mu\|\mathbf{B}_{htt}(s)\|_1^2 + \|\mathbf{u}_{httt}\|_{-1}^2 + \|\mathbf{B}_{httt}\|_{-1}^2]\,ds \\ &+ \int_0^t \sigma(t)[\|\mathbf{u}_{httt}\|_{-2}^2 + \|\mathbf{B}_{httt}\|_{-2}^2]\,ds \le \kappa,\end{aligned} \tag{3.31}$$

$$\sigma^2(t)[\|\mathbf{u}_{ht}(t)\|_2^2 + \|\mathbf{B}_{ht}(t)\|_2^2 + \|p_{ht}(t)\|_{0,\Omega}^2] \le \kappa. \tag{3.32}$$

Proof. Differentiating (3.18) and (3.19) with respect to t, we derive, for all $(\mathbf{v}_h, q_h, \mathbf{C}_h) \in \mathbf{X}_h \times M_h \times \mathbf{W}_h$,

$$\begin{aligned}(\mathbf{u}_{httt}, \mathbf{v}_h)_\Omega &+ \nu(\nabla\mathbf{u}_{htt}, \nabla\mathbf{v}_h)_\Omega - (\nabla\cdot\mathbf{v}_h, p_{htt})_\Omega + (\nabla\cdot\mathbf{u}_{htt}, q_h)_\Omega \\ &+ b(\mathbf{u}_{htt}, \mathbf{u}_h, \mathbf{v}_h) + 2b(\mathbf{u}_{ht}, \mathbf{u}_{ht}, \mathbf{v}_h) + b(\mathbf{u}_h, \mathbf{u}_{htt}, \mathbf{v}_h) \\ &+ sd(\mathbf{v}_h, \mathbf{B}_{htt}, \mathbf{B}_h) + 2sd(\mathbf{v}_h, \mathbf{B}_{ht}, \mathbf{B}_{ht}) + sd(\mathbf{v}_h, \mathbf{B}_h, \mathbf{B}_{htt}) = (\mathbf{f}_{tt}, \mathbf{v}_h)_\Omega,\end{aligned} \tag{3.33}$$

$$\begin{aligned}(\mathbf{B}_{httt}, \mathbf{C}_h)_\Omega &+ \mu(\nabla\times\mathbf{B}_{htt}, \nabla\times\mathbf{C}_h)_\Omega + \mu(\nabla\cdot\mathbf{B}_{htt}, \nabla\cdot\mathbf{C}_h)_\Omega \\ &- d(\mathbf{u}_{htt}, \mathbf{B}_h, \mathbf{C}_h) - 2d(\mathbf{u}_{ht}, \mathbf{B}_{ht}, \mathbf{C}_h) - d(\mathbf{u}_h, \mathbf{B}_{htt}, \mathbf{C}_h) = 0,\end{aligned} \tag{3.34}$$

Taking the sum of (3.33) with $(\mathbf{v}_h, q_h) = (\mathbf{u}_{htt}, p_{htt})$ and (3.34) with $\mathbf{C}_h = s\mathbf{B}_{htt}$ and using (3.6), we have

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\mathbf{u}_{htt}\|_0^2 &+ \nu\|\mathbf{u}_{htt}\|_1^2 + \frac{s}{2}\frac{d}{dt}\|\mathbf{B}_{htt}\|_0^2 + s\mu\|\mathbf{B}_{htt}\|_1^2 \\ &+ b(\mathbf{u}_{htt}, \mathbf{u}_h, \mathbf{u}_{htt}) + 2b(\mathbf{u}_{ht}, \mathbf{u}_{ht}, \mathbf{u}_{htt}) + sd(\mathbf{u}_{htt}, \mathbf{B}_{htt}, \mathbf{B}_h) \\ &+ 2sd(\mathbf{u}_{htt}, \mathbf{B}_{ht}, \mathbf{B}_{ht}) - 2sd(\mathbf{u}_{ht}, \mathbf{B}_{ht}, \mathbf{B}_{htt}) - sd(\mathbf{u}_h, \mathbf{B}_{htt}, \mathbf{B}_{htt}) = \frac{\nu}{16}\|\mathbf{u}_{tt}\|_1^2 + \frac{4}{\nu}\gamma^2\|\mathbf{f}_{tt}\|_{0,\Omega}^2.\end{aligned} \tag{3.35}$$

Using (3.1)–(3.2), we can estimate all the trilinear terms above as follows:

$$|b(\mathbf{u}_{htt}, \mathbf{u}_h, \mathbf{u}_{htt})| \le c_0\|\mathbf{u}_h\|_2\|\mathbf{u}_{htt}\|_0\|\mathbf{u}_{tt}\|_1 \le \frac{\nu}{16}\|\mathbf{u}_{tt}\|_1^2 + \frac{4}{\nu}c_0^2\|\mathbf{u}_h\|_2^2\|\mathbf{u}_{htt}\|_0^2,$$

$$2|b(\mathbf{u}_{ht}, \mathbf{u}_{ht}, \mathbf{u}_{htt})| \le c_0\|\mathbf{u}_{ht}\|_1^2\|\mathbf{u}_{htt}\|_1 \le \frac{\nu}{16}\|\mathbf{u}_{htt}\|_1^2 + \frac{4}{\nu}c_0^2\|\mathbf{u}_{ht}\|_1^4,$$

$$\begin{aligned}s|d(\mathbf{u}_{htt}, \mathbf{B}_{htt}, \mathbf{B}_h)| + s|d(\mathbf{u}_h, \mathbf{B}_{htt}, \mathbf{B}_{htt})| &\le sc_0(\|\mathbf{u}_{htt}\|_1\|\mathbf{B}_h\|_2 + \|\mathbf{B}_{htt}\|_1\|\mathbf{u}_h\|_2)\|\mathbf{B}_{htt}\|_0 \\ &\le \frac{\nu}{16}\|\mathbf{u}_{htt}\|_1^2 + \frac{s\mu}{16}\|\mathbf{B}_{htt}\|_1^2 + \Big(\frac{4}{\nu}s^2c_0^2\|\mathbf{B}_h\|_2^2 + \frac{4}{\mu}sc_0^2\|\mathbf{u}_h\|_2^2\Big)\|\mathbf{B}_{htt}\|_0^2,\end{aligned}$$

$$\begin{aligned}s|d(\mathbf{u}_{htt}, \mathbf{B}_{ht}, \mathbf{B}_{ht})| + s|d(\mathbf{u}_{ht}, \mathbf{B}_{ht}, \mathbf{B}_{htt})| &\le sc_0\|\mathbf{u}_{htt}\|_1\|\mathbf{B}_{ht}\|_1^2 + sc_0\|\mathbf{u}_{ht}\|_1\|\mathbf{B}_{ht}\|_1\|\mathbf{B}_{htt}\|_1 \\ &\le \frac{\nu}{16}\|\mathbf{u}_{htt}\|_1^2 + \frac{s\mu}{16}\|\mathbf{B}_{htt}\|_1^2 + \Big(\frac{4}{\nu}s^2c_0^2\|\mathbf{B}_{ht}\|_1^2 + \frac{4}{\mu}sc_0^2\|\mathbf{u}_{ht}\|_1^2\Big)\|\mathbf{B}_{ht}\|_1^2.\end{aligned}$$

Combining the above inequalities with (3.35) yields

$$\begin{aligned}\frac{d}{dt}(\|\mathbf{u}_{htt}\|_0^2 &+ s\|\mathbf{B}_{htt}\|_0^2) + \nu\|\mathbf{u}_{htt}\|_1^2 + s\mu\|\mathbf{B}_{htt}\|_1^2 \\ &\le c\|\mathbf{f}_{tt}\|_{0,\Omega}^2 + c(\mu + \|\mathbf{B}_h\|_2^2 + \|\mathbf{u}_h\|_2^2)(\|\mathbf{u}_{htt}\|_0^2 + s\|\mathbf{B}_{htt}\|_0^2) + c(\|\mathbf{u}_{ht}\|_1^2 + s\|\mathbf{B}_{ht}\|_1^2)^2.\end{aligned}$$

Now, multiplying by $\sigma^2(t)$, then integrating with respect to t and using Lemmas 3.5–3.6, we can get (3.31), along with the following negative-norm estimates by means of (3.33)–(3.34) and (3.1)–(3.2):

$$\|\mathbf{u}_{httt}\|_{-2} \le \nu\|\mathbf{u}_{htt}\|_0 + c\|\mathbf{f}_{tt}\|_{0,\Omega} + c\|\mathbf{u}_{htt}\|_0\|\mathbf{u}_h\|_1 + c\|\mathbf{u}_{ht}\|_0\|\mathbf{u}_{ht}\|_1 + c\|\mathbf{B}_{htt}\|_0\|\mathbf{B}_h\|_1 + c\|\mathbf{B}_{ht}\|_0\|\mathbf{B}_{ht}\|_1,$$
$$\|\mathbf{B}_{httt}\|_{-2} \le \mu\|\mathbf{B}_{htt}\|_0 + c\|\mathbf{u}_{htt}\|_0\|\mathbf{B}_h\|_1 + c\|\mathbf{u}_h\|_1\|\mathbf{B}_{htt}\|_0 + c\|\mathbf{u}_{ht}\|_0\|\mathbf{B}_{ht}\|_1,$$
$$\|\mathbf{u}_{httt}\|_{-1} \le \nu\|\mathbf{u}_{htt}\|_1 + c\|\mathbf{f}_{tt}\|_{0,\Omega} + c\|\mathbf{u}_{htt}\|_1\|\mathbf{u}_h\|_1 + c\|\mathbf{u}_{ht}\|_1\|\mathbf{u}_{ht}\|_1 + c\|\mathbf{B}_{htt}\|_1\|\mathbf{B}_h\|_1 + c\|\mathbf{B}_{ht}\|_1\|\mathbf{B}_{ht}\|_1,$$
$$\|\mathbf{B}_{httt}\|_{-1} \le \mu\|\mathbf{B}_{htt}\|_1 + c\|\mathbf{u}_{htt}\|_1\|\mathbf{B}_h\|_1 + c\|\mathbf{u}_h\|_1\|\mathbf{B}_{htt}\|_1 + c\|\mathbf{u}_{ht}\|_1\|\mathbf{B}_{ht}\|_1.$$

Bound (3.32) can be obtained directly from (3.18)–(3.19), (3.1)–(3.2) and Assumption (A3) as follows:

$$\begin{aligned}
\nu\|\mathbf{u}_{ht}\|_2 &\le \|\mathbf{f}_t\|_{0,\Omega} + \|\mathbf{u}_{htt}\|_0 + c\|(\mathbf{u}_{ht}\cdot\nabla)\mathbf{u}_h\|_{0,\Omega} + c\|(\mathbf{u}_h\cdot\nabla)\mathbf{u}_{ht}\|_{0,\Omega}\\
&\qquad + c\|\mathbf{B}_{ht}\times(\nabla\times\mathbf{B}_h)\|_{0,\Omega} + c\|\mathbf{B}_h\times(\nabla\times\mathbf{B}_{ht})\|_{0,\Omega}\\
&\le \|\mathbf{f}_t\|_{0,\Omega} + \|\mathbf{u}_{htt}\|_0 + c\|\mathbf{u}_{ht}\|_1\|\mathbf{u}_h\|_2 + c\|\mathbf{B}_{ht}\|_1\|\mathbf{B}_h\|_2,\\
\|p_{ht}\|_{0,\Omega} &\le c\|\mathbf{u}_{ht}\|_1 + c\|\mathbf{f}_t\|_{0,\Omega} + c\|\mathbf{u}_{htt}\|_0 + c\|\mathbf{u}_{ht}\|_1\|\mathbf{u}_h\|_2 + c\|\mathbf{B}_{ht}\|_1\|\mathbf{B}_h\|_2,\\
\mu\|\mathbf{B}_{ht}\|_2 &\le c\|\mathbf{B}_{htt}\|_0 + c\|\nabla\times(\mathbf{u}_{ht}\times\mathbf{B}_h)\|_{0,\Omega} + c\|\nabla\times(\mathbf{u}_h\times\mathbf{B}_{ht})\|_{0,\Omega}\\
&\le c\|\mathbf{B}_{htt}\|_0 + c\|\mathbf{u}_{ht}\|_1\|\mathbf{B}_h\|_2 + c\|\mathbf{u}_h\|_2\|\mathbf{B}_{ht}\|_1.
\end{aligned}$$
□

Lemma 3.8. *Under Assumptions* (A0)–(A3)*, the solution* $(\mathbf{u}_h(t), p_h(t), \mathbf{B}_h(t))$ *of the discrete problem* (2.1)–(2.2) *satisfies the estimates*

$$\sigma^3(t)[\|\mathbf{u}_{htt}(t)\|_1^2 + s\|\mathbf{B}_{htt}(t)\|_1^2] + \int_0^t \sigma^3(t)[\nu\|A_h\mathbf{u}_{htt}(s)\|_{0,\Omega}^2 + s\mu\|A_{2h}\mathbf{B}_{htt}(s)\|_{0,\Omega}^2]\,ds \le \kappa, \tag{3.36}$$
$$\int_0^t \sigma^3(t)[\|\mathbf{u}_{httt}(s)\|_0^2 + \|\mathbf{B}_{httt}(s)\|_0^2 + \|-\nu\Delta_h\mathbf{u}_{htt}(s) + \nabla_h p_{htt}(s)\|_{0,\Omega}^2]\,ds \le \kappa. \tag{3.37}$$

Proof. Taking the sum of (3.33) with $(\mathbf{v}_h, q_h) = (A_h\mathbf{u}_{htt}, 0)$ and (3.34) with $\mathbf{C}_h = sA_{2h}\mathbf{B}_{htt}$, we obtain

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|\mathbf{u}_{htt}\|_1^2 + \nu\|A_h\mathbf{u}_{htt}\|_{0,\Omega}^2 + \frac{s}{2}\frac{d}{dt}\|\mathbf{B}_{htt}\|_1^2 + s\mu\|A_{2h}\mathbf{B}_{htt}\|_{0,\Omega}^2\\
&\qquad + b(\mathbf{u}_{htt}, \mathbf{u}_h, A_h\mathbf{u}_{htt}) + 2b(\mathbf{u}_{ht}, \mathbf{u}_{ht}, A_h\mathbf{u}_{htt}) + b(\mathbf{u}_h, \mathbf{u}_{htt}, A_h\mathbf{u}_{htt})\\
&\qquad + sd(A_h\mathbf{u}_{htt}, \mathbf{B}_{htt}, \mathbf{B}_h) + 2sd(A_h\mathbf{u}_{htt}, \mathbf{B}_{ht}, \mathbf{B}_{ht}) + sd(A_h\mathbf{u}_{htt}, \mathbf{B}_h, \mathbf{B}_{htt})\\
&\qquad - sd(\mathbf{u}_{htt}, \mathbf{B}_h, A_{2h}\mathbf{B}_{htt}) - 2sd(\mathbf{u}_{ht}, \mathbf{B}_{ht}, A_{2h}\mathbf{B}_{htt}) - sd(\mathbf{u}_h, \mathbf{B}_{htt}, A_{2h}\mathbf{B}_{htt})\\
&= \frac{\nu}{16}\|A_h\mathbf{u}_{htt}\|_{0,\Omega}^2 + \frac{4}{\nu}\|\mathbf{f}_{tt}\|_{0,\Omega}^2.
\end{aligned} \tag{3.38}$$

Using (3.1)–(3.2), we can bound all the trilinear terms as follows:

$$\begin{aligned}
|b(\mathbf{u}_{htt}, \mathbf{u}_h, A_h\mathbf{u}_{htt})| + |b(\mathbf{u}_h, \mathbf{u}_{htt}, A_h\mathbf{u}_{htt})| &\le c_0\|\mathbf{u}_h\|_2\|\mathbf{u}_{htt}\|_1\|A_h\mathbf{u}_{htt}\|_{0,\Omega}\\
&\le \frac{\nu}{16}\|A_h\mathbf{u}_{tt}\|_{0,\Omega}^2 + \frac{4}{\nu}c_0^2\|\mathbf{u}_h\|_2^2\|\mathbf{u}_{htt}\|_1^2,\\
2|b(\mathbf{u}_{ht}, \mathbf{u}_{ht}, A_h\mathbf{u}_{htt})| &\le c_0\|\mathbf{u}_{ht}\|_1\|A_h\mathbf{u}_{ht}\|_{0,\Omega}\|A_h\mathbf{u}_{htt}\|_{0,\Omega}\\
&\le \frac{\nu}{16}\|A_h\mathbf{u}_{htt}\|_{0,\Omega}^2 + \frac{4}{\nu}c_0^2\|\mathbf{u}_{ht}\|_1^2\|A_h\mathbf{u}_{ht}\|_{0,\Omega}^2,\\
s|d(A_h\mathbf{u}_{htt}, \mathbf{B}_{htt}, \mathbf{B}_h)| + s|d(A_h\mathbf{u}_{htt}, \mathbf{B}_h, \mathbf{B}_{htt})| &\le sc_0\|A_h\mathbf{u}_{htt}\|_{0,\Omega}\|\mathbf{B}_h\|_2\|\mathbf{B}_{htt}\|_1\\
&\le \frac{\nu}{16}\|A_h\mathbf{u}_{htt}\|_{0,\Omega}^2 + \frac{4}{\nu}s^2c_0^2\|\mathbf{B}_h\|_2^2\|\mathbf{B}_{htt}\|_1^2,\\
2s|d(A_h\mathbf{u}_{htt}, \mathbf{B}_{ht}, \mathbf{B}_{ht})| &\le sc_0\|A_h\mathbf{u}_{htt}\|_{0,\Omega}\|A_{2h}\mathbf{B}_{ht}\|_{0,\Omega}\|\mathbf{B}_{ht}\|_1\\
&\le \frac{\nu}{16}\|A_h\mathbf{u}_{htt}\|_{0,\Omega}^2 + \frac{4}{\nu}s^2c_0^2\|A_{2h}\mathbf{B}_{ht}\|_{0,\Omega}^2\|\mathbf{B}_{ht}\|_1^2,
\end{aligned}$$

$$
\begin{aligned}
s|d(\mathbf{u}_h, \mathbf{B}_{htt}, A_{2h}\mathbf{B}_{htt})| + s|d(\mathbf{u}_{htt}, \mathbf{B}_h, A_{2h}\mathbf{B}_{htt})| &\le sc_0(\|\mathbf{u}_{htt}\|_1\|\mathbf{B}_h\|_2 + \|\mathbf{B}_{htt}\|_1\|\mathbf{u}_h\|_2)\|A_{2h}\mathbf{B}_{htt}\|_{0,\Omega} \\
&\le \frac{s\mu}{8}\|A_{2h}\mathbf{B}_{htt}\|_{0,\Omega}^2 + \frac{4}{\mu}sc_0^2(\|\mathbf{u}_{htt}\|_1^2\|\mathbf{B}_h\|_2^2 + \|\mathbf{B}_{htt}\|_1^2\|\mathbf{u}_h\|_2^2), \\
2s|d(\mathbf{u}_{ht}, \mathbf{B}_{ht}, A_{2h}\mathbf{B}_{htt})| &\le sc_0\|A_h\mathbf{u}_{ht}\|_{0,\Omega}\|\mathbf{B}_{ht}\|_1\|A_{2h}\mathbf{B}_{htt}\|_{0,\Omega} \\
&\le \frac{s\mu}{16}\|A_{2h}\mathbf{B}_{htt}\|_{0,\Omega}^2 + \frac{4}{\mu}sc_0^2\|A_h\mathbf{u}_{ht}\|_{0,\Omega}^2\|\mathbf{B}_{ht}\|_1^2.
\end{aligned}
$$

Combining all the above estimates with (3.38) gives

$$
\begin{aligned}
&\frac{d}{dt}(\|\mathbf{u}_{htt}\|_1^2 + s\|\mathbf{B}_{htt}\|_1^2) + \nu\|A_h\mathbf{u}_{htt}\|_{0,\Omega}^2 + s\mu\|A_{2h}\mathbf{B}_{htt}\|_{0,\Omega}^2 \\
&\quad\le c\|\mathbf{f}_{tt}\|_{0,\Omega}^2 + c(\mu + \|\mathbf{B}_h\|_2^2 + \|\mathbf{u}_h\|_2^2)(\|\mathbf{u}_{htt}\|_1^2 + s\|\mathbf{B}_{htt}\|_1^2) \\
&\qquad + c(\|A_h\mathbf{u}_{ht}\|_{0,\Omega}^2 + \|A_{2h}\mathbf{B}_{ht}\|_{0,\Omega}^2)(\|\mathbf{u}_{ht}\|_1^2 + s\|\mathbf{B}_{ht}\|_1^2).
\end{aligned}
$$

Then, multiplying both sides by $\sigma^2(t)$, integrating with respect to t and using Lemmas 3.5–3.7, we get (3.36). Similarly, we can derive (3.37) by using (3.36) and the following bounds from (3.33)–(3.34):

$$
\begin{aligned}
\|\mathbf{u}_{httt}\|_0 &\le \nu\|A_h\mathbf{u}_{htt}\|_{0,\Omega} + \|\mathbf{f}_{tt}\|_{0,\Omega} + c\|(\mathbf{u}_{htt}\cdot\nabla)\mathbf{u}_h\|_{0,\Omega} + c\|(\mathbf{u}_h\cdot\nabla)\mathbf{u}_{htt}\|_{0,\Omega} \\
&\quad + c\|(\mathbf{u}_{ht}\cdot\nabla)\mathbf{u}_{ht}\|_{0,\Omega} + c\|\mathbf{B}_{htt}\times(\nabla\times\mathbf{B}_h)\|_{0,\Omega} \\
&\quad + c\|\mathbf{B}_h\times(\nabla\times\mathbf{B}_{htt})\|_{0,\Omega} + c\|\mathbf{B}_{ht}\times(\nabla\times\mathbf{B}_{ht})\|_{0,\Omega} \\
&\le \|\mathbf{f}_{tt}\|_{0,\Omega} + \nu\|A_h\mathbf{u}_{htt}\|_0 + c\|\mathbf{u}_{htt}\|_1\|\mathbf{u}_h\|_2 + c\|\mathbf{B}_{htt}\|_1\|\mathbf{B}_h\|_2 \\
&\quad + c\|A_h\mathbf{u}_{ht}\|_{0,\Omega}\|\mathbf{u}_{ht}\|_1 + c\|A_{2h}\mathbf{B}_{ht}\|_{0,\Omega}\|\mathbf{B}_{ht}\|_1, \\
\|-\nu\Delta_h\mathbf{u}_{htt} + \nabla_h p_{htt}\|_{0,\Omega} &\le \|\mathbf{u}_{httt}\|_0 + \|\mathbf{f}_{tt}\|_{0,\Omega} + c\|\mathbf{u}_{htt}\|_1\|\mathbf{u}_h\|_2 + c\|\mathbf{B}_{htt}\|_1\|\mathbf{B}_h\|_2 \\
&\quad + c\|A_h\mathbf{u}_{ht}\|_{0,\Omega}\|\mathbf{u}_{ht}\|_1 + c\|A_{2h}\mathbf{B}_{ht}\|_{0,\Omega}\|\mathbf{B}_{ht}\|_1, \\
\|\mathbf{B}_{httt}\|_0 &\le \mu\|A_{2h}\mathbf{B}_{htt}\|_{0,\Omega} + \mu\|\mathbf{B}_{htt}\|_0 + c\|\nabla\times(\mathbf{u}_{htt}\times\mathbf{B}_h)\|_{0,\Omega} \\
&\quad + c\|\nabla\times(\mathbf{u}_h\times\mathbf{B}_{htt})\|_{0,\Omega} + c\|\nabla\times(\mathbf{u}_{ht}\times\mathbf{B}_{ht})\|_{0,\Omega} \\
&\le \mu\|A_{2h}\mathbf{B}_{htt}\|_{0,\Omega} + \mu\|\mathbf{B}_{htt}\|_0 + c\|\mathbf{u}_{htt}\|_1\|\mathbf{B}_h\|_2 \\
&\quad + c\|\mathbf{u}_h\|_2\|\mathbf{B}_{htt}\|_1 + c\|A_h\mathbf{u}_{ht}\|_{0,\Omega}\|\mathbf{B}_{ht}\|_1.
\end{aligned}
$$

□

For the error estimates of the fully discrete finite element solutions in the next section, we now introduce some Gronwall lemmas (see [19]).

Lemma 3.9. *For constant $C > 0$ and the positive sequences a_n, b_n, c_n, d_n satisfying*

$$
a_m + \tau\sum_{n=1}^{m} b_n \le \tau\sum_{n=0}^{m-1} d_n a_n \tau + \tau\sum_{n=0}^{m-1} c_n + C,
$$

the following Gronwall inequality holds:

$$
a_m + \tau\sum_{n=1}^{m} b_n \le \exp\Bigg(\tau\sum_{n=0}^{m-1} d_n\Bigg)\Bigg(\tau\sum_{n=0}^{m-1} c_n + C\Bigg).
$$

Lemma 3.10. *For constant $C > 0$ and the positive sequences a_n, b_n, c_n, d_n satisfying*

$$
a_m + \tau\sum_{n=1}^{m} b_n \le \tau\sum_{n=0}^{m} d_n a_n \tau + \tau\sum_{n=0}^{m} c_n + C,
$$

with $d_n\tau \le \frac{1}{2}$, the following Gronwall inequality holds:

$$
a_m + \tau\sum_{n=1}^{m} b_n \le \exp\Bigg(2\tau\sum_{n=0}^{m} d_n\Bigg)\Bigg(\tau\sum_{n=0}^{m} c_n + C\Bigg).
$$

4 Fully Discrete Finite Element Method with the Crank–Nicolson/Adams–Bashforth Scheme

In this section, we first discuss the time discretization of the semi-discrete finite element system (2.1)–(2.2) to get our interested fully discrete finite element scheme to the MHD system system (1.4)–(1.5). We start with the partition of the time interval $[0, T]$ and the triangulation of the physical domain Ω. We divide the time interval $[0, T]$ into N equally spaced subintervals using the nodal points

$$0 = t_0 < t_1 < t_2 < \cdots < t_N = \mathrm{T},$$

where $t_n = n\tau$ for $n = 0, 1, \dots, N$ and $\tau = \frac{T}{N}$. For any continuous function $f(t)$ or any discrete continuous function $\bar{\mathbf{u}}_h(t) \in \mathbf{X}_h$ in time, we shall use the notation

$$\bar{\mathbf{u}}_h^n = \frac{1}{2}(\mathbf{u}_h^n + \mathbf{u}_h^{n-1}), \quad \bar{\mathbf{f}}(t_n) = \frac{1}{2}(\mathbf{f}(t_n) + \mathbf{f}(t_{n-1})), \quad d_t\mathbf{u}_h^n = \frac{u_h^n - u_h^{n-1}}{\tau},$$

and all other subsequent notation for the time and space discretizations as well as the finite element spaces are carried over from the previous section.

We shall use the implicit second-order Crank–Nicolson scheme to handle the linear terms in system (2.1)–(2.2) and the second-order Adams–Bashforth scheme to take care of the nonlinear term. Since the Crank–Nicolson/Adams–Bashforth scheme involve three levels in times, we first define

$$(\mathbf{u}_h^0, \mathbf{B}_h^0) = (\mathbf{u}_h(0), \mathbf{B}_h(0)) \quad \text{and} \quad (\mathbf{u}_h^1, p_h^1, \mathbf{B}_h^1) \in \mathbf{X}_h \times M_h \times \mathbf{W}_h$$

to solve the following Euler-backward system for all $(\mathbf{v}_h, q_h, \mathbf{C}_h) \in \mathbf{X}_h \times M_h \times \mathbf{W}_h$:

$$\begin{aligned}(d_t\mathbf{u}_h^1, \mathbf{v}_h)_\Omega + \nu(\nabla\mathbf{u}_h^1, \nabla\mathbf{v}_h)_\Omega - (\nabla\cdot\mathbf{v}_h, p_h^1)_\Omega + (\nabla\cdot\mathbf{u}_h^1, q_h)_\Omega&\\ + b(\mathbf{u}_h^0, \mathbf{u}_h^0, \mathbf{v}_h) + sd(\mathbf{v}_h, \mathbf{B}_h^0, \mathbf{B}_h^0) &= (\mathbf{f}(t_1), \mathbf{v}_h),\end{aligned} \tag{4.1}$$

$$(d_tB_h^1, C_h)_\Omega + \mu(\nabla\times B_h^1, \nabla\times C_h)_\Omega + \mu(\nabla\cdot B_h^1, \nabla\cdot C_h)_\Omega - d(\mathbf{u}_h^0, \mathbf{B}_h^0, \mathbf{C}_h) = 0. \tag{4.2}$$

Then we can recursively define the finite element solutions $(\mathbf{u}_h^n, p_h^n, \mathbf{B}_h^n) \in \mathbf{X}_h \times M_h \times \mathbf{W}_h$ for $n = 2, \dots, N$ by the system

$$\begin{aligned}(d_t\mathbf{u}_h^n, \mathbf{v}_h)_\Omega + \nu(\nabla\bar{\mathbf{u}}_h^n, \nabla\mathbf{v}_h)_\Omega - (\nabla\cdot\mathbf{v}_h, p_h^n)_\Omega + (\nabla\cdot\mathbf{u}_h^n, q_h)_\Omega&\\ + \frac{3}{2}b(\mathbf{u}_h^{n-1}, \mathbf{u}_h^{n-1}, \mathbf{v}_h) - \frac{1}{2}b(\mathbf{u}_h^{n-2}, \mathbf{u}_h^{n-2}, \mathbf{v}_h)&\\ + \frac{3}{2}sd(\mathbf{v}_h, \mathbf{B}_h^{n-1}, \mathbf{B}_h^{n-1}) - \frac{1}{2}sd(\mathbf{v}_h, \mathbf{B}_h^{n-2}, \mathbf{B}_h^{n-2}) &= (\bar{\mathbf{f}}(t_n), \mathbf{v}_h)_\Omega,\end{aligned} \tag{4.3}$$

$$\begin{aligned}(d_t\mathbf{B}_h^n, \mathbf{C}_h)_\Omega + \mu(\nabla\times\bar{\mathbf{B}}_h^n, \nabla\times\mathbf{C}_h)_\Omega + \mu(\nabla\cdot\bar{\mathbf{B}}_h^n, \nabla\cdot\mathbf{C}_h)_\Omega&\\ - \frac{3}{2}d(\mathbf{u}_h^{n-1}, \mathbf{B}_h^{n-1}, \mathbf{C}_h) + \frac{1}{2}d(\mathbf{u}_h^{n-2}, \mathbf{B}_h^{n-2}, \mathbf{C}_h) &= 0.\end{aligned} \tag{4.4}$$

For the subsequent error estimates of the fully discrete finite element solution, we introduce the notation

$$e^n = \mathbf{u}_h(t_n) - \mathbf{u}_h^n, \quad \varepsilon^n = \mathbf{B}_h(t_n) - \mathbf{B}_h^n, \quad \eta^n = \bar{p}_h(t_n) - p_h^n, \quad n = 1, \dots, N.$$

We first study the approximation errors of the initial values $(\mathbf{u}_h^1, p_h^1, \mathbf{B}_h^1) \in \mathbf{X}_h \times M_h \times \mathbf{W}_h$ defined by (4.1)–(4.2).

Lemma 4.1. *Under Assumptions* (A0)–(A3), *for* $\tau \le \frac{1}{4}$ *and* $\alpha = -2, -1, 0, 1$, *we have*

$$\|e^1\|_\alpha^2 + \|d_te^1\|_\alpha^2\tau^2 + \nu\|e^1\|_{\alpha+1}^2\tau + \|\varepsilon^1\|_\alpha^2 + \|d_t\varepsilon^1\|_\alpha^2\tau^2 + \mu\|\varepsilon^1\|_{\alpha+1}^2\tau \le \kappa_1\tau^{2-\alpha}, \quad \|\eta^1\|_0^2 \le \kappa_1. \tag{4.5}$$

Proof. We first integrate (2.1) with $(\mathbf{v}_h, q_h) \in \mathbf{X}_h \times M_h$ and (2.2) with $\mathbf{C}_h \in \mathbf{W}_h$ from t_0 to t_1 to obtain

$$\begin{aligned}(d_t\mathbf{u}_h(t_1), \mathbf{v}_h)_\Omega + \frac{1}{\tau}\int_{t_0}^{t_1}(-\nu\Delta_h\mathbf{u}_h(t) + \nabla_hp_h(t), \mathbf{v}_h)_\Omega\,dt + (\nabla\cdot\mathbf{u}_h(t_1), q_h)_\Omega&\\ + \frac{1}{\tau}\int_{t_0}^{t_1} b(\mathbf{u}_h(t), \mathbf{u}_h(t), \mathbf{v}_h)\,dt + s\frac{1}{\tau}\int_{t_0}^{t_1} d(\mathbf{v}_h, \mathbf{B}_h(t), \mathbf{B}_h(t))\,dt &= \frac{1}{\tau}\int_{t_0}^{t_1}(\mathbf{f}(t), \mathbf{v}_h)\,dt,\end{aligned} \tag{4.6}$$

$$(d_t\mathbf{B}_h(t_1),\mathbf{C}_h)_\Omega+\mu\frac{1}{\tau}\int_{t_0}^{t_1}[(\nabla\times\mathbf{B}_h(t),\nabla\times\mathbf{C}_h)_\Omega+(\nabla\cdot\mathbf{B}_h(t),\nabla\cdot\mathbf{C}_h)_\Omega]\,dt$$
$$-\frac{1}{\tau}\int_{t_0}^{t_1}d(\mathbf{u}_h(t),\mathbf{B}_h(t),\mathbf{C}_h)\,dt=0. \tag{4.7}$$

To derive the error equations of e^1 and ε^1, we subtract (4.1) with $(\mathbf{v}_h,q_h)\in\mathbf{X}_h\times M_h$ and (4.2) with $\mathbf{C}_h\in\mathbf{W}_{0h}$ from (4.6) and (4.7), respectively, and then use integration by parts to deduce

$$(d_te^1,\mathbf{v}_h)_\Omega+\nu(\nabla e^1,\nabla\mathbf{v}_h)_\Omega-(\nabla\cdot\mathbf{v}_h,\eta^1)_\Omega+(\nabla\cdot e^1,q_h)_\Omega=(E_1,\mathbf{v}_h)_\Omega, \tag{4.8}$$
$$(d_t\varepsilon^1,\mathbf{C}_h)_\Omega+\mu(\nabla\times\varepsilon^1,\nabla\times\mathbf{C}_h)_\Omega+\mu(\nabla\cdot\varepsilon^1,\nabla\cdot\mathbf{C}_h)_\Omega=(F_1,\mathbf{C}_h)_\Omega, \tag{4.9}$$

where

$$(E_1,\mathbf{v}_h)_\Omega=-\frac{1}{\tau}\int_{t_0}^{t_1}(t-t_0)(\mathbf{f}_t(t),\mathbf{v}_h)_\Omega\,dt+\frac{1}{\tau}\int_{t_0}^{t_1}(t-t_0)(-\nu\Delta_h\mathbf{u}_{ht}(t)+\nabla_hp_{ht},\mathbf{v}_h)_\Omega\,dt$$
$$+\frac{1}{\tau}\int_{t_0}^{t_1}(t-t_1)b_t(\mathbf{u}_h(t),\mathbf{u}_h(t),\mathbf{v}_h)\,dt+s\frac{1}{\tau}\int_{t_0}^{t_1}(t-t_1)d_t(\mathbf{v}_h,\mathbf{B}_h(t),\mathbf{B}_h(t))\,dt, \tag{4.10}$$
$$(F_1,\mathbf{C}_h)_\Omega=\mu\frac{1}{\tau}\int_{t_0}^{t_1}(t-t_0)[(\nabla\times\mathbf{B}_{ht}(t),\nabla\times\mathbf{C}_h)_\Omega+(\nabla\cdot\mathbf{B}_{ht}(t),\nabla\cdot\mathbf{C}_h)_\Omega]\,dt$$
$$-\frac{1}{\tau}\int_{t_0}^{t_1}(t-t_1)d_t(\mathbf{u}_h(t),\mathbf{B}_h(t),\mathbf{C}_h)\,dt. \tag{4.11}$$

To continue the estimate, we introduce the L^2-orthogonal projection $P_{0h}\colon L^2(\Omega)^3\to\mathbf{X}_h$ defined by

$$(P_{0h}\mathbf{v},\mathbf{v}_h)=(\mathbf{v},\mathbf{v}_h)_\Omega\quad\text{for all }\mathbf{v}\in L^2(\Omega)^3,\ \mathbf{v}_h\in\mathbf{X}_h.$$

Then we can use (4.10)–(4.11) and (3.1)–(3.2) to bound three weighted norms of the truncation error E_1 by

$$\begin{aligned}\|P_hE_1\|_0\le\|P_{0h}E_1\|_{0,\Omega}&=\sup_{v_h\in\mathbf{X}_h}\frac{(E_1,\mathbf{v}_h)_\Omega}{\|\mathbf{v}_h\|_0}\\&\le c\sup_{0\le t\le t_1}[\|\mathbf{f}(t)\|_{0,\Omega}+\|-\nu\Delta_h\mathbf{u}_h(t)+\nabla_hp_h(t)\|_0+\|A_h\mathbf{u}_h(t)\|_0\|\mathbf{u}_h(t)\|_1]\\&\quad+c\sup_{0\le t\le t_1}[\|\mathbf{B}_h(t)\|_1\|\mathbf{B}_h(t)\|_2]\le\kappa,\\ \|A_h^{-\frac12}P_hE_1\|_{0,\Omega}&=\sup_{\mathbf{v}_h\in\mathbf{V}_h}\frac{(E_1,\mathbf{v}_h)_\Omega}{\|\mathbf{v}_h\|_1}\\&\le c\sup_{0\le t\le t_1}[\tau\|\mathbf{f}_t(t)\|_{0,\Omega}+\tau^{\frac12}\sigma^{\frac12}(t)\|\mathbf{u}_{ht}(t)\|_1+\tau\|A_h\mathbf{u}_h(t)\|_{0,\Omega}\|\mathbf{u}_{ht}(t)\|_0]\\&\quad+c\sup_{0\le t\le t_1}[\tau\|A_{2h}\mathbf{B}_h(t)\|_{0,\Omega}\|\mathbf{B}_{ht}(t)\|_0]\le\kappa\tau^{\frac12},\\ \|A_h^{-1}P_hE_1\|_{0,\Omega}&=\sup_{\mathbf{v}_h\in\mathbf{V}_h}\frac{(E_1,\mathbf{v}_h)_\Omega}{\|A_h\mathbf{v}_h\|_0}\\&\le c\sup_{0\le t\le t_1}[\tau\|\mathbf{f}_t(t)\|_{0,\Omega}+\nu\tau\|\mathbf{u}_{ht}(t)\|_0+\tau\|\mathbf{u}_h(t)\|_1\|u_{ht}(t)\|_0]\\&\quad+c\sup_{0\le t\le t_1}[\|\mathbf{B}_h(t)\|_1\|\mathbf{B}_{ht}(t)\|_0]\le\kappa\tau,\end{aligned} \tag{4.12}$$

and to bound three weighted norms of the truncation error F_1 by

$$\begin{aligned}\|R_{0h}F_1\|_{0,\Omega}&=\sup_{\mathbf{C}_h\in\mathbf{W}_h}\frac{(F_1,\mathbf{C}_h)_\Omega}{\|\mathbf{C}_h\|_0}\\&\le c\sup_{0\le t\le t_1}[\|A_{2h}\mathbf{B}_h(t)\|_{0,\Omega}+\|A_{2h}\mathbf{B}_h(t)\|_{0,\Omega}\|\mathbf{u}_h(t)\|_1]\le\kappa,\end{aligned}$$

$$
\begin{aligned}
\|A_{2h}^{-\frac{1}{2}} R_{0h} F_1\|_{0,\Omega} &= \sup_{\mathbf{C}_h \in \mathbf{W}_{0h}} \frac{(F_1, \mathbf{C}_h)_\Omega}{\|A_{2h}^{\frac{1}{2}} \mathbf{C}_h\|_{0,\Omega}} \\
&\leq c\tau^{\frac{1}{2}} \sigma^{\frac{1}{2}}(t)\|\mathbf{B}_{ht}(t)\|_1 + \tau\|A_h \mathbf{u}_h(t)\|_{0,\Omega}\|\mathbf{B}_{ht}(t)\|_0] \\
&\qquad + c \sup_{0\leq t\leq t_1} [\tau\|A_{2h}\mathbf{B}_h(t)\|_{0,\Omega}\|\mathbf{u}_{ht}(t)\|_0 + \tau\|A_h\mathbf{u}_h(t)\|_{0,\Omega}\|\mathbf{B}_{ht}(t)\|_0] \leq \kappa\tau^{\frac{1}{2}}, \\
\|A_{2h}^{-1} R_{0h} F_1\|_{0,\Omega} &= \sup_{\mathbf{C}_h \in \mathbf{W}_h} \frac{(F_1, \mathbf{C}_h)_\Omega}{\|A_{2h}\mathbf{C}_h\|_{0,\Omega}} \\
&\leq c \sup_{0\leq t\leq t_1} [\tau\|\mathbf{B}_{ht}(t)\|_0 + \tau\|\mathbf{u}_h(t)\|_1\|\mathbf{B}_{ht}(t)\|_0 + \tau\|\mathbf{u}_{ht}(t)\|_0\|\mathbf{B}_h(t)\|_1] \leq \kappa\tau.
\end{aligned}
\tag{4.13}
$$

To further our estimates, we recall some smooth properties of the semi-discrete solution $(\mathbf{u}_h(t), p_h(t), \mathbf{B}_h(t))$ to (2.1)–(2.2) (cf. Lemmas 3.4–3.8):

$$
\begin{aligned}
&\|\mathbf{u}_h(t)\|_2^2 + \|\mathbf{B}_h(t)\|_2^2 + \|-\nu\Delta_h\mathbf{u}_h(t) + \nabla_h p_h(t)\|_{0,\Omega}^2 \leq \kappa_0, \\
&\sigma^2(t)\|p_{ht}\|_{0,\Omega}^2 \leq \kappa_0, \quad \sigma^r(t)(\|u_{ht}(t)\|_r^2 + \|\mathbf{B}_{ht}(t)\|_r^2) \leq \kappa_0, \qquad r = 0, 1, 2, \\
&\sigma^{r+2}(t)(\|u_{htt}(t)\|_r^2 + \|\mathbf{B}_{htt}(t)\|_r^2) \leq \kappa_0, \qquad r = -1, 0, 1, \\
&\int_0^t \sigma^r(s)(\|\mathbf{u}_{ht}(s)\|_{r+1}^2 + \|\mathbf{B}_{ht}(s)\|_{r+1}^2)\, ds \leq \kappa_0, \qquad r = 0, 1, \\
&\int_0^t \sigma^{r+1}(s)(\|\mathbf{u}_{htt}(s)\|_r^2 + \|\mathbf{B}_{htt}\|_r^2)\, ds \leq \kappa_0, \qquad r = -1, 0, 1, 2, \\
&\int_0^t \sigma^{r+2}(s)(\|\mathbf{u}_{httt}(s)\|_{r-1}^2 + \|\mathbf{B}_{httt}(s)\|_{r-1})\, ds \leq \kappa_0, \qquad r = -1, 0, 1, \\
&\int_0^t [\sigma^3(s)\|-\nu\Delta_h\mathbf{u}_{htt}(s) + \nabla_h p_{htt}(s)\|_{0,\Omega}^2 + \sigma(s)\|-\nu\Delta_h\mathbf{u}_{ht}(s) + \nabla_h p_{ht}(s)\|_{0,\Omega}^2]\, ds \leq \kappa_0.
\end{aligned}
\tag{4.14}
$$

With the above preparations and for $\alpha = -1, 0, 1$, we now take $\mathbf{v}_h = 2A_h^\alpha e^1\tau \in \mathbf{V}_h$, $q_h = 0$ in (4.8) and $\mathbf{C}_h = 2A_{2h}^\alpha \varepsilon^1\tau$ in (4.9), respectively. Then, using (4.12), (4.13) and (4.14), we can deduce

$$
\begin{aligned}
\|e^1\|_\alpha^2 + \|d_t e^1\|_\alpha^2\tau^2 + \nu\|e^1\|_{\alpha+1}^2\tau &\leq \nu^{-1}\|A_h^{\frac{\alpha-1}{2}} P_h E_1\|_{0,\Omega}^2\tau \leq \kappa\tau^{2-\alpha}, \\
\|\varepsilon^1\|_\alpha^2 + \|d_t\varepsilon^1\|_\alpha^2\tau^2 + \mu\|\varepsilon^1\|_{\alpha+1}^2\tau - 2\mu\|\varepsilon^1\|_\alpha^2\tau &\leq \mu^{-1}\|A_{2h}^{\frac{\alpha-1}{2}} R_{0h} F_1\|_{0,\Omega}^2\tau \leq \kappa\tau^{2-\alpha}.
\end{aligned}
$$

To have the desired estimates of the errors in negative-norm, we take $\mathbf{v}_h = 2A_h^{-2}e^1\tau \in \mathbf{V}_h$, $q_h = 0$ in (4.8) and $\mathbf{C}_h = 2A_{2h}^{-2}\varepsilon^1\tau \in \mathbf{W}_h$ in (4.9), then use (4.14) to obtain

$$
\begin{aligned}
\|e^1\|_{-2}^2 + \|d_t e^1\|_{-2}^2\tau^2 + \nu\|e^1\|_{-1}^2\tau &\leq 4\|A_h^{-1} P_h E_1\|_0^2\tau^2 \leq \kappa\tau^4, \\
\|\varepsilon^1\|_{-2}^2 + \|d_t\varepsilon^1\|_{-2}^2\tau^2 + \mu\|\varepsilon^1\|_{-1}^2\tau - 2\mu\|\varepsilon\|_{-2}^2\tau &\leq 2\mu^{-1}\|A_{2h}^{-1} R_{0h} F_1\|_{0,\Omega}^2\tau^2 \leq \kappa\tau^4,
\end{aligned}
$$

Combining these above estimates, we can see the first desired estimate in (4.5). The second estimate in (4.5) can be achieved by using (3.1)–(3.2) and Assumption (A3):

$$
\|\eta^1\|_0 \leq c\|d_t e^1\|_0 + c\|e^1\|_1 + c\|P_{0h}E_1\|_0 \leq \kappa. \qquad \square
$$

5 Error Estimates Part I

In this section, we establish the equations of the errors $e^n = \mathbf{u}_h(t_n) - \mathbf{u}_h^n$, $\eta^n = \frac{1}{\tau}\int_{t_{n-1}}^{t_n} p_h(t)\, dt - p_h^n$ and $\varepsilon^n = \mathbf{B}_h(t_n) - \mathbf{B}_h^n$ and then deduce the bounds of the truncation errors E_n and F_n for $n = 2, \ldots, N$. The estimates of these errors in H^1- and L^2-norm will be developed in the next section. For simplicity, we always assume that $\mathbf{v}_h \in \mathbf{X}_h$ and $q_h \in M_h$ and $\mathbf{C}_h \in \mathbf{W}_h$ in this section.

We start by integrating (2.1) and (2.2) from t_{n-1} to t_n, respectively, to have

$$(d_t\mathbf{u}_h(t_n),\mathbf{v}_h)_\Omega+\frac{1}{\tau}\int_{t_{n-1}}^{t_n}(-\nu\Delta_h\mathbf{u}_h(t)+\nabla_h p_h(t),\mathbf{v}_h)_\Omega\,dt+(\nabla\cdot\mathbf{u}_h(t_n),q_h)_\Omega$$
$$+\frac{1}{\tau}\int_{t_{n-1}}^{t_n}b(\mathbf{u}_h(t),\mathbf{u}_h(t),\mathbf{v}_h)\,dt+s\frac{1}{\tau}\int_{t_{n-1}}^{t_n}d(\mathbf{v}_h,\mathbf{B}_h(t),\mathbf{B}_h(t))\,dt=\frac{1}{\tau}\int_{t_{n-1}}^{t_n}(\mathbf{f}(t),\mathbf{v}_h)_\Omega\,dt, \tag{5.1}$$

$$(d_t\mathbf{B}_h(t_n),\mathbf{C}_h)_\Omega+\frac{\mu}{\tau}\int_{t_{n-1}}^{t_n}[(\nabla\times\mathbf{B}_h(t),\nabla\times\mathbf{C}_h)_\Omega+(\nabla\cdot\mathbf{B}_h(t),\nabla\cdot\mathbf{C}_h)_\Omega]\,dt$$
$$-\frac{1}{\tau}\int_{t_{n-1}}^{t_n}d(\mathbf{u}_h(t),\mathbf{B}_h(t),\mathbf{C}_h)\,dt=0. \tag{5.2}$$

Subtracting (3.3) and (3.4) from (5.1) and (5.2), respectively, and using the formulas

$$\frac{1}{\tau}\int_{t_{n-1}}^{t_n}\phi(t)\,dt-\bar\phi(t_n)=-\frac{1}{2\tau}\int_{t_{n-1}}^{t_n}(t-t_{n-1})(t_n-t)\phi_{tt}(t)\,dt,$$
$$\bar\phi(t_n)-\frac{3}{2}\phi(t_{n-1})+\frac{1}{2}\phi(t_{n-2})=-\frac{1}{2}\int_{t_{n-1}}^{t_n}(t-t_n)\phi_{tt}(t)\,dt+\frac{1}{2}\int_{t_{n-2}}^{t_{n-1}}(t-t_{n-2})\phi_{tt}(t)\,dt$$

for all $\phi\in H^2(t_{n-1},t_n)$, we can derive

$$\begin{aligned}&(d_te^n,\mathbf{v}_h)_\Omega+\nu(\nabla\bar e^n,\nabla\mathbf{v}_h)_\Omega-(\nabla\cdot\mathbf{v}_h,\eta^n)_\Omega+(\nabla\cdot e^n,q_h)_\Omega\\
&\quad+\frac{3}{2}b(e^{n-1},\mathbf{u}_h(t_{n-1}),\mathbf{v}_h)+\frac{3}{2}b(\mathbf{u}_h^{n-1},e^{n-1},\mathbf{v}_h)\\
&\quad-\frac{1}{2}b(e^{n-2},\mathbf{u}_h(t_{n-2}),\mathbf{v}_h)-\frac{1}{2}b(\mathbf{u}_h^{n-2},e^{n-2},\mathbf{v}_h)\\
&\quad+\frac{3}{2}d(\mathbf{v}_h,\varepsilon^{n-1},\mathbf{B}_h(t_{n-1}))+\frac{3}{2}b(\mathbf{v}_h,\mathbf{B}_h^{n-1},\varepsilon^{n-1})\\
&\quad-\frac{1}{2}d(\mathbf{v}_h,\varepsilon^{n-2},\mathbf{B}_h(t_{n-2}))-\frac{1}{2}b(\mathbf{v}_h,\mathbf{B}_h^{n-2},\varepsilon^{n-2},v_h)=(E_n,\mathbf{v}_h)_\Omega,\end{aligned} \tag{5.3}$$

$$\begin{aligned}&(d_t\varepsilon^n,\mathbf{C}_h)_\Omega+\mu(\nabla\times\bar\varepsilon^n,\nabla\times\mathbf{C}_h)_\Omega+\mu(\nabla\cdot\bar\varepsilon^n,\nabla\cdot\mathbf{C}_h)_\Omega\\
&\quad-\frac{3}{2}d(e^{n-1},\mathbf{B}_h(t_{n-1}),\mathbf{C}_h)-\frac{3}{2}d(\mathbf{u}_h^{n-1},\varepsilon^{n-1},\mathbf{C}_h)\\
&\quad+\frac{1}{2}d(e^{n-2},\mathbf{B}_h(t_{n-2}),\mathbf{C}_h)+\frac{1}{2}b(\mathbf{u}_h^{n-2},\varepsilon^{n-2},\mathbf{C}_h)=(F_n,\mathbf{C}_h)_\Omega,\end{aligned} \tag{5.4}$$

where the right-hand side of (5.3) is defined by

$$\begin{aligned}(E_n,\mathbf{v}_h)_\Omega&=\frac{1}{2\tau}\int_{t_{n-1}}^{t_n}(t-t_{n-1})(t_n-t)(\mathbf{f}_{tt}(t),\mathbf{v}_h)_\Omega\,dt\\
&\quad+\frac{1}{\tau}\int_{t_{n-1}}^{t_n}(t-t_{n-1})(t_n-t)(-\nu\Delta_h\mathbf{u}_{htt}(t)+\nabla_h p_{htt}(t),\mathbf{v}_h)_\Omega\,dt\\
&\quad+\frac{1}{2\tau}\int_{t_{n-1}}^{t_n}(t-t_{n-1})(t_n-t)b_{tt}(\mathbf{u}_h(t),\mathbf{u}_h(t),\mathbf{v}_h)\,dt\\
&\quad+\frac{1}{2}\int_{t_{n-1}}^{t_n}(t-t_n)b_{tt}(\mathbf{u}_h(t),\mathbf{u}_h(t),\mathbf{v}_h)\,dt-\frac{1}{2}\int_{t_{n-2}}^{t_{n-1}}(t-t_{n-2})b_{tt}(\mathbf{u}_h(t),\mathbf{u}_h(t),\mathbf{v}_h)\,dt\\
&\quad+\frac{1}{2\tau}\int_{t_{n-1}}^{t_n}(t-t_{n-1})(t_n-t)d_{tt}(\mathbf{v}_h,\mathbf{B}_h(t),\mathbf{B}_h(t))\,dt\\
&\quad+\frac{1}{2}\int_{t_{n-1}}^{t_n}(t-t_n)d_{tt}(\mathbf{v}_h,\mathbf{B}_h(t),\mathbf{B}_h(t))\,dt-\frac{1}{2}\int_{t_{n-2}}^{t_{n-1}}(t-t_{n-2})d_{tt}(\mathbf{v}_h,\mathbf{B}_h(t),\mathbf{B}_h(t))\,dt,\end{aligned} \tag{5.5}$$

and the right-hand side of (5.4) is defined by

$$
\begin{aligned}
(F_n, \mathbf{C}_h)_\Omega = \frac{\mu}{\tau}\int_{t_{n-1}}^{t_n} (t - t_{n-1})(t_n - t)(\nabla \times \mathbf{B}_{htt}(t), \nabla \times \mathbf{C}_h)_\Omega\,dt + \frac{\mu}{\tau}\int_{t_{n-1}}^{t_n} (t - t_{n-1})(t_n - t)(\nabla \cdot \mathbf{B}_{htt}(t), \nabla \cdot \mathbf{C}_h)_\Omega\,dt \\
- \frac{1}{2\tau}\int_{t_{n-1}}^{t_n} (t - t_{n-1})(t_n - t) d_{tt}(\mathbf{u}_h(t), \mathbf{B}_h(t), \mathbf{C}_h)\,dt \\
- \frac{1}{2}\int_{t_{n-1}}^{t_n} (t - t_n) d_{tt}(\mathbf{u}_h(t), \mathbf{B}_h(t), \mathbf{C}_h)\,dt + \frac{1}{2}\int_{t_{n-2}}^{t_{n-1}} (t - t_{n-2}) d_{tt}(\mathbf{u}_h(t), \mathbf{B}_h(t), \mathbf{C}_h)\,dt,
\end{aligned}
\tag{5.6}
$$

while the three b_{tt}- and d_{tt}-terms in (5.5)–(5.6) are defined by

$$
\begin{aligned}
b_{tt}(\mathbf{u}_h(t), \mathbf{u}_h(t), \mathbf{v}_h) &= b(\mathbf{u}_{htt}(t), \mathbf{u}_h(t), \mathbf{v}_h) + b(\mathbf{u}_h(t), \mathbf{u}_{htt}(t), \mathbf{v}_h) + 2b(\mathbf{u}_{ht}, \mathbf{u}_{ht}, \mathbf{v}_h), \\
d_{tt}(\mathbf{v}_h, \mathbf{B}_h(t), \mathbf{B}_h(t)) &= d(\mathbf{v}_h, \mathbf{B}_{htt}(t), \mathbf{B}_h(t)) + d(\mathbf{v}_h, \mathbf{B}_h(t), \mathbf{B}_{htt}(t)) + 2b(\mathbf{v}_h, \mathbf{B}_{ht}, \mathbf{B}_{ht}), \\
d_{tt}(\mathbf{u}_h(t), \mathbf{B}_h(t), \mathbf{C}_h) &= d(\mathbf{u}_{htt}(t), \mathbf{B}_h(t), \mathbf{C}_h) + d(\mathbf{u}_h(t), \mathbf{B}_{htt}(t), \mathbf{C}_h) + 2d(\mathbf{u}_{ht}, \mathbf{B}_{ht}, \mathbf{C}_h).
\end{aligned}
$$

We end this section with the bounds of the truncation errors E_n and F_n for $n = 2, \dots, N$.

Lemma 5.1. *Under Assumptions* (A0)–(A3), *the truncation errors E_n and F_n satisfy the bounds*

$$
\begin{aligned}
\tau \sum_{n=2}^{m} \|A_h^{-\frac{3}{2}} P_h E_n\|_{0,\Omega}^2 &\le \kappa_2 \tau^4, \\
\tau \sum_{n=2}^{m} \sigma^i(t_n) \|A_h^{-1} P_h E_n\|_{0,\Omega}^2 &\le \kappa_2 \tau^{3+i}, \quad i = 0, 1, \\
\tau \sum_{n=2}^{n} \sigma^i(t_n) \|A_h^{-\frac{1}{2}} P_h E_h\|_{0,\Omega}^2 &\le \kappa_2 \tau^{2+i}, \quad i = 0, 1, 2, \\
\tau \sum_{n=2}^{m} \sigma^i(t_n) \|P_{0h} E_n\|_{0,\Omega}^2 &\le \kappa_2 \tau^{1+i}, \quad i = 0, 1, 2, 3,
\end{aligned}
\tag{5.7}
$$

$$
\begin{aligned}
\tau \sum_{n=2}^{m} \|A_{1h}^{-\frac{3}{2}} R_{0h} F_n\|_{0,\Omega}^2 &\le \kappa_2 \tau^4, \\
\tau \sum_{n=2}^{m} \sigma^i(t_n) \|A_{1h}^{-1} R_{0h} F_n\|_{0,\Omega}^2 &\le \kappa_2 \tau^{3+i}, \quad i = 0, 1, \\
\tau \sum_{n=2}^{n} \sigma^i(t_n) \|A_{1h}^{-\frac{1}{2}} R_{0h} F_h\|_{0,\Omega}^2 &\le \kappa_2 \tau^{2+i}, \quad i = 0, 1, 2, \\
\tau \sum_{n=2}^{m} \sigma^i(t_n) \|R_{0h} F_n\|_{0,\Omega}^2 &\le \kappa_2 \tau^{1+i}, \quad i = 0, 1, 2, 3,
\end{aligned}
\tag{5.8}
$$

for all $2 \le m \le N$.

Proof. Using Lemmas 3.2–3.3, (3.1)–(3.2), we can derive the weighted norms of E_n from (5.5) as follows:

$$
\begin{aligned}
\|A_h^{-\frac{3}{2}} P_h E_n\|_{0,\Omega} &= \sup_{\mathbf{v}_h \in \mathbf{V}_h} \frac{|(E_n, \mathbf{v}_h)_\Omega|}{\|A_h^{\frac{3}{2}} \mathbf{v}_h\|_{0,\Omega}} \\
&\le c\tau^{\frac{3}{2}} \bigg(\int_{t_{n-1}}^{t_n} \|\mathbf{f}_{tt}(t)\|_{0,\Omega}^2\,dt \bigg)^{\frac{1}{2}} + c\tau^{\frac{3}{2}} \bigg(\int_{t_{n-1}}^{t_n} \|\mathbf{u}_{htt}(t)\|_{-1}^2\,dt \bigg)^{\frac{1}{2}} \\
&\quad + c\tau^{\frac{3}{2}} \bigg(\int_{t_{n-2}}^{t_n} [\|\mathbf{u}_{ht}(t)\|_0^2 \|\mathbf{u}_{ht}(t)\|_1^2 + \|A_h \mathbf{u}_h(t)\|_{0,\Omega}^2 \|\mathbf{u}_{htt}(t)\|_{-1}^2]\,dt \bigg)^{\frac{1}{2}} \\
&\quad + c\tau^{\frac{3}{2}} \bigg(\int_{t_{n-2}}^{t_n} [\|\mathbf{B}_{ht}(t)\|_0^2 \|\mathbf{B}_{ht}(t)\|_1^2 + \|A_h \mathbf{u}_h(t)\|_{0,\Omega}^2 \|\mathbf{B}_{htt}(t)\|_{-1}^2]\,dt \bigg)^{\frac{1}{2}},
\end{aligned}
$$

$$
\begin{aligned}
\|A_h^{-1}P_hE_n\|_{0,\Omega} &= \sup_{\mathbf{v}_h\in\mathbf{V}_h}\frac{|(E_n,\mathbf{v}_h)_\Omega|}{\|A_h\mathbf{v}_h\|_{0,\Omega}}\\
&\le c\tau^{\frac{3}{2}}\Big(\int_{t_{n-1}}^{t_n}\|\mathbf{f}_{tt}(t)\|_{0,\Omega}^2\,dt\Big)^{\frac{1}{2}}+c\tau\Big(\int_{t_{n-1}}^{t_n}(t-t_{n-1})\|\mathbf{u}_{htt}(t)\|_0^2\,dt\Big)^{\frac{1}{2}}\\
&\quad+c\tau^{\frac{3}{2}}\Big(\int_{t_{n-2}}^{t_n}[\|A_h\mathbf{u}_h(t)\|_{0,\Omega}^2\|\mathbf{u}_{htt}(t)\|_{-1}^2+\|\mathbf{u}_{ht}\|_1^2\|\mathbf{u}_{ht}\|_0^2]\,dt\Big)^{\frac{1}{2}}\\
&\quad+c\tau^{\frac{3}{2}}\Big(\int_{t_{n-2}}^{t_n}[\|A_{2h}\mathbf{B}_h(t)\|_{0,\Omega}^2\|\mathbf{B}_{htt}(t)\|_{-1}^2+\|\mathbf{B}_{ht}(t)\|_1^2\|\mathbf{B}_{ht}(t)\|_0^2]\,dt\Big)^{\frac{1}{2}}.
\end{aligned}
$$

Continuing with similar arguments, we can further derive

$$
\begin{aligned}
\|A_h^{-\frac{1}{2}}P_hE_n\|_{0,\Omega} &= \sup_{\mathbf{v}_h\in\mathbf{V}_h}\frac{|(E_n,\mathbf{v}_h)_\Omega|}{\|\mathbf{v}_h\|_1}\\
&\le c\tau^{\frac{3}{2}}\Big(\int_{t_{n-1}}^{t_n}\|\mathbf{f}_{tt}(t)\|_{0,\Omega}^2\,dt\Big)^{\frac{1}{2}}+c\tau^{\frac{1}{2}}\Big(\int_{t_{n-1}}^{t_n}(t-t_{n-1})^2\|\mathbf{u}_{htt}(t)\|_1^2\,dt\Big)^{\frac{1}{2}}\\
&\quad+c\tau\Big(\int_{t_{n-2}}^{t_n}(t-t_{n-2})[\|A_h\mathbf{u}_h(t)\|_{0,\Omega}^2\|\mathbf{u}_{htt}(t)\|_0^2+\|\mathbf{u}_{ht}(t)\|_1^2\|\mathbf{u}_{ht}(t)\|_1^2]\,dt\Big)^{\frac{1}{2}}\\
&\quad+c\tau\Big(\int_{t_{n-2}}^{t_n}(t-t_{n-2})[\|A_{2h}\mathbf{B}_h(t)\|_{0,\Omega}^2\|\mathbf{B}_{htt}(t)\|_0^2+\|\mathbf{B}_{ht}(t)\|_1^2\|\mathbf{B}_{ht}(t)\|_1^2]\,dt\Big)^{\frac{1}{2}},
\end{aligned}
$$

and

$$
\begin{aligned}
\|P_hE_n\|_0 \le \|P_{0h}E_n\|_{0,\Omega} &= \sup_{\mathbf{v}_h\in\mathbf{X}_h}\frac{|(E_n,\mathbf{v}_h)_\Omega|}{\|\mathbf{v}_h\|_0}\\
&\le c\tau^{\frac{3}{2}}\Big(\int_{t_{n-1}}^{t_n}\|\mathbf{f}_{tt}(t)\|_{0,\Omega}^2\,dt\Big)^{\frac{1}{2}}+c\Big(\int_{t_{n-1}}^{t_n}(t_n-t)(t-t_{n-1})^2\|-\nu\Delta_h\mathbf{u}_{htt}(t)+\nabla_hp_{htt}(t)\|_{0,\Omega}^2\,dt\Big)^{\frac{1}{2}}\\
&\quad+c\tau^{\frac{1}{2}}\Big(\int_{t_{n-1}}^{t_n}(t_n-t)^2[\|A_h\mathbf{u}_h(t)\|_{0,\Omega}^2\|\mathbf{u}_{htt}(t)\|_1^2+\|\mathbf{u}_{ht}(t)\|_1^2\|A_h\mathbf{u}_{ht}(t)\|_{0,\Omega}^2]\,dt\Big)^{\frac{1}{2}}\\
&\quad+c\tau^{\frac{1}{2}}\Big(\int_{t_{n-2}}^{t_{n-1}}(t-t_{n-2})^2(\|A_h\mathbf{u}_h(t)\|_{0,\Omega}^2\|\mathbf{u}_{htt}(t)\|_1^2+\|\mathbf{u}_{ht}\|_1^2\|A_h\mathbf{u}_{ht}(t)\|_{0,\Omega}^2)\,dt\Big)^{\frac{1}{2}}\\
&\quad+c\tau^{\frac{1}{2}}\Big(\int_{t_{n-1}}^{t_n}(t_n-t)^2[\|A_{2h}\mathbf{B}_h(t)\|_{0,\Omega}^2\|\mathbf{B}_{htt}(t)\|_1^2+\|\mathbf{B}_{ht}(t)\|_1^2\|A_{2h}\mathbf{B}_{ht}(t)\|_{0,\Omega}^2]\,dt\Big)^{\frac{1}{2}}\\
&\quad+c\tau^{\frac{1}{2}}\Big(\int_{t_{n-2}}^{t_{n-1}}(t-t_{n-2})^2(\|A_{2h}\mathbf{B}_h(t)\|_{0,\Omega}^2\|\mathbf{B}_{htt}(t)\|_1^2+\|\mathbf{B}_{ht}\|_1^2\|A_{2h}\mathbf{B}_{ht}(t)\|_{0,\Omega}^2)\,dt\Big)^{\frac{1}{2}}.
\end{aligned}
$$

Combining all the above estimates with (4.14), we readily get the desired estimate (5.7). Similarly, by using Lemmas 3.2–3.3 and (3.1)–(3.2), we can establish (5.8) from (5.6). □

6 Error Estimates Part II

With the preparations from the previous section, we can now establish the H^1- and L^2-estimates of the errors $e^n = \mathbf{u}_h(t_n) - \mathbf{u}_h^n$, $\varepsilon^n = \mathbf{B}_h(t_n) - \mathbf{B}_h^n$ and the L^2-bound of the error $\eta^n = \bar{p}_h(t_n) - p_h^n$ for all $1 \le n \le N$.

First, for $\alpha = -2, -1, 0, 1$, we add up equation (5.3) with $\mathbf{v}_h = 2A_h^\alpha e^n \tau \in \mathbf{V}_h$ and $q_h = 0$ and equation (5.4) with $\mathbf{C}_h = 2A_{2h}^\alpha \varepsilon^n \tau \in \mathbf{W}_h$ to deduce

$$
\begin{aligned}
\|e^n\|_\alpha^2 - \|e^{n-1}\|_\alpha^2 + \|d_t e^n\|_\alpha^2 \tau^2 + \frac{\nu}{2} + (\|e^n\|_{\alpha+1}^2 - \|e^{n-1}\|_{\alpha+1}^2 + 4\|\bar{e}^n\|_{\alpha+1}^2)\tau \\
+ \|\varepsilon^n\|_\alpha^2 - \|\varepsilon^{n-1}\|_\alpha^2 + \|d_t \varepsilon^n\|_\alpha^2 \tau^2 + \frac{\mu}{2}(\|\varepsilon^n\|_{\alpha+1}^2 - \|\varepsilon^{n-1}\|_{\alpha+1}^2 + 4\|\bar{\varepsilon}^n\|_{\alpha+1}^2)\tau + \sum_{i=1}^{12} K_i \\
\leq 2\tau\Big(E_n, A_h^\alpha \bar{e}^n + \frac{1}{2}A_h^\alpha d_t e^n \tau\Big)_\Omega + 2\tau\Big(F_n, A_{2h}^\alpha \bar{\varepsilon}^n + \frac{1}{2}A_{2h}^\alpha d_t \varepsilon^n \tau\Big)_\Omega,
\end{aligned}
\tag{6.1}
$$

where $K_1, K_2, \dots, K_{12}$ are defined by

$$
\begin{aligned}
K_1 &=: 3b(e^{n-1}, \mathbf{u}_h(t_{n-1}), A_h^\alpha e^n)\tau + 3b(\mathbf{u}_h(t_{n-1}), e^{n-1}, A_h^\alpha e^n)\tau, \\
K_2 &=: -b(e^{n-2}, \mathbf{u}_h(t_{n-2}), A_h^\alpha e^n)\tau - b(\mathbf{u}_h(t_{n-2}), e^{n-2}, A_h^\alpha e^n)\tau, \\
K_3^1 + K_3^2 &=: -2b(\bar{e}^{n-1}, \bar{e}^{n-1}, A_h^\alpha e^n)\tau + \frac{1}{2}b(d_t e^{n-1}, d_t e^{n-1}, A_h^\alpha e^n)\tau^3, \\
K_4 &=: -2b(\bar{e}^{n-1}, d_t e^{n-1}, A_h^\alpha e^n)\tau^2 - 2b(d_t e^{n-1}, \bar{e}^{n-1}, A_h^\alpha e^n)\tau^2, \\
K_5 &=: 3sd(A_h^\alpha e^n, \varepsilon^{n-1}, \mathbf{B}_h(t_{n-1}))\tau + 3sd(A_h^\alpha e^n, \mathbf{B}_h(t_{n-1}), \varepsilon^{n-1})\tau, \\
K_6 &=: -sd(A_h^\alpha e^n, \varepsilon^{n-2}, \mathbf{B}_h(t_{n-2}))\tau - sd(A_h^\alpha e^n, \mathbf{B}_h(t_{n-2}), \varepsilon^{n-2})\tau, \\
K_7^1 + K_7^2 &=: -2sd(A_h^\alpha e^n, \bar{\varepsilon}^{n-1}, \bar{\varepsilon}^{n-1})\tau + \frac{1}{2}sd(A_h^\alpha e^n, d_t\varepsilon^{n-1}, d_t\varepsilon^{n-1})\tau^3, \\
K_8 &=: -2sd(A_h^\alpha e^n, \bar{\varepsilon}^{n-1}, d_t\varepsilon^{n-1})\tau^2 - 2sd(A_h^\alpha e^n, d_t\varepsilon^{n-1}, \bar{\varepsilon}^{n-1})\tau^2, \\
K_9 &=: -3d(e^{n-1}, \mathbf{B}_h(t_{n-1}), A_{2h}^\alpha \varepsilon^n)\tau - 3d(\mathbf{u}_h(t_{n-1}), \varepsilon^{n-1}, A_{2h}^\alpha \varepsilon^n)\tau, \\
K_{10} &=: d(e^{n-2}, \mathbf{B}_h(t_{n-2}), A_{2h}^\alpha \varepsilon^n)\tau + d(\mathbf{u}_h(t_{n-2}), \varepsilon^{n-2}, A_{2h}^\alpha \varepsilon^n)\tau, \\
K_{11}^1 + K_{11}^2 &=: 2d(\bar{e}^{n-1}, \bar{\varepsilon}^{n-1}, A_{2h}^\alpha \varepsilon^n)\tau - \frac{1}{2}d(d_t e^{n-1}, d_t\varepsilon^{n-1}, A_{2h}^\alpha \varepsilon^n)\tau^3, \\
K_{12} &=: 2d(\bar{e}^{n-1}, d_t\varepsilon^{n-1}, A_{2h}^\alpha \varepsilon^n)\tau^2 + 2d(d_t e^{n-1}, \bar{\varepsilon}^{n-1}, A_{2h}^\alpha \varepsilon^n)\tau^2.
\end{aligned}
\tag{6.2}
$$

We will divide the entire analysis of the errors e^n and ε^n in several lemmas for different α-norm.

Lemma 6.1. *Under Assumptions* (A0)–(A3) *and for τ satisfying*

$$
64c_0^2\kappa_4^2\tau \leq \kappa_2, \quad 64c_0^2\kappa_4^2 T^{\frac{1}{2}}(1 + \nu^{-\frac{3}{2}} + \mu^{-\frac{3}{2}})\tau \leq \kappa_1, \quad \tau \leq \frac{1}{4\mu}, \tag{6.3}
$$

there holds that

$$
\begin{aligned}
\|e^m\|_\alpha^2 + \frac{\nu}{2}\|e^m\|_{\alpha+1}^2\tau + \tau\sum_{n=1}^m \Big(\frac{1}{2}\|d_t e^n\|_\alpha^2\tau + \nu\|\bar{e}^n\|_{\alpha+1}^2\Big) \\
+ \|\varepsilon^m\|_\alpha^2 + \frac{\mu}{2}\|\varepsilon^m\|_{\alpha+1}^2\tau + \tau\sum_{n=1}^m \Big(\frac{1}{2}\|d_t \varepsilon^n\|_\alpha^2\tau + \mu\|\bar{\varepsilon}^n\|_{\alpha+1}^2\Big) \leq \kappa_4\tau^{2-\alpha}
\end{aligned}
\tag{6.4}
$$

for $\alpha = -1, 0, 1$ and $1 \leq m \leq N$, where

$$
\begin{aligned}
d_n &= 48c_0^2(\|A_h\mathbf{u}_h(t_n)\|_{0,\Omega}^2 + \|A_{2h}\mathbf{B}_h(t_n)\|_{0,\Omega}^2), \\
\kappa_3 &= \max_{1\leq n\leq N-1}\{d_n\}, \quad \kappa_4 = e^{\kappa_3 T}[3\kappa_1 + 18(1 + \nu^{-1} + \mu^{-1})\kappa_2].
\end{aligned}
\tag{6.5}
$$

Proof. For the desired error estimate (6.4), we will bound all the terms K_i for $1 \leq i \leq 12$ defined in (6.2) and also frequently use the fact that $e^n = \bar{e}^n + \frac{1}{2}\tau d_t e^n$ and $\varepsilon^n = \bar{\varepsilon}^n + \frac{1}{2}\tau d_t\varepsilon^n$.

For K_1, K_2, K_5 and K_6 and $\alpha = -1, 0, 1$, we get readily, from (3.1) and Lemmas 3.1–3.3,

$$
\begin{aligned}
|K_1| &\leq 3\Big|b\Big(e^{n-1}, \mathbf{u}_h(t_{n-1}), A_h^\alpha \bar{e}^n + \frac{1}{2}\tau A_h^\alpha d_t e^n\Big)\Big|\tau + 3\Big|b\Big(\mathbf{u}_h(t_{n-1}), e^{n-1}, A_h^\alpha \bar{e}^n + \frac{1}{2}\tau A_h^\alpha d_t e^n\Big)\Big|\tau \\
&\leq \sqrt{\nu}c_0\|A_h\mathbf{u}_h(t_{n-1})\|_{0,\Omega}\Big(\|e^{n-1}\|_\alpha\|\bar{e}^n\|_{\alpha+1} + \frac{1}{2}\tau\|e^{n-1}\|_{\alpha+1}\|d_t e^n\|_\alpha\Big)\tau \\
&\leq \frac{1}{32}\|d_t e^n\|_\alpha^2\tau^2 + \frac{\nu}{32}\|\bar{e}^n\|_{\alpha+1}^2\tau + 8c_0^2\|A_h\mathbf{u}_h(t_{n-1})\|_{0,\Omega}^2\Big(\|e^{n-1}\|_\alpha^2 + \frac{\nu}{2}\|e^{n-1}\|_{\alpha+1}^2\tau\Big)\tau, \\
|K_2| &\leq \Big|b\Big(e^{n-2}, \mathbf{u}_h(t_{n-2}), A_h^\alpha \bar{e}^n + \frac{1}{2}\tau A_h^\alpha d_t e^n\Big)\Big|\tau + \Big|b\Big(\mathbf{u}_h(t_{n-2}), e^{n-2}, A_h^\alpha \bar{e}^n + \frac{1}{2}\tau A_h^\alpha d_t e^n\Big)\Big|\tau \\
&\leq \frac{1}{32}\|d_t e^n\|_\alpha^2\tau^2 + \frac{\nu}{32}\|\bar{e}^n\|_{\alpha+1}^2\tau + 8c_0^2\|A_h\mathbf{u}_h(t_{n-2})\|_{0,\Omega}^2\Big(\|e^{n-2}\|_\alpha^2 + \frac{\nu}{2}\|e^{n-2}\|_{\alpha+1}^2\tau\Big)\tau,
\end{aligned}
$$

$$
\begin{aligned}
|K_5| &\le 3s\Big|d\Big(A_h^\alpha\bar{e}^n+\frac{1}{2}A_h^\alpha d_te^n\tau,\varepsilon^{n-1},\mathbf{B}_h(t_{n-1})\Big)\Big|\tau+3s\Big|d\Big(A_h^\alpha\bar{e}^n+\frac{1}{2}A_h^\alpha d_te^n\tau,\mathbf{B}_h(t_{n-1}),\varepsilon^{n-1}\Big)\Big|\tau\\
&\le c_0\|A_{2h}\mathbf{B}_h(t_{n-1})\|_{0,\Omega}\Big(\sqrt{\nu}\|\varepsilon^{n-1}\|_\alpha\|\bar{e}^n\|_{\alpha+1}+\frac{\sqrt{\mu}}{2}\|\varepsilon^{n-1}\|_{\alpha+1}\|d_te^n\|_\alpha\tau\Big)\tau\\
&\le \frac{1}{32}\|d_te^n\|_\alpha^2\tau^2+\frac{\nu}{32}\|\bar{e}^n\|_{\alpha+1}^2\tau+8c_0^2\|A_{2h}\mathbf{B}_h(t_{n-1})\|_{0,\Omega}^2\Big(\|\varepsilon^{n-1}\|_\alpha^2+\frac{\mu}{2}\|\varepsilon^{n-1}\|_{\alpha+1}^2\tau\Big)\tau,\\
|K_6| &\le s\Big|d\Big(A_h^\alpha\bar{e}^n+\frac{1}{2}A_h^\alpha d_te^n\tau,\varepsilon^{n-2},\mathbf{B}_h(t_{n-2})\Big)\Big|\tau+s\Big|d\Big(A_h^\alpha\bar{e}^n+\frac{1}{2}A_h^\alpha d_te^n\tau,\mathbf{B}_h(t_{n-2}),\varepsilon^{n-2}\Big)\Big|\tau\\
&\le \frac{1}{32}\|d_te^n\|_\alpha^2\tau^2+\frac{\nu}{32}\|\bar{e}^n\|_{\alpha+1}^2\tau+8c_0^2\|A_{2h}\mathbf{B}_h(t_{n-2})\|_{0,\Omega}^2\Big(\|\varepsilon^{n-2}\|_\alpha^2+\frac{\mu}{2}\|\varepsilon^{n-2}\|_{\alpha+1}^2\tau\Big)\tau.
\end{aligned}
$$

For K_9, K_{10}, K_3^1 and K_4 and $\alpha=-1,0,1$, we can estimate, by means of Lemmas 3.2–3.3,

$$
\begin{aligned}
|K_9| &\le 3\Big|d\Big(e^{n-1},\mathbf{B}_h(t_{n-1}),A_{2h}^\alpha\bar{\varepsilon}^n+\frac{1}{2}A_{2h}^\alpha d_t\varepsilon^n\tau\Big)\Big|\tau+3\Big|d\Big(\mathbf{u}_h(t_{n-1}),\varepsilon^{n-1},A_{2h}^\alpha\bar{\varepsilon}^n+\frac{1}{2}A_{2h}^\alpha d_t\varepsilon^n\tau\Big)\Big|\tau\\
&\le c_0\|A_{2h}\mathbf{B}_h(t_{n-1})\|_{0,\Omega}\Big(\sqrt{\mu}\|e^{n-1}\|_\alpha\|\bar{\varepsilon}^n\|_{\alpha+1}+\frac{\sqrt{\nu}}{2}\|e^{n-1}\|_{\alpha+1}\|d_t\varepsilon^n\|_\alpha\tau\Big)\tau\\
&\quad+c_0\|A_h\mathbf{u}_h(t_{n-1})\|_{0,\Omega}\sqrt{\mu}\Big(\|\varepsilon^{n-1}\|_\alpha\|\bar{\varepsilon}^n\|_{\alpha+1}+\frac{1}{2}\|\varepsilon^{n-1}\|_{\alpha+1}\|d_t\varepsilon^n\|_\alpha\tau\Big)\tau\\
&\le \frac{1}{16}\|d_t\varepsilon^n\|_\alpha^2\tau^2+\frac{\mu}{16}\|\bar{\varepsilon}^n\|_{\alpha+1}^2\tau+8c_0^2\|A_{2h}\mathbf{B}_h(t_{n-1})\|_{0,\Omega}^2\Big(\|e^{n-1}\|_\alpha^2+\frac{\nu}{2}\|e^{n-1}\|_{\alpha+1}^2\tau\Big)\\
&\quad+8c_0^2\|A_h\mathbf{u}_h(t_{n-1})\|_{0,\Omega}^2\Big(\|\varepsilon^{n-1}\|_\alpha^2+\frac{\mu}{2}\|\varepsilon^{n-1}\|_{\alpha+1}^2\tau\Big)\tau,\\
|K_{10}| &\le \Big|d\Big(e^{n-2},\mathbf{B}_h(t_{n-2}),A_{2h}^\alpha\bar{\varepsilon}^n+\frac{1}{2}A_{2h}^\alpha d_t\varepsilon^n\tau\Big)\Big|\tau+\Big|d\Big(\mathbf{u}_h(t_{n-2}),\varepsilon^{n-2},A_{2h}^\alpha\bar{\varepsilon}^n+\frac{1}{2}A_{2h}^\alpha d_t\varepsilon^n\tau\Big)\Big|\tau\\
&\le \frac{1}{16}\|d_t\varepsilon^n\|_\alpha^2\tau^2+\frac{\mu}{16}\|\bar{\varepsilon}^n\|_{\alpha+1}^2\tau+8c_0^2\|A_{2h}\mathbf{B}_h(t_{n-2})\|_{0,\Omega}^2\Big(\|e^{n-2}\|_\alpha^2+\frac{\nu}{2}\|e^{n-2}\|_{\alpha+1}^2\tau\Big)\\
&\quad+8c_0^2\|A_h\mathbf{u}_h(t_{n-2})\|_{0,\Omega}^2\Big(\|\varepsilon^{n-2}\|_\alpha^2+\frac{\mu}{2}\|\varepsilon^{n-2}\|_{\alpha+1}^2\tau\Big)\tau,\\
|K_3^1| &\le 2\Big|b\Big(\bar{e}^{n-1},\bar{e}^{n-1},A_h^\alpha\bar{e}^n+\frac{1}{2}A_h^\alpha d_te^n\tau\Big)\Big|\tau\\
&\le c_0\sqrt{\nu}\|A_h\bar{e}^{n-1}\|_{0,\Omega}\Big(\|\bar{e}^{n-1}\|_\alpha\|\bar{e}^n\|_{\alpha+1}+\frac{1}{2}\|\bar{e}^{n-1}\|_{\alpha+1}\|d_te^n\|_\alpha\tau\Big)\tau\\
&\le \frac{\nu}{32}\|\bar{e}^n\|_{\alpha+1}^2\tau+\frac{1}{32}\|d_te^n\|_\alpha^2\tau^2+8c_0^2\|A_h\bar{e}^{n-1}\|_{0,\Omega}^2\Big(\|\bar{e}^{n-1}\|_\alpha^2+\frac{\nu}{2}\|\bar{e}^{n-1}\|_{\alpha+1}^2\tau\Big)\tau,\\
|K_4| &\le 2\Big|b\Big(\bar{e}^{n-1},d_te^{n-1},A_h^\alpha\bar{e}^n+\frac{1}{2}A_h^\alpha d_te^n\tau\Big)\Big|\tau^2+2\Big|b\Big(d_te^{n-1},\bar{e}^{n-1},A_h^\alpha\bar{e}^n+\frac{1}{2}A_h^\alpha d_te^n\tau\Big)\Big|\tau^2\\
&\le c_0\|A_h\bar{e}^{n-1}\|_{0,\Omega}(\sqrt{\nu}\|d_te^{n-1}\|_\alpha\|\bar{e}^n\|_{\alpha+1}+\|d_te^{n-1}\|_{\alpha+1}\|d_te^n\|_\alpha\tau)\tau^2\\
&\le \frac{\nu}{32}\|\bar{e}^n\|_{\alpha+1}^2\tau+\frac{1}{32}\|d_te^n\|_\alpha^2\tau^2+8c_0^2\|A_h\bar{e}^{n-1}\|_{0,\Omega}^2(\|d_te^{n-1}\|_\alpha^2+\|d_te^{n-1}\|_{\alpha+1}^2\tau)\tau^3.
\end{aligned}
$$

For K_3^2 with different α, K_7 and K_8, we can estimate, by means of Lemmas 3.2–3.3,

$$
\begin{aligned}
|K_3^2| &\le \frac{1}{2}\Big|b\Big(d_te^{n-1},d_te^{n-1},A_h^\alpha\bar{e}^n+\frac{1}{2}A_h^\alpha d_te^n\tau\Big)\Big|\tau^3\quad(\text{for }\alpha=-1,0)\\
&\le c_0\|A_hd_te^{n-1}\|_{0,\Omega}\Big(\sqrt{\nu}\|d_te^{n-1}\|_\alpha\|\bar{e}^n\|_{\alpha+1}+\frac{1}{2}\|d_te^{n-1}\|_{\alpha+1}\|d_te^n\|_\alpha\tau\Big)\tau^3\\
&\le \frac{\nu}{32}\|\bar{e}^n\|_{\alpha+1}^2\tau+\frac{1}{32}\|d_te^n\|_\alpha^2\tau^2+8c_0^2\|A_hd_te^{n-1}\|_{0,\Omega}^2(\|d_te^{n-1}\|_\alpha^2+\|d_te^{n-1}\|_{\alpha+1}^2\tau)\tau^5,\\
|K_3^2| &\le \frac{1}{2}\Big|b\Big(d_te^{n-1},d_te^{n-1},A_h\bar{e}^n+\frac{1}{2}A_hd_te^n\tau\Big)\Big|\tau^3\quad(\text{for }\alpha=1)\\
&\le c_0\|A_hd_te^{n-1}\|_{0,\Omega}\Big(\sqrt{\nu}\|d_te^{n-1}\|_1\|\bar{e}^n\|_2+\frac{1}{2}\|d_te^{n-1}\|_1^{\frac{1}{2}}\|A_hd_te^{n-1}\|_{0,\Omega}^{\frac{1}{2}}\|d_te^n\|_1\tau\Big)\tau^3\\
&\le \frac{\nu}{32}\|\bar{e}^n\|_2^2\tau+\frac{1}{32}\|d_te^n\|_1^2\tau^2+8c_0^2\|A_hd_te^{n-1}\|_{0,\Omega}^2(\|d_te^{n-1}\|_1^2+\|A_hd_te^{n-1}\|_{0,\Omega}\|d_te^{n-1}\|_1\tau)\tau^5,\\
|K_7^1| &\le 2s\Big|d\Big(A_h^\alpha\bar{e}^n+\frac{1}{2}A_h^\alpha d_te^n\tau,\bar{\varepsilon}^{n-1},\bar{\varepsilon}^{n-1}\Big)\Big|\tau\\
&\le c_0\|A_{2h}\bar{\varepsilon}^{n-1}\|_{0,\Omega}\Big(\sqrt{\nu}\|\bar{\varepsilon}^{n-1}\|_\alpha\|\bar{e}^n\|_{\alpha+1}+\frac{\sqrt{\mu}}{2}\|\bar{\varepsilon}^{n-1}\|_{\alpha+1}\|d_te^n\|_\alpha\tau\Big)\tau\\
&\le \frac{\nu}{32}\|\bar{e}^n\|_{\alpha+1}^2\tau+\frac{1}{32}\|d_te^n\|_\alpha^2\tau^2+8c_0^2\|A_{2h}\bar{\varepsilon}^{n-1}\|_{0,\Omega}^2\Big(\|\bar{\varepsilon}^{n-1}\|_\alpha^2+\frac{\mu}{2}\|\bar{\varepsilon}^{n-1}\|_{\alpha+1}^2\tau\Big)\tau,
\end{aligned}
$$

$$
\begin{aligned}
|K_8| &\le 2s\Big|d\Big(A_h^\alpha \bar{e}^n + \frac{1}{2}A_h^\alpha d_t e^n\tau, \bar{\varepsilon}^{n-1}, d_t\varepsilon^{n-1}\Big)\Big|\tau^2 + 2s\Big|d\Big(A_h^\alpha \bar{e}^n + \frac{1}{2}A_h^\alpha d_t e^n\tau, d_t\varepsilon^{n-1}, \bar{\varepsilon}^{n-1}\Big)\Big|\tau^2 \\
&\le c_0\|A_{2h}\bar{\varepsilon}^{n-1}\|_{0,\Omega}(\sqrt{\nu}\|d_t\varepsilon^{n-1}\|_\alpha\|\bar{e}^n\|_{\alpha+1} + \|d_t\varepsilon^{n-1}\|_{\alpha+1}\|d_t e^n\|_\alpha\tau)\tau^2 \\
&\le \frac{\nu}{32}\|\bar{e}^n\|_{\alpha+1}^2\tau + \frac{1}{32}\|d_t e^n\|_\alpha^2\tau^2 + 8c_0^2\|A_{2h}\bar{\varepsilon}^{n-1}\|_{0,\Omega}^2(\|d_t\varepsilon^{n-1}\|_\alpha^2 + \|d_t\varepsilon^{n-1}\|_{\alpha+1}\tau)\tau^3, \\
|K_7^2| &\le \frac{1}{2}s\Big|d\Big(A_h^\alpha \bar{e}^n + \frac{1}{2}A_h^\alpha d_t e^n\tau, d_t\varepsilon^{n-1}, d_t\varepsilon^{n-1}\Big)\Big|\tau^3 \quad (\text{for } \alpha = -1, 0) \\
&\le c_0\|A_{2h}d_t\varepsilon^{n-1}\|_{0,\Omega}(\sqrt{\nu}\|d_t\varepsilon^{n-1}\|_\alpha\|\bar{e}^n\|_{\alpha+1} + \|d_t\varepsilon^{n-1}\|_{\alpha+1}\|d_t e^n\|_\alpha\tau)\tau^3 \\
&\le \frac{\nu}{32}\|\bar{e}^n\|_{\alpha+1}^2\tau + \frac{1}{32}\|d_t e^n\|_\alpha^2\tau^2 + 8c_0^2\|A_{2h}d_t\varepsilon^{n-1}\|_{0,\Omega}^2(\|d_t\varepsilon^{n-1}\|_\alpha^2 + \|d_t\varepsilon^{n-1}\|_{\alpha+1}^2\tau)\tau^5, \\
|K_7^2| &\le \frac{1}{2}s\Big|d\Big(A_h^\alpha \bar{e}^n + \frac{1}{2}A_h^\alpha d_t e^n\tau, d_t\varepsilon^{n-1}, d_t\varepsilon^{n-1}\Big)\Big|\tau^3 \quad (\text{for } \alpha = 1) \\
&\le sc_0\|A_{2h}d_t\varepsilon^{n-1}\|_{0,\Omega}(\sqrt{\nu}\|d_t\varepsilon^{n-1}\|_1\|\bar{e}^n\|_2 + \|d_t\varepsilon^{n-1}\|_1^{\frac{1}{2}}\|A_{2h}d_t\varepsilon^{n-1}\|_{0,\Omega}^{\frac{1}{2}}\|d_t e^n\|_1\tau)\tau^3 \\
&\le \frac{\nu}{32}\|\bar{e}^n\|_2^2\tau + \frac{1}{32}\|d_t e^n\|_\alpha^2\tau^2 + 8c_0^2\|A_{2h}d_t\varepsilon^{n-1}\|_{0,\Omega}^2(\|d_t\varepsilon^{n-1}\|_1^2 + \|A_{2h}d_t\varepsilon^{n-1}\|_{0,\Omega}\|d_t\varepsilon^{n-1}\|_1\tau)\tau^5.
\end{aligned}
$$

For K_{11}^1 and $\alpha = -1, 0, 1$, we get readily, from (3.1) and Lemmas 3.1–3.3,

$$
\begin{aligned}
|K_{11}^1| &\le 2\Big|d\Big(\bar{e}^{n-1}, \bar{\varepsilon}^{n-1}, A_{2h}^\alpha\bar{\varepsilon}^n + \frac{1}{2}A_{2h}^\alpha d_t\varepsilon^n\tau\Big)\Big|\tau \\
&\le c_0\sqrt{\mu}\|A_h\bar{e}^{n-1}\|_{0,\Omega}(\|\bar{\varepsilon}^{n-1}\|_\alpha\|\bar{\varepsilon}^n\|_{\alpha+1} + \frac{1}{2}\|\bar{\varepsilon}^{n-1}\|_{\alpha+1}\|d_t\varepsilon^n\|_\alpha\tau)\tau \\
&\le \frac{\mu}{32}\|\bar{\varepsilon}^n\|_{\alpha+1}^2\tau + \frac{1}{32}\|d_t\varepsilon^n\|_\alpha^2\tau^2 + 8c_0^2\|A_h\bar{e}^{n-1}\|_{0,\Omega}^2\Big(\|\bar{\varepsilon}^{n-1}\|_\alpha^2 + \frac{\mu}{2}\|\bar{\varepsilon}^{n-1}\|_{\alpha+1}^2\tau\Big)\tau, \\
|K_{12}| &\le 2\Big|d\Big(\bar{e}^{n-1}, d_t\varepsilon^{n-1}, A_{2h}^\alpha\bar{\varepsilon}^n + \frac{1}{2}A_{2h}^\alpha d_t\varepsilon^n\tau\Big)\Big|\tau^2 + 2\Big|d\Big(d_t e^{n-1}, \bar{\varepsilon}^{n-1}, A_{2h}^\alpha\bar{\varepsilon}^n + \frac{1}{2}A_{2h}^\alpha d_t\varepsilon^n\tau\Big)\Big|\tau^2 \\
&\le c_0\|A_h\bar{e}^{n-1}\|_{0,\Omega}(\sqrt{\mu}\|d_t\varepsilon^{n-1}\|_\alpha\|\bar{\varepsilon}^n\|_{\alpha+1} + \|d_t\varepsilon^{n-1}\|_{\alpha+1}\|d_t\varepsilon^n\|_\alpha\tau)\tau \\
&\qquad + c_0\|A_{2h}\bar{\varepsilon}^{n-1}\|_{0,\Omega}(\sqrt{\mu}\|d_t e^{n-1}\|_\alpha\|\bar{\varepsilon}^n\|_{\alpha+1} + \|d_t e^{n-1}\|_{\alpha+1}\|d_t\varepsilon^n\|_\alpha\tau)\tau^2 \\
&\le \frac{\mu}{16}\|\bar{\varepsilon}^n\|_{\alpha+1}^2\tau + \frac{1}{16}\|d_t\varepsilon^n\|_\alpha^2\tau^2 + 8c_0^2\|A_h\bar{e}^{n-1}\|_{0,\Omega}^2(\|d_t\varepsilon^{n-1}\|_\alpha^2 + \|d_t\varepsilon^{n-1}\|_{\alpha+1}^2\tau)\tau^2 \\
&\qquad + 8c_0^2\|A_{2h}\bar{\varepsilon}^{n-1}\|_{0,\Omega}^2(\|d_t e^{n-1}\|_\alpha^2 + \|d_t e^{n-1}\|_{\alpha+1}^2)\tau)\tau^2, \\
|K_{11}^2| &\le \frac{1}{2}\Big|d\Big(d_t e^{n-1}, d_t\varepsilon^{n-1}, A_{2h}^\alpha\bar{\varepsilon}^n + \frac{1}{2}A_{2h}^\alpha d_t\varepsilon^n\tau\Big)\Big|\tau^3 \quad (\text{for } \alpha = -1, 0) \\
&\le c_0\|A_h d_t e^{n-1}\|_{0,\Omega}(\sqrt{\mu}\|d_t\varepsilon^{n-1}\|_\alpha\|\bar{\varepsilon}^n\|_{\alpha+1} + \|d_t\varepsilon^{n-1}\|_{\alpha+1}\|d_t\varepsilon^n\|_\alpha\tau)\tau^3 \\
&\le \frac{\mu}{32}\|\bar{\varepsilon}^n\|_{\alpha+1}^2\tau + \frac{1}{32}\|d_t\varepsilon^n\|_\alpha^2\tau^2 + 8c_0^2\|A_h d_t e^{n-1}\|_{0,\Omega}^2(\|d_t\varepsilon^{n-1}\|_\alpha^2 + \|d_t\varepsilon^{n-1}\|_{\alpha+1}^2\tau)\tau^5, \\
|K_{11}^2| &\le \frac{1}{2}\Big|d\Big(d_t e^{n-1}, d_t\varepsilon^{n-1}, A_{2h}^\alpha\bar{\varepsilon}^n + \frac{1}{2}A_{2h}^\alpha d_t\varepsilon^n\tau\Big)\Big|\tau^3 \quad (\text{for } \alpha = 1) \\
&\le c_0\|A_h d_t e^{n-1}\|_{0,\Omega}(\sqrt{\mu}\|d_t\varepsilon^{n-1}\|_1\|\bar{\varepsilon}^n\|_2 + \|d_t\varepsilon^{n-1}\|_1^{\frac{1}{2}}\|A_{2h}d_t\varepsilon^{n-1}\|_{0,\Omega}^{\frac{1}{2}}\|d_t\varepsilon^n\|_1\tau)\tau^3 \\
&\le \frac{\mu}{32}\|\bar{\varepsilon}^n\|_2^2\tau + \frac{1}{32}\|d_t\varepsilon^n\|_1^2\tau^2 + 8c_0^2\|A_h d_t e^{n-1}\|_{0,\Omega}^2(\|d_t\varepsilon^{n-1}\|_1^2 + \|d_t\varepsilon^{n-1}\|_1\|A_{2h}d_t\varepsilon^{n-1}\|_{0,\Omega}\tau)\tau^5.
\end{aligned}
$$

It remains to bound the two terms on the right-hand side of (6.2):

$$
\begin{aligned}
2\Big|\Big(E_n, A_h^\alpha\bar{e}^n + \frac{1}{2}A_h^\alpha d_t e^n\tau\Big)_\Omega\Big|\tau &\le \frac{\nu}{16}\|\bar{e}^n\|_{\alpha+1}^2\tau + \frac{1}{16}\|d_t e^n\|_\alpha^2\tau^2 + \frac{16}{\nu}\|A_h^{\frac{\alpha-1}{2}}P_h E_n\|_{0,\Omega}^2\tau + 8\|A_h^{\frac{\alpha}{2}}P_h E_n\|_{0,\Omega}^2\tau^2, \\
2|(F_n, A_{2h}^\alpha\bar{\varepsilon}^n + \frac{1}{2}A_{2h}^\alpha d_t e^n\tau)_\Omega|\tau &\le \frac{\mu}{16}\|\bar{\varepsilon}^n\|_{\alpha+1}^2\tau + \frac{1}{32}\|d_t\varepsilon^n\|_\alpha^2\tau^2 + \frac{16}{\mu}\|A_{2h}^{\frac{\alpha-1}{2}}R_{0h}F_n\|_{0,\Omega}^2\tau + 8\|A_{2h}^{\frac{\alpha}{2}}R_{0h}F_n\|_{0,\Omega}^2\tau^2.
\end{aligned}
$$

Applying these bounds and the above inequalities of K_i, then using (6.3) and (6.5), we obtain from (6.2) that

$$
\begin{aligned}
a_n - a_{n-1} + b_n\tau &\le \frac{1}{2}d_{n-1}a_{n-1}\tau + \frac{1}{2}d_{n-2}a_{n-2}\tau + c_n^\alpha\tau + \frac{16}{\nu}\|A_h^{\frac{\alpha-1}{2}}P_h E_n\|_{0,\Omega}^2\tau \\
&\quad + 8\|A_h^{\frac{\alpha}{2}}P_h E_n\|_{0,\Omega}^2\tau^2 + 16\|A_{2h}^{\frac{\alpha-1}{2}}R_{0h}F_n\|_{0,\Omega}^2\tau + 16\|A_{2h}^{\frac{\alpha}{2}}R_{0h}F_n\|_{0,\Omega}^2\tau^2
\end{aligned} \tag{6.6}
$$

for all $2 \le n \le N$, where

$$a_n =: \|e^n\|_\alpha^2 + \frac{\nu}{2}\|e^n\|_{\alpha+1}^2\tau + (1+\mu\tau)\|\varepsilon^n\|_\alpha^2 + \frac{\mu}{2}\|\varepsilon^n\|_{\alpha+1}^2\tau,$$

$$b_n =: \nu\|\bar e^n\|_{\alpha+1}^2 + \mu\|\bar\varepsilon^n\|_{\alpha+1}^2 + \frac{1}{2}[\|d_t e^n\|_\alpha^2\tau + \|d_t\varepsilon^n\|_\alpha^2\tau],$$

$$\begin{aligned} c_{n-1}^\alpha =:\ & 8c_0^2(\|A_h\bar e^{n-1}\|_{0,\Omega}^2 + \|A_{2h}\bar\varepsilon^{n-1}\|_{0,\Omega}^2)\Big(\|\bar e^{n-1}\|_\alpha^2 + \|\bar\varepsilon^{n-1}\|_\alpha^2 + \frac{\nu}{2}\|\bar e^{n-1}\|_{\alpha+1}^2\tau + \frac{\mu}{2}\|\bar\varepsilon^{n-1}\|_{\alpha+1}^2\tau\Big) \\ & + 8c_0^2(\|A_h\bar e^{n-1}\|_{0,\Omega}^2 + \|A_{2h}\bar\varepsilon^{n-1}\|_{0,\Omega}^2)(\|d_te^{n-1}\|_\alpha^2 + \|d_t\varepsilon^{n-1}\|_\alpha^2 + \|d_te^{n-1}\|_{\alpha+1}^2\tau + \|d_t\varepsilon^{n-1}\|_{\alpha+1}^2\tau)\tau^2 \\ & + 8c_0^2(\|A_hd_te^{n-1}\|_{0,\Omega}^2 + \|A_{2h}d_t\varepsilon^{n-1}\|_{0,\Omega}^2)(\|d_te^{n-1}\|_\alpha^2 + \|d_t\varepsilon^{n-1}\|_\alpha^2)\tau^4 + G_{n-1}^\alpha, \end{aligned}$$

$$\begin{aligned} G_{n-1}^\alpha =:\ & 8c_0^2\|A_hd_te^{n-1}\|_{0,\Omega}^2(\|d_te^{n-1}\|_{\alpha+1}^2 + \|d_t\varepsilon^{n-1}\|_{\alpha+1}^2)\tau^5 \\ & + 8c_0^2\|A_{2h}d_t\varepsilon^{n-1}\|_{0,\Omega}^2\|d_t\varepsilon^{n-1}\|_{\alpha+1}^2\tau^5 \quad (\alpha = -1, 0), \end{aligned}$$

$$\begin{aligned} G_{n-1}^1 =:\ & 8c_0^2\|A_hd_te^{n-1}\|_{0,\Omega}^3\|d_te^{n-1}\|_1\tau^5 + 8c_0^2\|A_{2h}d_t\varepsilon^{n-1}\|_{0,\Omega}^3\|d_t\varepsilon^{n-1}\|_1\tau^5 \\ & + 8c_0^2\|A_hd_te^{n-1}\|_{0,\Omega}^2\|A_{2h}d_t\varepsilon^{n-1}\|_{0,\Omega}\|d_t\varepsilon^{n-1}\|_1\tau^5 \quad (\alpha = 1). \end{aligned}$$

Now, summing up (6.6) from 2 to m and using Lemma 5.1, we derive for $1 \le m \le N$ that

$$\begin{aligned} a_m + \tau\sum_{n=1}^m b_n &\le 2a_1 + b_1\tau + \tau\sum_{n=1}^{m-1} d_na_n + \tau\sum_{n=1}^{m-1} c_n^\alpha + 16(1+\nu^{-1}+\mu^{-1})\kappa_2\tau^{2-\alpha} \\ &\le \tau\sum_{n=1}^{m-1} d_na_n + \tau\sum_{n=1}^{m-1} c_n^\alpha + [2\kappa_1 + 16(1+\nu^{-1}+\mu^{-1})\kappa_2]\tau^{2-\alpha}. \end{aligned}$$

Now we apply Lemma 3.9 to get for all $1 \le m \le N$ that

$$a_m + \tau\sum_{n=1}^m b_n \le e^{\kappa_3 T}\Big\{(2\kappa_1 + 16(1+\nu^{-1}+\mu^{-1})\kappa_2)\tau^{2-\alpha} + \tau\sum_{n=1}^{m-1} c_n^\alpha\Big\}. \tag{6.7}$$

We can now end the proof of (6.4) by induction. First, we know (6.4) holds for $m = 0, 1$ by Lemma 4.1. Then we prove (6.4) for $m = J+1$ under the assumption that it holds for $m = 0, 1, \dots, J$. Based on (6.3) and the induction assumption, we have

$$\begin{aligned} &\tau\sum_{n=1}^{m-1} c_n^\alpha \le 64c_0^2\kappa_4^2(1+\nu^{-1}+\mu^{-1})\tau^{3-\alpha} \quad (\text{for } \alpha = -1, 0), \\ &\tau\sum_{n=1}^{m-1} c_n^1 \le 64c_0^2\kappa_4^2(1+\nu^{-1}+\mu^{-1})\tau^{3-\alpha} + 64c_0^2\kappa_4^2T^{\frac12}(1+\nu^{-\frac32}+\mu^{-\frac32})\tau^{3-\alpha}. \end{aligned} \tag{6.8}$$

Combining these estimates with (6.7) and using (6.3), we get the desired estimate (6.4) for $m = J+1$. □

Lemma 6.2. *Under Assumptions* (A0)–(A3) *and for τ satisfying*

$$64c_0^2\kappa_4^2\tau \le \kappa_2, \quad 64c_0^2\kappa_4^2T^{\frac12}(1+\nu^{-\frac32}+\mu^{-\frac32})\tau \le \kappa_1, \quad \tau \le \frac{1}{4\mu}, \quad \kappa_5\tau \le \frac12, \tag{6.9}$$

there holds that

$$\begin{aligned} &\|e^m\|_{-2}^2 + \frac{\nu}{2}\|e^m\|_{-1}^2\tau + \tau\sum_{n=1}^m(\|d_te^n\|_{-2}^2\tau + \nu\|\bar e^n\|_{-1}^2) \\ &\quad + \|\varepsilon^m\|_{-2}^2 + \frac{\mu}{2}\|\varepsilon^m\|_{-1}^2\tau + \tau\sum_{n=1}^m(\|d_t\varepsilon^n\|_{-2}^2\tau + \mu\|\bar\varepsilon^n\|_{-1}^2) \le \kappa_6e^{\kappa_5T}\tau^4 \end{aligned} \tag{6.10}$$

for all $1 \le m \le N$, where $\kappa_5 = \max_{1\le n\le N} d_n$, and

$$\begin{aligned} d_n = &\ 16c_0^2(1+\nu^{-1}+\mu^{-1})(\|A_h\mathbf{u}_h(t_{n-1})\|_{0,\Omega}^2 + \|A_h\mathbf{u}_h(t_{n-2})\|_{0,\Omega}^2) \\ & + 16c_0^2(1+\nu^{-1}+\mu^{-1})(\|A_{2h}\mathbf{B}_h(t_{n-1})\|_{0,\Omega}^2 + \|A_{2h}\mathbf{B}_h(t_{n-2})\|_{0,\Omega}^2) \\ & + 16c_0^2(\nu^{-1}\|A_h\bar e^{n-1}\|_{0,\Omega}^2 + \mu^{-1}\|A_{2h}\bar\varepsilon^{n-1}\|_{0,\Omega}^2) \\ & + 16c_0^2(\|\bar e^{n-1}\|_1^2\tau^{-1} + \|\bar\varepsilon^{n-1}\|_1^2\tau^{-1} + \|d_te^{n-1}\|_1^2\tau + \|d_t\varepsilon^{n-1}\|_1^2\tau). \end{aligned}$$

Proof. We can estimate all the terms K_i in (6.2) with $\alpha = -2$, by using the identities

$$e^{n-1} = \bar{e}^{n-1} + \frac{1}{2} d_t e^{n-1} \tau, \quad e^{n-2} = \bar{e}^{n-1} - \frac{1}{2} d_t e^{n-1} \tau,$$
$$\varepsilon^{n-1} = \bar{\varepsilon}^{n-1} + \frac{1}{2} d_t \varepsilon^{n-1} \tau, \quad \varepsilon^{n-2} = \bar{\varepsilon}^{n-1} - \frac{1}{2} d_t \varepsilon^{n-1} \tau$$

and Lemmas 3.2–3.3 and (3.1)–(3.2) as follows:

$$
\begin{aligned}
|K_1| &\le \frac{1}{32}\|d_t e^{n-1}\|_{-1}^2 \tau^3 + \frac{\nu}{32}\|\bar{e}^{n-1}\|_{-1}^2 \tau + 8c_0^2(1+\nu^{-1})\|A_h \mathbf{u}_h(t_{n-1})\|_{0,\Omega}^2 \|e^n\|_{-2}^2 \tau, \\
|K_2| &\le \frac{1}{32}\|d_t e^{n-1}\|_{-1}^2 \tau^3 + \frac{\nu}{32}\|\bar{e}^{n-1}\|_{-1}^2 \tau + 8c_0^2(1+\nu^{-1})\|A_h \mathbf{u}_h(t_{n-2})\|_{0,\Omega}^2 \|e^n\|_{-2}^2 \tau, \\
|K_5| &\le \frac{1}{32}\|d_t \varepsilon^{n-1}\|_{-1}^2 \tau^3 + \frac{\mu}{32}\|\bar{\varepsilon}^{n-1}\|_{-1}^2 \tau + 8c_0^2(1+\mu^{-1})\|A_{2h} \mathbf{B}_h(t_{n-1})\|_{0,\Omega}^2 \|e^n\|_{-2}^2 \tau, \\
|K_6| &\le \frac{1}{32}\|d_t \varepsilon^{n-1}\|_{-1}^2 \tau^3 + \frac{\mu}{32}\|\bar{\varepsilon}^{n-1}\|_{-1}^2 \tau + 8c_0^2(1+\mu^{-1})\|A_{2h} \mathbf{B}_h(t_{n-2})\|_{0,\Omega}^2 \|e^n\|_{-2}^2 \tau, \\
|K_9| &\le \frac{1}{32}\|d_t e^{n-1}\|_{-1}^2 \tau^3 + \frac{\nu}{32}\|\bar{e}^{n-1}\|_{-1}^2 \tau + \frac{1}{32}\|d_t \varepsilon^{n-1}\|_{-1}^2 \tau^3 + \frac{\mu}{32}\|\bar{\varepsilon}^{n-1}\|_{-1}^2 \tau \\
&\quad + 8c_0^2[(1+\nu^{-1})\|A_{2h} \mathbf{B}_h(t_{n-1})\|_{0,\Omega}^2 + (1+\mu^{-1})\|A_h \mathbf{u}_h(t_{n-1})\|_{0,\Omega}^2]\|e^n\|_{-2}^2 \tau, \\
|K_{10}| &\le \frac{1}{32}\|d_t e^{n-1}\|_{-1}^2 \tau^3 + \frac{\nu}{32}\|\bar{e}^{n-1}\|_{-1}^2 \tau + \frac{1}{32}\|d_t \varepsilon^{n-1}\|_{-1}^2 \tau^3 + \frac{\mu}{32}\|\bar{\varepsilon}^{n-1}\|_{-1}^2 \tau \\
&\quad + 8c_0^2((1+\nu^{-1})\|A_{2h} \mathbf{B}_h(t_{n-2})\|_{0,\Omega}^2 + (1+\mu^{-1})\|A_h \mathbf{u}_h(t_{n-2})\|_{0,\Omega}^2)\|e^n\|_{-2}^2 \tau, \\
|K_3^1| &\le \frac{\nu}{32}\|\bar{e}^{n-1}\|_{-1}^2 \tau + 8c_0^2 \nu^{-1}\|A_h \bar{e}^{n-1}\|_{0,\Omega}^2 \|e^n\|_{-2}^2 \tau, \\
|K_4| &\le \frac{1}{32}\|d_t e^{n-1}\|_0^2 \tau^4 + 8c_0^2\|\bar{e}^{n-1}\|_1^2 \|e^n\|_{-2}^2, \\
|K_3^2| &\le \frac{1}{32}\|d_t e^{n-1}\|_0^2 \tau^4 + 8c_0^2\|d_t e^{n-1}\|_1^2 \|e^n\|_{-2}^2 \tau^2, \\
|K_7^1| &\le \frac{\mu}{32}\|\bar{\varepsilon}^{n-1}\|_{-1}^2 \tau + 8c_0^2 \mu^{-1}\|A_{2h} \bar{\varepsilon}^{n-1}\|_{0,\Omega}^2 \|e^n\|_{-2}^2 \tau, \\
|K_8| &\le \frac{1}{32}\|d_t \varepsilon^{n-1}\|_0^2 \tau^4 + 8c_0^2\|\bar{\varepsilon}^{n-1}\|_1^2 \|e^n\|_{-2}^2, \\
|K_{11}^2| &\le \frac{1}{32}\|d_t \varepsilon^{n-1}\|_0^2 \tau^4 + 8c_0^2\|d_t \varepsilon^{n-1}\|_1^2 \|e^n\|_{-2}^2 \tau^2, \\
|K_{11}^1| &\le \frac{\mu}{32}\|\bar{\varepsilon}^{n-1}\|_{-1}^2 \tau + 8c_0^2 \mu^{-1}\|A_h \bar{e}^{n-1}\|_{0,\Omega}^2 \|\varepsilon^n\|_{-2}^2 \tau, \\
|K_{12}| &\le \frac{1}{32}\|d_t \varepsilon^{n-1}\|_0^2 \tau^4 + \frac{1}{32}\|d_t e^{n-1}\|_0^2 \tau^4 + 8c_0^2(\|\bar{e}^{n-1}\|_1^2 + \|\bar{\varepsilon}^{n-1}\|_1^2)\|\varepsilon^n\|_{-2}^2, \\
|K_7^2| &\le \frac{1}{32}\|d_t \varepsilon^{n-1}\|_0^2 \tau^4 + 8c_0^2\|d_t e^{n-1}\|_1^2 \|e^n\|_{-2}^2 \tau^2.
\end{aligned}
$$

Furthermore, the two terms in the right-hand side of (6.2) can be bounded by

$$
\begin{aligned}
2|(E_n, A^{-2} e^n)_\Omega|\tau &\le \frac{\nu}{16}\|\bar{e}^n\|_{-1}^2 \tau + \frac{1}{16}\|d_t e^n\|_{-2}^2 \tau^2 + \frac{16}{\nu}\|A_h^{-\frac{3}{2}} P_h E_n\|_{0,\Omega}^2 \tau + 8\|A_h^{-1} P_h E_n\|_{0,\Omega}^2 \tau^2, \\
2|(F_n, A_{2h}^{-2} \varepsilon^n)_\Omega|\tau &\le \frac{\mu}{16}\|\bar{\varepsilon}^n\|_{-1}^2 \tau + \frac{1}{32}\|d_t \varepsilon^n\|_{-2}^2 \tau^2 + \frac{16}{\mu}\|A_{2h}^{-\frac{3}{2}} R_{0h} F_n\|_{0,\Omega}^2 \tau + 8\|A_{2h}^{-1} R_{0h} F_n\|_{0,\Omega}^2 \tau^2.
\end{aligned}
$$

Now we can combine the above inequalities with (6.2) for $\alpha = -2$ and derive

$$
\begin{aligned}
a_n - a_{n-1} + b_n \tau &+ \frac{\nu}{2}[\|e^n\|_{-1}^2 - \|e^{n-1}\|_{-1}^2]\tau + \frac{\mu}{2}[\|\varepsilon^n\|_{-1}^2 - \|\varepsilon^{n-1}\|_{-1}^2]\tau \\
&\le d_n a_n \tau + [\|d_t e^{n-1}\|_0^2 + \|d_t \varepsilon^{n-1}\|_0^2]\tau^4 + [\|d_t e^{n-1}\|_{-1}^2 + \|d_t \varepsilon^{n-1}\|_{-1}^2]\tau^3 \\
&\quad + \frac{16}{\nu}\|A_h^{-\frac{3}{2}} P_h E_n\|_{0,\Omega}^2 \tau + 8\|A_h^{-1} P_h E_n\|_{0,\Omega}^2 \tau^2 + \frac{16}{\mu}\|A_{2h}^{-\frac{3}{2}} R_{0h} F_n\|_{0,\Omega}^2 \tau + 8\|A_{2h}^{-1} R_{0h} F_n\|_{0,\Omega}^2 \tau^2 \quad (6.11)
\end{aligned}
$$

for all $2 \le n \le N$, where

$$a_n = \|e^n\|_{-2}^2 + (1+\mu\tau)\|\varepsilon^n\|_{-2}^2, \quad b_n = \nu\|\bar{e}^n\|_{-1}^2 + \mu\|\bar{\varepsilon}^n\|_{-1}^2.$$

We then sum up (6.11) over $n = 2, \dots, m$, use Lemmas 4.1 and 5.1 to achieve, for $1 \le m \le N$,

$$\begin{aligned} a_m + \tau \sum_{n=1}^{m} b_n &\le a_1 + b_1\tau + \tau \sum_{n=0}^{m} d_n a_n \\ &\quad + \tau \sum_{n=1}^{m} \Big[\frac{16}{\nu} \|A_h^{-\frac{3}{2}} P_h E_n\|_{0,\Omega}^2 + 8\|A_h^{-1} P_h E_n\|_{0,\Omega}^2 \tau \Big] \\ &\quad + \tau \sum_{n=1}^{m} \Big[\frac{16}{\mu} \|A_{2h}^{-\frac{3}{2}} R_{0h} F_n\|_{0,\Omega}^2 + 8\|A_{2h}^{-1} R_{0h} F_n\|_{0,\Omega}^2 \tau \Big] \\ &\le \tau \sum_{n=0}^{m} d_n a_n + \kappa_6 \tau^4. \end{aligned}$$

Noting that $d_n\tau \le \kappa_5\tau \le \frac{1}{2}$ from (6.9), we can apply Lemma 3.10 to get the desired estimate (6.10). □

Lemma 6.3. *Under the assumptions of Lemma 6.2, we have for $\alpha = -1, 0$ and $1 \le m \le N$ that*

$$\begin{aligned} &\sigma(t_m)\|e^m\|_\alpha^2 + \nu\sigma(t_m)\|e^m\|_{\alpha+1}^2\tau + \tau \sum_{n=1}^{m} \sigma(t_n)\Big(\frac{1}{2}\|d_t e^n\|_\alpha^2 \tau + \nu\|\bar{e}^n\|_{\alpha+1}^2\Big) \\ &\quad + \sigma(t_m)\|\varepsilon^m\|_\alpha^2 + \mu\sigma(t_m)\|\varepsilon^m\|_{\alpha+1}^2\tau + \tau \sum_{n=1}^{m} \sigma(t_n)\Big(\frac{1}{2}\|d_t \varepsilon^n\|_\alpha^2 \tau + \mu\|\bar{\varepsilon}^n\|_{\alpha+1}^2\Big) \le \kappa\tau^{3-\alpha}. \end{aligned} \tag{6.12}$$

Proof. We multiply (6.6) with $\alpha = -1, 0$ by $\sigma(t_n)$ and use $\sigma(t_n) \le \sigma(t_{n-1}) + \tau$ to get for $n \ge 2$ that

$$\begin{aligned} &\sigma(t_n)a_n - \sigma(t_{n-1})a_{n-1} + \sigma(t_n)b_n\tau \\ &\quad \le \frac{3}{2}a_{n-1}\tau + a_{n-2}\tau + \frac{1}{2}d_{n-1}\sigma(t_{n-1})a_{n-1}\tau + \frac{1}{2}d_{n-2}\sigma(t_{n-2})a_{n-2}\tau + c_n^\alpha\tau \\ &\qquad + \frac{16}{\nu}\sigma(t_n)\|A_h^{\frac{\alpha-1}{2}} P_h E_n\|_{0,\Omega}^2\tau + 8\sigma(t_n)\|A^{\frac{\alpha}{2}} P_h E_n\|_{0,\Omega}^2\tau^2 \\ &\qquad + \frac{16}{\mu}\sigma(t_n)\|A_{2h}^{\frac{\alpha-1}{2}} R_{0h} F_n\|_{0,\Omega}^2\tau + 8\sigma(t_n)\|A_{2h}^{\frac{\alpha}{2}} R_{0h} F_n\|_{0,\Omega}^2\tau^2. \end{aligned} \tag{6.13}$$

Then, summing up (6.13) over $n = 2, 3, \dots, m$ and using (6.9), Lemma 4.1 and Lemma 5.1, we further get

$$\begin{aligned} \sigma(t_m)a_m + \tau \sum_{n=1}^{m} \sigma(t_n)b_n &\le 2a_1\tau + b_1\tau^2 + \frac{5}{2}\tau \sum_{n=1}^{m-1} a_n + \tau \sum_{n=1}^{m-1} d_n\sigma(t_n)a_n + \tau \sum_{n=1}^{m} c_n^\alpha + \kappa\tau^{3-\alpha} \\ &\le \tau \sum_{n=1}^{m-1} a_n + \tau \sum_{n=1}^{m-1} d_n\sigma(t_n)a_n + \tau \sum_{n=1}^{m} c_n^\alpha + \kappa\tau^{3-\alpha} \end{aligned} \tag{6.14}$$

for $1 \le m \le N$. Since $d_n \le \kappa_3$, we can apply Lemma 3.9 to (6.14) to obtain

$$\sigma(t_m)a_m + \tau \sum_{n=1}^{m} \sigma(t_n)b_n \le e^{\kappa_3 T}\Big(\frac{5}{2}\tau \sum_{n=1}^{m-1} a_n + \tau \sum_{n=1}^{m} c_n^\alpha + \kappa\tau^{3-\alpha}\Big). \tag{6.15}$$

But, using Lemma 6.1, we have

$$\begin{aligned} \frac{5}{2}\tau \sum_{n=1}^{m-1} a_n &= \frac{5}{2}\tau \sum_{n=1}^{m-1} \Big[\|e^n\|_\alpha^2 + \frac{\nu}{2}\|e^n\|_{\alpha+1}^2\tau + (1+\mu\tau)\|\varepsilon^n\|_\alpha^2 + \frac{\mu}{2}\|\varepsilon^n\|_{\alpha+1}^2\tau\Big] \\ &\le 5\tau \sum_{n=1}^{m-1} \Big[\|\bar{e}^n\|_\alpha^2 + \frac{\nu}{2}\|\bar{e}^n\|_{\alpha+1}^2\tau + \|\bar{\varepsilon}^n\|_\alpha^2 + \frac{\mu}{2}\|\bar{\varepsilon}^n\|_{\alpha+1}^2\tau\Big] \\ &\quad + 5\tau^3 \sum_{n=1}^{m-1} \Big[\|d_t e^n\|_\alpha^2 + \frac{\nu}{2}\|d_t e^n\|_{\alpha+1}^2\tau + \|d_t\varepsilon^n\|_\alpha^2 + \frac{\mu}{2}\|d_t\varepsilon^n\|_{\alpha+1}^2\tau\Big] \le \kappa\tau^{3-\alpha}. \end{aligned} \tag{6.16}$$

Combining (6.15) with (6.16) and using (6.8), we derive the desired estimate (6.12). □

Lemma 6.4. *Under the assumptions of Lemma 6.2, we have, for all $1 \le m \le N$,*

$$\sigma^2(t_m)\|e^m\|_0^2 + \tau\nu \sum_{n=1}^{m} \sigma^2(t_n)\|\bar{e}^n\|_1^2 + \sigma^2(t_m)\|\varepsilon^m\|_0^2 + \mu\tau \sum_{n=1}^{m} \sigma^2(t_n)\|\bar{\varepsilon}^n\|_1^2 \le \kappa\tau^4.$$

Proof. Taking $\mathbf{v}_h = 2\bar{e}^n\tau \in \mathbf{V}_h$ and $q_h = 0$ in (5.3) and $\mathbf{C}_h = 2\bar{\varepsilon}^n\tau \in \mathbf{W}_{0h}$ in (5.4), then adding up the two equations, we obtain

$$\begin{aligned}
&\|e^n\|_0^2 - \|e^{n-1}\|_0^2 + 2\nu\|\bar{e}^n\|_1^2\tau + \|\varepsilon^n\|_0^2 - \|\varepsilon^{n-1}\|_0^2 + 2\mu\|\bar{\varepsilon}^n\|_1^2\tau\\
&\quad + 3b(e^{n-1}, \mathbf{u}_h(t_{n-1}), \bar{e}^n)\tau + 3b(\mathbf{u}_h(t_{n-1}), e^{n-1}, \bar{e}^n)\tau\\
&\quad - b(e^{n-2}, \mathbf{u}_h(t_{n-2}), \bar{e}^n)\tau - b(\mathbf{u}_h(t_{n-2}), e^{n-2}, \bar{e}^n)\tau\\
&\quad - 3b(e^{n-1}, e^{n-1}, \bar{e}^n)\tau + b(e^{n-2}, e^{n-2}, \bar{e}^n)\tau\\
&\quad + 3sd(\bar{e}^n, \varepsilon^{n-1}, \mathbf{B}_h(t_{n-1}))\tau + 3sd(\bar{e}^n, \mathbf{B}_h(t_{n-1}), \varepsilon^{n-1})\tau\\
&\quad - sd(\bar{e}^n, \varepsilon^{n-2}, \mathbf{B}_h(t_{n-2}))\tau - sd(\bar{e}^n, \mathbf{B}_h(t_{n-2}), \varepsilon^{n-2})\tau\\
&\quad - 3sd(\bar{e}^n, \varepsilon^{n-1}, \varepsilon^{n-1})\tau + sd(\bar{e}^n, \varepsilon^{n-2}, \varepsilon^{n-2})\tau\\
&\quad - 3d(e^{n-1}, \mathbf{B}_h(t_{n-1}), \bar{\varepsilon}^n)\tau - 3d(\mathbf{u}_h(t_{n-1}), \varepsilon^{n-1}, \bar{\varepsilon}^n)\tau\\
&\quad + d(e^{n-2}, \mathbf{B}_h(t_{n-2}), \bar{\varepsilon}^n)\tau + d(\mathbf{u}_h(t_{n-2}), \varepsilon^{n-2}, \bar{\varepsilon}^n)\tau\\
&\quad + 3d(e^{n-1}, \varepsilon^{n-1}, \bar{\varepsilon}^n)\tau - d(e^{n-2}, \varepsilon^{n-2}, \bar{\varepsilon}^n)\tau = 2(E_n, \bar{e}^n)_\Omega\tau + 2(F_n, \bar{\varepsilon}^n)_\Omega\tau. \qquad (6.17)
\end{aligned}$$

Then we can use (3.1)–(3.2) and Lemmas 3.2–3.3 to estimate all the terms in (6.17) as follows:

$$\begin{aligned}
3|b(e^{n-1}, \mathbf{u}_h(t_{n-1}), \bar{e}^n)|\tau + 3|b(\mathbf{u}_h(t_{n-1}), e^{n-1}, \bar{e}^n)|\tau &\le c_0\|e^{n-1}\|_0\|A_h\mathbf{u}_h(t_{n-1})\|_{0,\Omega}\|\bar{e}^n\|_1\tau\\
&\le \frac{\nu}{32}\|\bar{e}^n\|_1^2\tau + \frac{8}{\nu}c_0^2\|A_h\mathbf{u}_h(t_{n-1})\|_{0,\Omega}^2\|e^{n-1}\|_0^2\tau,\\
|b(e^{n-2}, \mathbf{u}_h(t_{n-2}), \bar{e}^n)|\tau + |b(\mathbf{u}_h(t_{n-2}), e^{n-2}, \bar{e}^n)|\tau &\le \frac{\nu}{32}\|\bar{e}^n\|_1^2\tau + \frac{8}{\nu}c_0^2\|A_h\mathbf{u}_h(t_{n-2})\|_{0,\Omega}^2\|e^{n-2}\|_0^2\tau,\\
3|b(e^{n-1}, e^{n-1}, \bar{e}^n)|\tau &\le \frac{\nu}{32}\|\bar{e}^n\|_1^2\tau + \frac{8}{\nu}c_0^2\|A_he^{n-1}\|_{0,\Omega}^2\|e^{n-1}\|_0^2\tau,\\
|b(e^{n-2}, e^{n-2}, \bar{e}^n)|\tau &\le \frac{\nu}{16}\|\bar{e}^n\|_1^2\tau + \frac{4}{\nu}c_0^2\|A_he^{n-2}\|_{0,\Omega}^2\|e^{n-2}\|_0^2\tau,\\
3s|d(\bar{e}^n, \mathbf{B}_h(t_{n-1}), \varepsilon^{n-1})|\tau + 3s|d(\bar{e}^n, \varepsilon^{n-1}, \mathbf{B}_h(t_{n-1}))|\tau &\le \frac{\nu}{32}\|\bar{e}^n\|_1^2\tau + \frac{8}{\nu}c_0^2\|A_{2h}\mathbf{B}_h(t_{n-1})\|_{0,\Omega}^2\|\varepsilon^{n-1}\|_0^2\tau,\\
s|d(\bar{e}^n, \mathbf{B}_h(t_{n-2}), \varepsilon^{n-2})|\tau + s|d(\bar{e}^n, \varepsilon^{n-2}, \mathbf{B}_h(t_{n-2}))|\tau &\le \frac{\nu}{32}\|\bar{e}^n\|_1^2\tau + \frac{8}{\nu}c_0^2\|A_{2h}\mathbf{B}_h(t_{n-2})\|_{0,\Omega}^2\|\varepsilon^{n-2}\|_0^2\tau,\\
3s|d(\bar{e}^n, \varepsilon^{n-1}, \varepsilon^{n-1})|\tau &\le \frac{\nu}{32}\|\bar{e}^n\|_1^2\tau + \frac{8}{\nu}c_0^2\|A_{2h}\varepsilon^{n-1}\|_{0,\Omega}^2\|\varepsilon^{n-1}\|_0^2\tau,\\
s|d(\bar{e}^n, \varepsilon^{n-2}, \varepsilon^{n-2})|\tau &\le \frac{\nu}{32}\|\bar{e}^n\|_1^2\tau + \frac{8}{\nu}c_0^2\|A_{2h}\varepsilon^{n-2}\|_{0,\Omega}^2\|\varepsilon^{n-2}\|_0^2\tau,\\
3|d(e^{n-1}, \mathbf{B}_h(t_{n-1}), \bar{\varepsilon}^n)|\tau + 3|d(\mathbf{u}_h(t_{n-1}), \varepsilon^{n-1}, \bar{\varepsilon}^n)|\tau &\le \frac{\mu}{16}\|\bar{\varepsilon}^n\|_1^2\tau + \frac{8}{\mu}c_0^2\|A_h\mathbf{u}_h(t_{n-1})\|_{0,\Omega}^2\|\varepsilon^{n-1}\|_0^2\tau\\
&\quad + \frac{8}{\mu}c_0^2\|A_{2h}\mathbf{B}_h(t_{n-1})\|_{0,\Omega}^2\|e^{n-1}\|_0^2\tau,\\
|d(e^{n-2}, \mathbf{B}_h(t_{n-2}), \bar{\varepsilon}^n)|\tau + |d(\mathbf{u}_h(t_{n-2}), \varepsilon^{n-2}, \bar{\varepsilon}^n)|\tau &\le \frac{\mu}{16}\|\bar{\varepsilon}^n\|_1^2\tau + \frac{8}{\mu}c_0^2\|A_h\mathbf{u}_h(t_{n-2})\|_{0,\Omega}^2\|\varepsilon^{n-1}\|_0^2\tau\\
&\quad + \frac{8}{\mu}c_0^2\|A_{2h}\mathbf{B}_h(t_{n-2})\|_{0,\Omega}^2\|e^{n-2}\|_0^2\tau,\\
3|d(e^{n-1}, \varepsilon^{n-1}, \bar{\varepsilon}^n)|\tau &\le \frac{\mu}{32}\|\bar{\varepsilon}^n\|_1^2\tau + \frac{8}{\mu}c_0^2\|A_he^{n-1}\|_{0,\Omega}^2\|\varepsilon^{n-1}\|_0^2\tau,\\
|d(e^{n-2}, \varepsilon^{n-2}, \bar{\varepsilon}^n)|\tau &\le \frac{\mu}{32}\|\bar{\varepsilon}^n\|_1^2\tau + \frac{8}{\mu}c_0^2\|A_he^{n-2}\|_{0,\Omega}^2\|\varepsilon^{n-2}\|_0^2\tau,
\end{aligned}$$

while the two terms on the right-hand side of (6.17) can be estimated by

$$\begin{aligned}
2|(E_n, \bar{e}^n)_\Omega|\tau &\le \frac{\nu}{16}\|\bar{e}^n\|_1^2\tau + \frac{16}{\nu}\|A_h^{-\frac{1}{2}}P_hE_n\|_{0,\Omega}^2\tau,\\
2|(F_n, \bar{\varepsilon})_\Omega| &\le \frac{\mu}{16}\|\bar{\varepsilon}^n\|_1^2\tau + 16\|A_{2h}^{-\frac{1}{2}}R_{0h}F_n\|_{0,\Omega}^2\tau.
\end{aligned}$$

Now we can combine all the above estimates to obtain from (6.17) that

$$a_n - a_{n-1} + \nu\|\bar{e}^n\|_1^2\tau + \mu\|\bar{\varepsilon}^n\|_1^2\tau \le 2\|\bar{\varepsilon}^n\|_0^2\tau + \frac{1}{2}d_{n-1}a_{n-1}\tau + \frac{1}{2}d_{n-2}a_{n-2}\tau + \frac{16}{\nu}\|A_h^{-\frac{1}{2}}P_hE_n\|_{0,\Omega}^2\tau + \frac{16}{\mu}\|A_{2h}^{-\frac{1}{2}}R_{0h}F_n\|_{0,\Omega}^2\tau \tag{6.18}$$

for all $2 \le n \le N$, where

$$\begin{aligned} a_n &:= \|e^n\|_0^2 + \|\varepsilon^n\|_0^2, \\ d_{n-1} &:= 16c_0^2(\nu^{-1} + \mu^{-1})(\|A_h\mathbf{u}_h(t_{n-1})\|_{0,\Omega}^2 + \|A_{2h}\mathbf{B}_h(t_{n-1})\|_{0,\Omega}^2) \\ &\quad + 16c_0^2(\nu^{-1} + \mu^{-1})(\|A_he^{n-1}\|_{0,\Omega}^2 + \|A_{2h}\varepsilon^{n-1}\|_{0,\Omega}^2). \end{aligned}$$

Multiplying both sides of (6.18) by $\sigma^2(t_n)$, we can derive

$$\begin{aligned} &\sigma^2(t_n)a_n - \sigma^2(t_{n-1})a_{n-1} + \nu\sigma^2(t_n)\|\bar{e}^n\|_1^2\tau + \mu\sigma^2(t_n)\|\bar{\varepsilon}^n\|_1^2\tau \\ &\quad \le 2\tau\sigma(t_{n-1})a_{n-1} + 2\sigma(t_n)\|\bar{\varepsilon}^n\|_0^2\tau + d_{n-1}\sigma(t_{n-1})a_{n-1}\tau + 2d_{n-2}\sigma(t_{n-2})a_{n-2}\tau \\ &\quad\quad + \sigma^2(t_n)\Big[\frac{16}{\nu}\|A_h^{-\frac{1}{2}}P_hE_n\|_{0,\Omega}^2\tau + \frac{16}{\mu}\|A_{2h}^{-\frac{1}{2}}R_{0h}F_n\|_{0,\Omega}^2\tau\Big] \end{aligned} \tag{6.19}$$

for all $2 \le n \le N$. Then the desired error estimates follow for $1 \le m \le N$ by summing up (6.19) from $n = 2$ to $n = m$, using Lemmas 4.1 and 5.1 and Lemmas 6.1–6.3:

$$\begin{aligned} \sigma^2(t_m)a_m + \tau\sum_{n=1}^{m}\sigma^2(t_n)(\nu\|\bar{e}^n\|_1^2 + \mu\|\bar{\varepsilon}^n\|_1^2) &\le \kappa\tau^4 + 3\tau\sum_{n=1}^{m-1}d_n\sigma(t_n)a_n \\ &\le \kappa\tau^4 + 6\tau\sum_{n=1}^{m-1}d_n\sigma(t_n)[\|\bar{e}^n\|_0^2 + \|\bar{\varepsilon}^n\|_0^2] \\ &\quad + 6\tau\sum_{n=1}^{m-1}d_n\sigma(t_n)[\|d_te^n\|_0^2\tau^2 + \|d_t\varepsilon^n\|_0^2\tau^2] \le \kappa\tau^4. \end{aligned}$$

□

With the previous estimates of the errors $e^n = \mathbf{u}_h(t_n) - \mathbf{u}_h^n$, $\varepsilon^n = \mathbf{B}_h(t_n) - \mathbf{B}_h^n$, we are now ready to establish the L^2-bound of the error $\eta^n = \bar{p}_h(t_n) - p_h^n$ for all $1 \le n \le N$.

It follows from Assumption (A3), (3.1)–(3.2) and Lemmas 4.1, 5.1 and 6.1–6.4 that

$$\|\eta^m\|_0 \le \kappa(\|d_te^m\|_0 + \|e^m\|_1 + \|e^{m-1}\|_1 + \|e^{m-2}\|_1) + \kappa(\|\varepsilon^{m-1}\|_1^2 + \|\varepsilon^{m-2}\|_1^2) + c\|P_{0h}E_n\|_{0,\Omega},$$

which yields

$$\begin{aligned} \sigma^2(t_m)\|\eta^m\|_0^2 &\le c\sigma^2(t_m)\|d_te^m\|_0^2 + c\sigma(t_m)\|e^m\|_1^2 + \kappa(\sigma(t_{m-1})\|e^{m-1}\|_1 + \sigma(t_{m-2})\|e^{m-2}\|_1) \\ &\quad + \kappa(\sigma(t_{m-1})\|\varepsilon^{m-1}\|_1^2 + \sigma(t_{m-2})\|\varepsilon^{m-2}\|_1^2) + c\sigma^2(t_m)\|P_{0h}E_m\|_{0,\Omega}^2 \le \kappa\tau^2. \end{aligned} \tag{6.20}$$

Now it follows readily from (6.20) and (4.14) that, for all $2 \le m \le N$ and some $\theta_m \in [t_{m-1}, t_m]$,

$$\begin{aligned} \sigma^2(t_m)\|p_h(t_m) - p_h^m\|_0^2 &= \sigma^2(t_m)\Big\|\frac{1}{2}(p_h(t_m) - p_h(t_{m-1})) + \eta^m\Big\|_{0,\Omega}^2 \\ &\le 2\sigma^2(t_{m-1})\|p_{ht}(\theta_m)\|_{0,\Omega}^2\tau^2 + 2\sigma^2(t_m)\|\eta^m\|_{0,\Omega}^2 \\ &\le 2\sigma^2(\theta_m)\|p_{ht}(\theta_m)\|_{0,\Omega}^2\tau^2 + \kappa\tau^2 \le \kappa\tau^2. \end{aligned} \tag{6.21}$$

In summary, we can now conclude the following error estimates by combining (6.20)–(6.21) with Lemmas 6.3–6.4 and using Theorem 2.1 and Lemma 4.1.

Theorem 6.1. *Under Assumptions* (A1)–(A3) *and for τ satisfying* (6.9), *the finite element solution* $(\mathbf{u}_h^n, p_h^n, \mathbf{B}_h^n)$ *to scheme* (4.3)–(4.4) *has the following error estimates for all* $t_m \in (0, T]$*:*

$$\begin{aligned} \|\mathbf{u}(t_m) - \mathbf{u}_h^m\|_{0,\Omega} + \|\mathbf{B}(t_m) - \mathbf{B}_h^m\|_{0,\Omega} &\le \kappa(\sigma^{-\frac{1}{2}}(t_m)h^3 + \sigma^{-1}(t_m)\tau^2), \\ \|\nabla(\mathbf{u}(t_m) - \mathbf{u}_h^m)\|_{0,\Omega} + \|\nabla(\mathbf{B}(t_m) - \mathbf{B}_h^m)\|_{0,\Omega} &\le \kappa\sigma^{-\frac{1}{2}}(t_m)(h^2 + \tau), \\ \|p(t_m) - p_h^m\|_{0,\Omega} &\le \kappa\sigma^{-1}(t_m)(h^2 + \tau). \end{aligned}$$

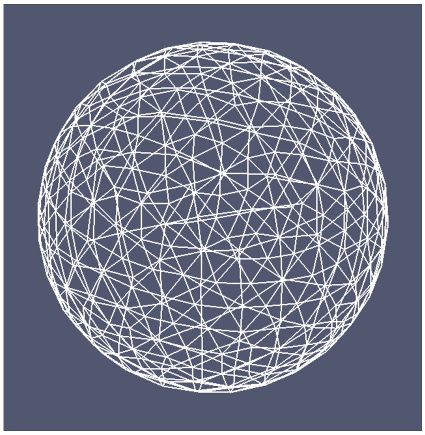


Figure 1: A sample mesh of $h = \frac{1}{8}$.

7 Numerical Experiments

In this section, we present some numerical experiments to verify the convergence orders of scheme (4.3)–(4.4). We take a spherical domain of radius $\frac{1}{2}$ with the center at $(0, 0, 0)$. The conforming and shape-regular triangulation with tetrahedral elements is generated by the software Gmsh [8]; see Figure 1 for a sample mesh with $h = \frac{1}{8}$.

For solving system (4.3), we may change the sign of the term $(\nabla \cdot \mathbf{u}_h^n, q_h)_\Omega$ to see this system is symmetric. Then we can apply the MINRES method to solve it, preconditioned by the diagonal block operator

$$\mathcal{P}_{NS} = \begin{pmatrix} \frac{1}{\tau} I - \nu\Delta & 0 \\ 0 & I \end{pmatrix}^{-1}.$$

Equation (4.4) is symmetric positive definite, so we can apply the conjugate gradient (CG) method to solve it. The two outer iterations terminate when the relative residual reaches 10^{-10}. All experiments are implemented using FEniCS software [22]. We set physical parameters $\nu = s = \mu = 1$ and apply the P_2, P_1 and P_2 finite elements to discretize the velocity $\mathbf{u}$, pressure p and magnetic field $\mathbf{B}$, respectively.

7.1 Verification of Temporal Accuracies

In this subsection, we check the temporal convergence orders of scheme (4.3)–(4.4). The right-hand sides and the boundary conditions of the MHD equations (1.1) are chosen such that the true solution is given by

$$\mathbf{B} = (y\cos(t), z\sin(t), xe^{-t}), \quad p = 0, \quad \mathbf{u} = (ye^{-t}, z\cos(t), x\sin(t)). \tag{7.1}$$

Note that this solution (7.1) is linear in space direction; hence the computational errors mainly come from the time discretization. We fix the mesh size $h = \frac{1}{16}$ and refine the time step size τ to run the scheme. The errors and convergence orders at $t = 2$ are illustrated in Table 1, where e_w is given by $e_w = w(t_n) - w_h^n$ for any function w at recorded time t_n. Table 1 demonstrates that scheme (4.3)–(4.4) has the second-order temporal accuracy. In addition, we have also plotted the exact solution (7.1) and the numerical solution in Figure 2. We see the computational solution matches the exact solution very well.

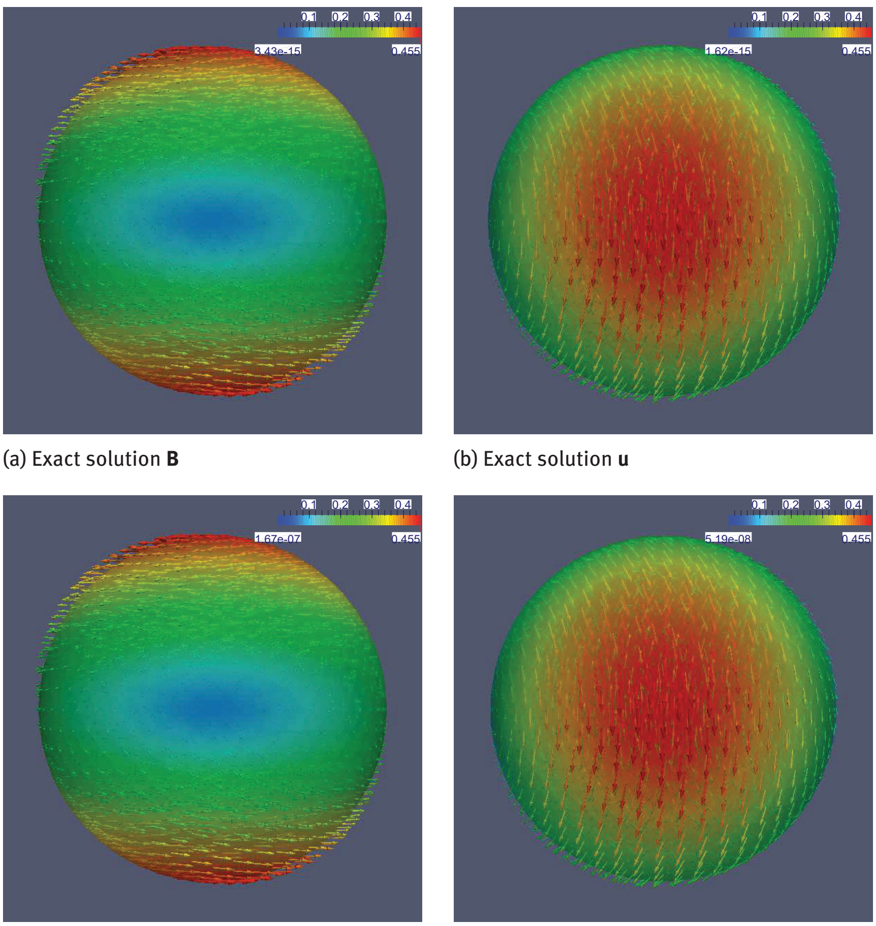
7.2 Verification of Spatial Accuracies

In this subsection, we check the spatial approximation orders of scheme (4.3)–(4.4). To this end, we choose the exact solution to the MHD equations (1.1) as follows:

$$\mathbf{B} = (\cos(y+t), \cos(z+t), \cos(x+t)), \quad p = e^{-t}xyz, \quad \mathbf{u} = ((y^2+z^2)e^{-t}, (x^2+z^2)\cos(t), (x^2+y^2)\sin(t)). \tag{7.2}$$

τ	$\|e_u\|_{L^2}$	Order	$\|e_u\|_{H^1}$	Order	$\|e_p\|_{L^2}$	Order	$\|e_b\|_{L^2}$	Order	$\|e_b\|_{H^1}$	Order
$\frac{1}{4}$	2.099e−4	—	2.559e−3	—	1.714e−3	—	3.857e−4	—	4.469e−3	—
$\frac{1}{8}$	2.134e−5	3.29	4.523e−4	2.50	4.111e−4	2.05	4.027e−5	3.25	7.574e−4	2.56
$\frac{1}{16}$	3.869e−6	2.46	7.946e−5	2.50	9.885e−5	2.05	7.415e−6	2.44	1.286e−4	2.55
$\frac{1}{32}$	9.295e−7	2.05	1.312e−5	2.59	2.416e−5	2.03	1.746e−6	2.08	2.165e−5	2.57
$\frac{1}{64}$	2.345e−7	1.99	2.393e−6	2.45	5.967e−6	2.01	4.320e−7	2.01	4.368e−6	2.30
$\frac{1}{128}$	5.911e−8	1.99	5.664e−7	2.07	1.482e−6	2.00	1.076e−7	2.00	1.032e−6	2.08
$\frac{1}{256}$	1.495e−8	1.98	1.420e−7	2.00	3.736e−7	1.99	2.689e−8	2.00	2.565e−7	2.00

Table 1: Errors and orders of $\|e_u\|_{L^2}$, $\|e_u\|_{H^1}$, $\|e_p\|_{L^2}$, $\|e_b\|_{L^2}$ and $\|e_b\|_{H^1}$ with the fixed grid size $h = \frac{1}{16}$.



(a) Exact solution **B**

(b) Exact solution **u**

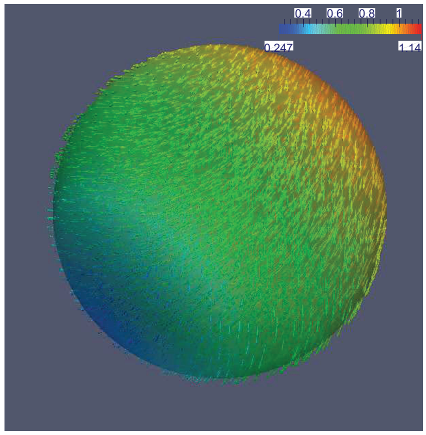
(c) Numerical solution **B**

(d) Numerical solution **u**

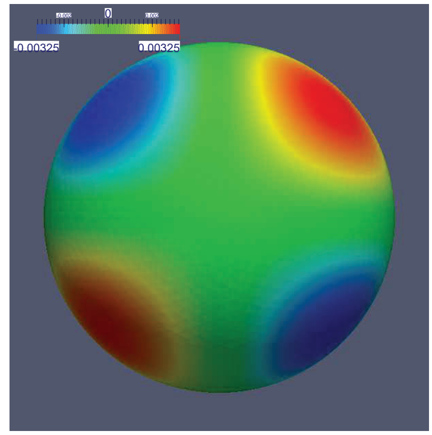
Figure 2: Comparisons of the exact solution (7.1) and the computational solution at $t = 2$.

We set $\tau = h$ and refine both parameters simultaneously to get the approximation errors and convergence orders. The recorded data at $t = 2.0$ are presented in Table 2. We see from Table 2 that the convergence orders of $\|e_u\|_{H^1}$, $\|e_p\|_{L^2}$, $\|e_b\|_{L^2}$ and $\|e_b\|_{H^1}$ have the second-order accuracies, while the error order of $\|e_u\|_{L^2}$ seems to yield a higher-order accuracy for this experiment. In addition, we also plot the exact solution and the numerical solution at $t = 2$ in Figure 3, from which we can see that the numerical solution captures the exact solution well.

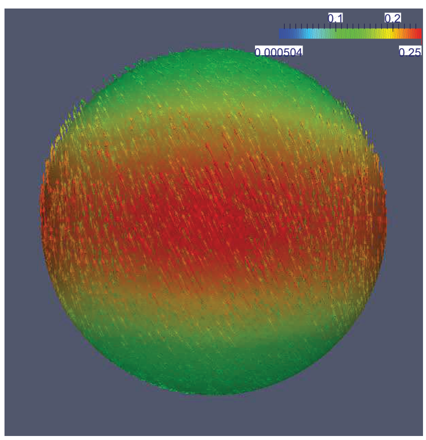
Furthermore, we have also checked the convergence orders of $\|e_u\|_{L^2}$ and $\|e_b\|_{L^2}$. To this end, we set $\tau = h^{\frac{3}{2}}$ to implement the fully discrete scheme (4.3)–(4.4). The corresponding numerical errors and convergence orders of $\|e_u\|_{L^2}$ and $\|e_b\|_{L^2}$ at $t = 2$ are displayed in Table 3, from which we can see that $\|e_u\|_{L^2}$ and $\|e_b\|_{L^2}$ achieve approximately the asymptotical third-order accuracies.



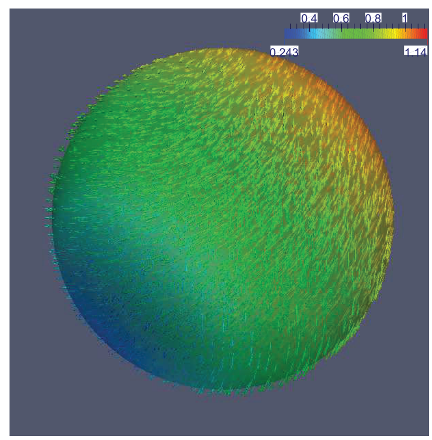
(a) Exact solution **B**



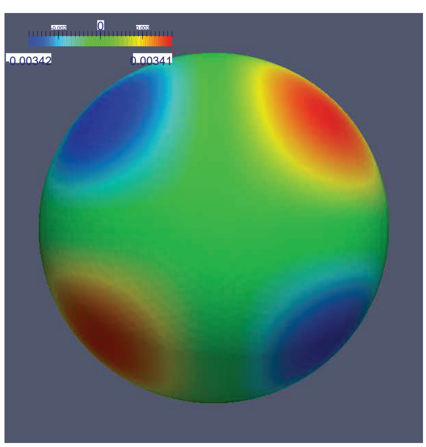
(b) Exact solution p



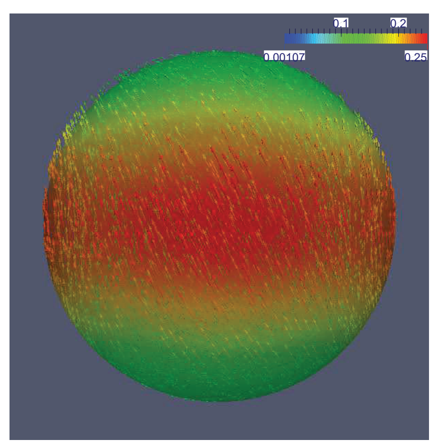
(c) Exact solution **u**



(d) Numerical solution **B**



(e) Numerical solution p



(f) Numerical solution **u**

Figure 3: Comparisons of the exact solution (7.2) and our computational solution at $t = 2$.

h	τ	$\|e_u\|_{L^2}$	Order	$\|e_u\|_{H^1}$	Order	$\|e_p\|_{L^2}$	Order	$\|e_b\|_{L^2}$	Order	$\|e_b\|_{H^1}$	Order
$\frac{1}{2}$	$\frac{1}{2}$	4.250e−4	—	6.476e−3	—	1.166e−2	—	2.929e−3	—	2.647e−2	—
$\frac{1}{4}$	$\frac{1}{4}$	3.566e−4	0.25	5.028e−3	0.36	3.805e−3	1.61	5.434e−4	2.43	6.492e−3	2.02
$\frac{1}{8}$	$\frac{1}{8}$	4.476e−5	2.99	1.243e−3	2.01	9.169e−4	2.05	8.763e−5	2.63	1.592e−3	2.02
$\frac{1}{16}$	$\frac{1}{16}$	4.583e−6	3.28	2.661e−4	2.22	2.118e−4	2.11	1.878e−5	2.22	3.699e−4	2.10
$\frac{1}{32}$	$\frac{1}{32}$	4.605e−7	3.31	5.253e−5	2.34	5.073e−5	2.06	4.570e−6	2.03	8.349e−5	2.14

Table 2: Errors and orders of $\|e_u\|_{L^2}$, $\|e_u\|_{H^1}$, $\|e_p\|_{L^2}$, $\|e_b\|_{L^2}$ and $\|e_b\|_{H^1}$ at $t = 2$ that are computed using $\tau = h$.

h	τ	$\|e_u\|_{L^2}$	Order	$\|e_b\|_{L^2}$	Order
$\frac{1}{2}$	$\frac{1}{3}$	3.54195e−4	—	1.15922e−3	—
$\frac{1}{4}$	$\frac{1}{8}$	4.63818e−5	2.93	1.00930e−4	3.52
$\frac{1}{8}$	$\frac{1}{23}$	1.19882e−6	5.27	1.55159e−5	2.70
$\frac{1}{16}$	$\frac{1}{64}$	8.32105e−8	3.84	2.23924e−6	2.79
$\frac{1}{32}$	$\frac{1}{181}$	1.06173e−8	2.97	2.61821e−7	3.09

Table 3: Errors and orders of $\|e_u\|_{L^2}$ and $\|e_b\|_{L^2}$ using $\tau = h^{\frac{3}{2}}$.

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