Solution 7

Supplementary Exercises

1 Let
$f(x)=$Thomae's function, which is integrable.
$g(x)=\begin{cases} 1 & , \quad x\neq 0 \\ 0 & , \quad x=0 \end{cases}$ g is integrable.
Then
$(g\circ f)(x)=\begin{cases} 1 & , \quad x\in\mathbb{Q} \\ 0 & , \quad x\in[0,1]\backslash\mathbb{Q} \end{cases}$
$g\circ f$ is not integrable.

2 (a) When $x\geq 0$, $\int |x|dx=\int xdx=\frac{x^2}{2}+C$.
When $x<0$, $\int |x|dx=-\int xdx=-\frac{x^2}{2}+C$.
Hence,
$\int |x|dx=\begin{cases} \frac{x^2}{2}+C & , \quad x\neq 0 \\ -\frac{x^2}{2}+C & , \quad x<0 \end{cases}$

(b) When $x\geq 0$, $\int x|x|dx=\int x^2dx=\frac{x^3}{3}+C$.
When $x<0$, $\int x|x|dx=-\int x^2dx=-\frac{x^3}{3}+C$.
Hence,
$\int x|x|dx=\begin{cases} \frac{x^3}{2}+C & , \quad x\neq 0 \\ -\frac{x^3}{2}+C & , \quad x<0 \end{cases}$

(c) When $x\geq 0$, $\int |\sin x|dx=\int \sin xdx=-\cos x+C$.
When $x<0$, $\int |\sin x|dx=-\int \sin xdx=\cos x+C$.
Hence,
$\int x|x|dx=\begin{cases} -\cos x+C & , \quad x\neq 0 \\ \cos x+C & , \quad x<0 \end{cases}$

3. (a) $\dfrac{1}{n+1}+\dfrac{1}{n+2}+\cdots+\dfrac{1}{n+n}=\dfrac{1}{n}\sum_{j=1}^{n}\dfrac{n}{n+j}=\dfrac{1}{n}\sum_{j=1}^{n}\dfrac{1}{1+j/n}$

Let $f(x)=\dfrac{1}{1+x}$, $x\in[0,1]$. Since $f\in C[0,1]$, $f\in\mathcal{R}[0,1]$.

Take $\mathcal{P}_n=\left\{\dfrac{j}{n}\right\}_{j=0}^{n}$, hence $\parallel\mathcal{P}_n\parallel\to 0$. Then

$$\begin{aligned}\lim\left(\frac{1}{n+1}+\frac{1}{n+2}+\cdots+\frac{1}{n+n}\right)&=\lim\frac{1}{n}\sum_{j=1}^{n}\frac{1}{1+j/n}=\lim\underline{S}(f,\mathcal{P}_n)\\&=\int_0^1\frac{dx}{1+x}=\log(1+x)|_0^1=\log 2\end{aligned}$$

(b) $\ln\dfrac{(n!)^{\frac{1}{n}}}{n}=\dfrac{1}{n}\ln n!-\ln n=\dfrac{1}{n}\sum_{k=1}^{n}(\ln k-\ln n)=\dfrac{1}{n}\sum_{k=1}^{n}\ln\dfrac{k}{n}$

Note that the improper integral $\displaystyle\int_0^1\ln xdx$ is convergent and equal to -1.

Now consider the difference

$$\ln \frac{(n!)^{\frac{1}{n}}}{n} - \int_0^1 \ln x dx = \frac{1}{n}\sum_{k=1}^{n} \ln \frac{k}{n} - \int_0^1 \ln x dx.$$

We claim that it tends to zero as n tends to infinity.
But this is equal to

$$\sum_{k=1}^{n} \int_{\frac{k}{n}}^{\frac{k+1}{n}} \left(\ln \frac{k}{n} - \ln x\right) dx - \int_0^{\frac{1}{n}} \ln x dx + \int_1^{\frac{n+1}{n}} \ln x dx,$$

the last two terms here obviously tends to 0 as n tends to infinity.
For the first term, by the mean-value theorem,

$$\left|\ln \frac{k}{n} - \ln x\right| \le \frac{1}{n}\left(\frac{1}{k/n}\right) = \frac{1}{k}.$$

So

$$\left|\sum_{k=1}^{n} \int_{\frac{k}{n}}^{\frac{k+1}{n}} \left(\ln \frac{k}{n} - \ln x\right) dx\right| \le \sum_{k=1}^{n} \int_{\frac{k}{n}}^{\frac{k+1}{n}} \frac{1}{k} dx$$

$$= \frac{1}{n}\left(1 + \sum_{k=2}^{n} \frac{1}{k}\right) \le \frac{1}{n}\left(1 + \int_1^n \frac{1}{x} dx\right) = \frac{1}{n}(1 + \ln n).$$

Since $\lim_n \frac{1}{n}(1 + \ln n) = 0$, we get the desired claim.
Hence

$$\lim_n \frac{(n!)^{\frac{1}{n}}}{n} = \exp\left(\lim_n \ln \frac{(n!)^{\frac{1}{n}}}{n}\right) = \exp\left(\int_0^1 \ln x dx\right) = \exp(-1).$$

4.

$$\begin{aligned}
& \int_0^{\pi} \frac{x\sin x}{1+\cos^2 x} dx \qquad \text{Let } t = x - \pi \\
= & \int_0^{-\pi} \frac{(t+\pi)\sin t}{1+\cos^2 t} dt \qquad \text{Let } x = -t \\
= & -\int_0^{\pi} \frac{x\sin x}{1+\cos^2 x} dx + \int_0^{-\pi} \frac{\pi \sin t}{1+\cos^2 t} dt \\
= & -\int_0^{\pi} \frac{x\sin x}{1+\cos^2 x} dx - \pi \int_0^{-\pi} \frac{1}{1+\cos^2 t} d\cos t \\
= & -\int_0^{\pi} \frac{x\sin x}{1+\cos^2 x} dx - \pi \int_0^{-\pi} d\arctan\cos t \\
= & -\int_0^{\pi} \frac{x\sin x}{1+\cos^2 x} dx + \frac{\pi^2}{2}.
\end{aligned}$$

Hence,

$$\int_0^{\pi} \frac{x\sin x}{1+\cos^2 x} dx = \frac{\pi^2}{4}.$$

5. (a) The improper integral $\int_0^\infty e^{-x}dx$ converges, since

$$\lim_{c\to\infty}\int_0^c e^{-x}dx = \lim_{c\to\infty}[1-e^{-c}]$$

exists and equals 1. Hence the value of the improper integral is $\int_0^\infty e^{-x}dx = 1$.

(b) The improper integral $\int_0^\infty xe^{-x}dx$ converges, since (integrating by parts)

$$\int_0^c xe^{-x}dx = 1-(1+c)e^{-c},$$

which converges to 1 as $c \to +\infty$. Hence the value of the improper integral is given by $\int_0^\infty xe^{-x}dx = 1$.

(c) The improper integral $\int_1^\infty \frac{1}{\sqrt{x}}dx$ diverges, since

$$\int_1^c \frac{1}{\sqrt{x}}dx = 2\sqrt{c}-2,$$

which diverges as $c \to +\infty$.

(d) The improper integral $\int_0^1 \frac{1}{\sqrt{x}}dx$ converges, since

$$\int_c^1 \frac{1}{\sqrt{x}}dx = 2-2\sqrt{c},$$

which converges to 2 as $c \to 0^+$. Hence the value of the improper integral is given by $\int_0^1 \frac{1}{\sqrt{x}}dx = 2$.

(e) The improper integral $\int_0^1 \frac{1}{x}dx$ diverges, since

$$\int_c^1 \frac{1}{x}dx = \ln 1 - \ln c = -\ln c,$$

which diverges as $c \to +\infty$.

(f) The improper integral $\int_0^{1/e} \frac{1}{x(\ln x)^2}dx$ converges, since (by substitution)

$$\int_c^{1/e} \frac{1}{x(\ln x)^2}dx = -\frac{1}{\ln\frac{1}{e}} + \frac{1}{\ln c} = 1 + \frac{1}{\ln c},$$

which converges to 1 as $c \to 0^+$. Hence the value of the improper integral is given by $\int_0^{1/e} \frac{1}{x(\ln x)^2}dx = 1$.

(g) The improper integral $\int_0^1 \frac{1}{\sqrt{1-x^2}}dx$ converges, since

$$\int_c^1 \frac{1}{\sqrt{1-x^2}}dx = \arcsin c,$$

which converges to $\pi/2$ as $c \to 1^-$. Hence the value of the improper integral is given by $\int_0^1 \frac{1}{\sqrt{1-x^2}}dx = \frac{\pi}{2}$.

6. Use Cauchy's criterion: Suppose $\int_a^b |f|$ exists. Then given $\varepsilon > 0$, there exists $\delta > 0$ such that for any c_1, c_2 with $a < c_1 < c_2 < a+\delta$, we have

$$\int_{c_1}^{c_2} |f| < \varepsilon.$$

But since

$$\left|\int_{c_1}^{c_2} f\right| \leq \int_{c_1}^{c_2} |f|,$$

this implies

$$\left|\int_{c_1}^{c_2} f\right| < \varepsilon$$

whenever $a < c_1 < c_2 < a + \delta$. Thus Cauchy's criterion again implies $\int_a^b f$ is convergent.

7. Again use Cauchy's criteria: If the improper integral $\int_a^b g$ converges, then $\varepsilon > 0$, there exists $\delta > 0$ such that for any c_1, c_2 with $a < c_1 < c_2 < a + \delta$, we have

$$\int_{c_1}^{c_2} g < \varepsilon.$$

But since

$$0 \leq \int_{c_1}^{c_2} f \leq \int_{c_1}^{c_2} g,$$

this implies

$$\left|\int_{c_1}^{c_2} f\right| < \varepsilon$$

whenever $a < c_1 < c_2 < a + \delta$. Thus Cauchy's criterion again implies $\int_a^b f$ is convergent.

8. (a) Since

$$0 \leq \frac{x+1}{x^3+x^2} \leq \frac{2x}{x^3} = \frac{2}{x^2} \quad \text{whenever } x \geq 1,$$

and since $\int_1^\infty \frac{2}{x^2} dx$ is convergent, by the comparison test,

$$\int_1^\infty \frac{x+1}{x^3+x^2} dx$$

is convergent.

(b) It is easy to check, for instance, that $\ln x \leq \sqrt{x}$ for all $x \geq 1$. Hence

$$0 \leq \frac{\ln x}{x^2} \leq \frac{\sqrt{x}}{x^2} = \frac{1}{x^{3/2}} \quad \text{whenever } x \geq 1,$$

and since $\int_1^\infty \frac{1}{x^{3/2}} dx$ is convergent, by the comparison test,

$$\int_1^\infty \frac{\ln x}{x^2} dx$$

is convergent.

(c) First note that

$$0 \leq \frac{1}{x|\ln x|} \leq \frac{e^x}{x|\ln x|} \quad \text{for } x \in (0, 1/e].$$

Next note that $\int_0^{1/e} \frac{1}{x|\ln x|} dx$ diverges: this is because

$$\int_c^{1/e} \frac{1}{x|\ln x|} dx = (-\ln|\ln x|)_{x=c}^{x=1/e} = \ln|\ln c| - \ln|-1| = \ln|\ln c|$$

which diverges as $c \to 0^+$. Hence by comparison test, $\int_0^{1/e} \frac{e^x}{x|\ln x|} dx$ diverges.

(d) Since

$$0 \leq \frac{|\sin x|}{x\sqrt{1+x^2}} \leq \frac{1}{x\sqrt{x^2}} = \frac{1}{x^2} \quad \text{whenever } x \geq 1,$$

and since $\int_1^\infty \frac{2}{x^2} dx$ is convergent, by the comparison test,

$$\int_1^\infty \frac{|\sin x|}{x\sqrt{1+x^2}} dx$$

is convergent. It follows that

$$\int_1^\infty \frac{\sin x}{x\sqrt{1+x^2}} dx$$

is also convergent.

(e) Note that for $c > 1$,

$$\int_1^c \frac{\sin x}{x} dx = -\frac{\cos c}{c} + \cos 1 - \int_1^c \frac{\cos x}{x^2} dx.$$

Now $\frac{|\sin x|}{x} \frac{\cos c}{c} = 0$ by sandwich theorem, and $\frac{|\sin x|}{x} \int_1^c \frac{\cos x}{x^2} dx$ exists; the latter is because

$$0 \leq \frac{|\cos x|}{x^2} \leq \frac{1}{x^2} \quad \text{for all } x \geq 1,$$

and $\int_1^\infty \frac{1}{x^2} dx$ converges. Hence altogether,

$$\int_1^c \frac{\sin x}{x} dx = -\frac{\cos c}{c} + \cos 1 - \int_1^c \frac{\cos x}{x^2} dx$$

converges as $c \to \infty$. Hence the improper integral $\int_1^\infty \frac{\sin x}{x} dx$ converges.