Solution 8

Section 8.1

1. Since we are only concerned about pointwise convergence here for $x \geq 0$, let $x \geq 0$ be fixed. Then as $n \to \infty$, we have
$$\lim_{n\to\infty} x = x$$
which is a finite real number, and
$$\lim_{n\to\infty}(x+n) = +\infty.$$
Hence
$$\lim_{n\to\infty} \frac{x}{x+n} = 0,$$
as desired.

2. Let $x \geq 0$ be fixed. Then for all $n \in \mathbb{N}$, we have
$$e^{nx} \geq 1 + nx > 0.$$
(Why? Either show this by differentiation, or use the power series expansion
$$e^{nx} = 1 + nx + \left(\frac{(nx)^2}{2!} + \frac{(nx)^3}{3!} + \dots\right)$$
where the bracketed terms are non-negative since $nx \geq 0$.) This says
$$0 \leq xe^{-nx} \leq \frac{x}{1+nx}$$
for all $n \in \mathbb{N}$. Since
$$\lim_{n\to\infty} \frac{x}{1+nx} = 0,$$
by Sandwich theorem, we see that
$$\lim_{n\to\infty} xe^{-nx}$$
exists and is equal to 0.

8. Let $x \geq 0$ be fixed. Then for all $n \in \mathbb{N}$, we have
$$e^{nx} \geq 1 + nx > 0.$$
(Why? Either show this by differentiation, or use the power series expansion
$$e^{nx} = 1 + nx + \left(\frac{(nx)^2}{2!} + \frac{(nx)^3}{3!} + \dots\right)$$
where the bracketed terms are non-negative since $nx \geq 0$.) This says
$$0 \leq xe^{-nx} \leq \frac{x}{1+nx}$$
for all $n \in \mathbb{N}$. Since
$$\lim_{n\to\infty} \frac{x}{1+nx} = 0,$$
by Sandwich theorem, we see that
$$\lim_{n\to\infty} xe^{-nx}$$
exists and is equal to 0.

11. Define the functions f_n and f on $[0,\infty)$ by $f_n(x) = \frac{x}{x+n}$ and $f(x) = 0$. We already know from Question 1 that f_n converges pointwisely to f on $[0,\infty)$.

    Suppose first $a > 0$ is given. To prove the uniform convergence of f_n to f on $[0,a]$, simply observe that the sup-norm of $f_n - f$ over $[0,a]$ is

    $$\left\| \frac{x}{x+n} - 0 \right\|_{[0,a]} = \frac{1}{1+\frac{n}{a}},$$

    which tends to 0 as $n \to \infty$. Hence f_n converges to f uniformly on $[0,a]$.

    Next, to show that f_n does not converge to f uniformly on $[0,\infty)$, note that

    $$\|f_n - f\|_{[0,\infty)} = \sup_{x\in[0,\infty)} \frac{x}{x+n} = 1,$$

    the last equality following since

    $$0 \le \frac{x}{x+n} \le 1$$

    for all $x \in [0,\infty)$, and

    $$\lim_{x\to\infty} \frac{x}{x+n} = 1.$$

    Hence $\|f_n - f\|_{[0,\infty)}$ does not tend to 0 as $n \to \infty$ (in fact it tends to 1 instead), and from this we see that the convergence of f_n to f is not uniform on $[0,\infty)$.

12. Define the functions f_n and f on $[0,\infty)$ by $f_n(x) = \frac{nx}{1+n^2x^2}$ and $f(x) = 0$. We already know from Question 2 that f_n converges pointwisely to f on $[0,\infty)$.

    To prove that this convergence is uniform on $[a,\infty)$ for all $a > 0$, let $a > 0$ be given. Then the sup-norm of $f_n - f$ over $[a,\infty)$ is

    $$\left\| \frac{nx}{1+n^2x^2} - 0 \right\|_{[a,\infty)} \le \left\| \frac{1}{nx} \right\|_{[a,\infty)} = \frac{1}{na},$$

    which tends to 0 as $n \to \infty$. Hence f_n converges to f uniformly on $[a,\infty)$.

    To prove that the convergence of f_n to f is not uniform on $[0,\infty)$, note that for all $n \in \mathbb{N}$, we have

    $$\|f_n - f\|_{[0,\infty)} = \sup_{x\in[0,\infty)} |f_n(x)| \ge f_n\left(\frac{1}{n}\right) = \frac{1}{2}.$$

    Hence $\|f_n - f\|_{[0,\infty)}$ does not tend to 0 as $n \to \infty$. This shows f_n does not converge uniformly to f on $[0,\infty)$.

18. Define the functions f_n and f on $[0,\infty)$ by $f_n(x) = xe^{-nx}$ and $f(x) = 0$. We already know from Question 8 that f_n converges pointwisely to f on $[0,\infty)$. To prove that this convergence is uniform on $[0,\infty)$, observe that the sup-norm of $f_n - f$ over $[0,\infty)$ is

    $$\left\| xe^{-nx} - 0 \right\|_{[0,\infty)} = \frac{1}{n}e^{-1},$$

    because the maximum value of xe^{-nx} on $[0,\infty)$ is attained when $x = 1/n$. (Check by differentiation!) Since

    $$\lim_{n\to\infty} \|f_n - f\|_{[0,\infty)} = \lim_{n\to\infty} \frac{1}{n}e^{-1} = 0,$$

    we see that f_n converges to f uniformly on $[0,\infty)$.

22. Recall $f_n(x) = x + \frac{1}{n}$ and $f(x) = x$ for all $x \in \mathbb{R}$, $n \in \mathbb{N}$. Thus

$$\|f_n - f\|_{\mathbb{R}} = \sup_{x\in\mathbb{R}} \left| \left(x + \frac{1}{n} \right) - x \right| = \frac{1}{n},$$

which tends to 0 as $n \to \infty$. So f_n converges to f uniformly on $\mathbb{R}$.

On the other hand,

$$\|f_n^2 - f^2\|_{\mathbb{R}} = \sup_{x\in\mathbb{R}} \left| \left(x + \frac{1}{n} \right)^2 - x^2 \right| = \sup_{x\in\mathbb{R}} \left| \frac{2x}{n} + \frac{1}{n}^2 \right| = +\infty$$

for all $n \in \mathbb{N}$. So f_n^2 does not converge uniformly to f^2 on $\mathbb{R}$.

Remark. Hence if f_n converges uniformly to f on a set A, and g_n converges uniformly to g on the same set A, it is not necessarily true that $f_n g_n$ converges uniformly to fg. The problem is that the functions involved could be unbounded. If we work only with bounded functions, then the product will still uniformly converge, as we see in the next question.

23. Suppose f_n and g_n are sequences of bounded functions that converge uniformly on A to f and g respectively.

First we claim that there exists a constant $M_1 \in \mathbb{R}$, such that

$$\|f_n\|_A \leq M_1 \quad \text{for all } n \in \mathbb{N}:$$

This is because using Cauchy's criterion, taking $\varepsilon = 1$, we see that there exists $N \in \mathbb{N}$, such that

$$\|f_n - f_m\|_A \leq 1 \quad \text{for all } n, m \geq N,$$

which implies that

$$\|f_n\|_A \leq \|f_N\|_A + 1 \quad \text{for all } n \geq N.$$

Hence taking $M_1 = \max\{\|f_1\|_A, \dots, \|f_{N-1}\|_A, \|f_N\|_A + 1\}$, we get the desired claim. (Or else: since the triangle inequality implies

$$|\|f_n\|_A - \|f_m\|_A| \leq \|f_n - f_m\|_A,$$

which shows that $\{\|f_n\|_A\}$ is a Cauchy sequence of real numbers, we see that $\{\|f_n\|_A\}$ is a convergent sequence of real numbers, which in particular implies that it is a bounded sequence of real numbers.) (We sometimes say that f_n is a sequence of uniformly bounded functions.)

Similarly, there exists a number $M_2 \in \mathbb{R}$, such that

$$\|g_n\|_A \leq M_2 \quad \text{for all } n \in \mathbb{N}.$$

It follows that

$$\begin{aligned} \|f_n g_n - fg\|_A &= \|f_n(g_n - g) + (f_n - f)g\|_A \\ &\leq \|f_n\|_A \|g_n - g\|_A + \|f_n - f\|_A \|g\|_A \\ &\leq M_1 \|g_n - g\|_A + M_2 \|f_n - f\|_A \\ &\to 0 \end{aligned}$$

as $n \to \infty$, by uniform convergence of f_n and g_n to f and g respectively. Hence $f_n g_n$ converges uniformly to fg on A.

24. Since g is continuous on $[-M, M]$, we see that g is uniformly continuous on $[-M, M]$. Hence given $\varepsilon > 0$, there exists $\delta > 0$, such that whenever $y_1, y_2 \in [-M, M]$ with $|y_1 - y_2| \leq \delta$, we have
$$|g(y_1) - g(y_2)| < \varepsilon.$$
Now f_n converges uniformly to f on A, and $|f_n(x)| \leq M$ for all $x \in A$ and $n \in \mathbb{N}$. Thus $|f(x)| \leq M$ for all $x \in A$ as well. In addition, uniform convergence of f_n to f implies that there exists $N \in \mathbb{N}$, such that for all $n \geq N$, we have $\|f_n(x) - f(x)\|_A \leq \delta$, i.e.
$$|f_n(x) - f(x)| \leq \delta \quad \text{for all } x \in A.$$
Hence using the earlier inequality for g with $y_1 = f_n(x)$, $y_2 = f(x)$, we get
$$|g(f_n(x)) - g(f(x))| < \varepsilon \quad \text{for all } x \in A \text{ and all } n \in \mathbb{N}.$$
Hence
$$\|g \circ f_n - g \circ f\|_A < \varepsilon,$$
and since this is true for all $n \geq N$, we see that $g \circ f_n$ converges uniformly to $g \circ f$ on A.

Supplementary Exercises

1. (a) Just note that
$$\lim_{n\to\infty} \frac{\cos 5x}{n + x^2} = 0$$
for all $x \in \mathbb{R}$, by sandwich theorem. Also,
$$\left\| \frac{\cos 5x}{n + x^2} \right\|_{\mathbb{R}} = \sup_{x\in\mathbb{R}} \frac{|\cos 5x|}{n + x^2} \leq \sup_{x\in\mathbb{R}} \frac{1}{n + x^2} = \frac{1}{n},$$
which tends to 0 as $n \to \infty$. Thus $\frac{\cos 5x}{n+x^2}$ converges uniformly to 0 on $\mathbb{R}$.

(b) Fix x. Then $0 \leq \cos^2 \pi x \leq 1$. If $\cos^2 \pi x = 1$, then $\cos^{2n} \pi x = 1 \;\Rightarrow\; \lim \cos^{2n} \pi x = 1$.

Otherwise, $\lim \cos^{2n} \pi x = 0$.

Hence $\lim \cos^{2n} \pi x = \begin{cases} 1, & \text{if } \cos^2 \pi x = 1 \\ 0, & \text{o.w.} \end{cases} = \begin{cases} 1, & x \in \mathbb{Z} \\ 0, & x \notin \mathbb{Z} \end{cases} =: f(x),$

since $\cos^2 \pi x = 1 \;\Leftrightarrow\; \pi x = k\pi, \text{ for some } k \in \mathbb{Z} \;\Leftrightarrow\; x \in \mathbb{Z}$.

Now for any positive integer n, we have
$$\left\|\cos^{2n} \pi x - f(x)\right\|_{\mathbb{R}} \geq \left\|\cos^{2n} \pi x - 0\right\|_{(0,1)} \geq \lim_{x\to 0^+} \cos^{2n} \pi x = 1,$$
which does not converge to 0 as $n \to \infty$. So $(\cos^{2n} \pi x)$ does not converge to f uniformly on $\mathbb{R}$ as $n \to \infty$.

(c) Now $\lim \dfrac{|x|}{n^2 + x^2} = 0$, we have $\left\| \dfrac{|x|}{n^2 + x^2 - 0} \right\|_{\mathbb{R}} = \sup_{x\in\mathbb{R}} \left| \dfrac{|x|}{n^2 + x^2} \right| = \sup_{x\in[0,+\infty)} \dfrac{x}{n^2 + x^2}.$

Hence we have
$$\begin{aligned} \left\| \frac{|x|}{n^2 + x^2} - 0 \right\|_{\mathbb{R}} &= \sup_{x\in[0,+\infty)} \frac{x}{n^2 + x^2} \leq \sup_{x\in[0,n)} \frac{x}{n^2 + x^2} + \sup_{x\in[n,+\infty)} \frac{x}{n^2 + x^2} \\ &\leq \sup_{x\in[0,n)} \frac{x}{n^2} + \sup_{x\in[n,+\infty)} \frac{x}{x^2} \leq \frac{2}{n} \to 0 \end{aligned}$$

Hence, the sequence $\left(\dfrac{|x|}{n^2 + x^2} \right)$ converges uniformly to 0 on $\mathbb{R}$.