Solution 6

Section 7.2

8. Suppose that the conclusion is not true, then there exists some point $c \in [a, b]$ such that $f(c) > 0$. By continuity of f, there exists a $\delta > 0$ such that $f > \frac{1}{2}f(c)$ on $[c - \delta, c + \delta]$. Then
$$0 = \int_a^b f \geq \int_{c-\delta}^{c+\delta} f \geq f(c)\delta > 0.$$
Contradiction!

9. Taking function f to be 0 on $(0, 1]$ and $f(0) = 1$. Obviously, f is not continuous on $[0, 1]$ and $f \geq 0$. Since $f(x) = 0$ except one point $x = 0$, therefore, $f \in \mathcal{R}[0, 1]$ and $\int_0^1 f = 0$. This is a counterexample if we drop the continuity assumption.

10. Let $F(x) = f(x) - g(x)$, then F is continuous on $[a, b]$ and $\int_a^b F = 0$. It suffices to show that there exists a point $c \in [a, b]$ such that $F(c) = 0$.

    If F changes sign on $[a, b]$, then by continuity of F, there must be a point c such that $F(c) = 0$.

    If F doesn't change sign, W.L.O.G, we can assume that $F > 0$. Then $\int_a^b F > 0$. Contradiction!

13. Taking function f to be $\frac{1}{x}$ on $(0, 1]$ and $f(0) = 0$. For any $c \in (0, 1)$, since f is continuous on $[c, 1]$, thus $f \in \mathcal{R}[c, 1]$. However, f is unbounded on $[0, 1]$, hence $f \notin \mathcal{R}[0, 1]$.

16. Since f is a continuous function on $[a, b]$, thus f attains its maximum and minimum. Let $M = \max_{[a,b]} f$ and $m = \min_{[a,b]} f$, then $m \leq f(x) \leq M$ for any $x \in [a, b]$. We have the following inequality:
$$m \leq \frac{1}{b-a}\int_a^b f \leq M.$$
Again by the continuity of f, there exists $c \in [a, b]$ such that $f(c) = \frac{1}{b-a}\int_a^b f$.

17. Let $M = \max_{[a,b]} f$ and $m = \min_{[a,b]} f$, then $m \leq f(x) \leq M$ for any $x \in [a, b]$. Since $g > 0$, by Q8 above,
$$\int_a^b g > 0,$$
and hence
$$m = \frac{\int_a^b mg}{\int_a^b g} \leq \frac{\int_a^b fg}{\int_a^b g} \leq \frac{\int_a^b Mg}{\int_a^b g} = M.$$
By the continuity of f, there exists $c \in [a, b]$ such that $f(c) = \frac{\int_a^b fg}{\int_a^b g}$, i.e. $\int_a^b fg = f(c)\int_a^b g$.

Section 7.3

9. (a) $G(x) = F(x) - \int_a^c f$.

   (b) $H(x) = \int_a^b f - F(x)$.

   (c) $S(x) = F(\sin x) - F(x)$.

11. (a)
$$F'(x) = \frac{1}{1+x^6}(x^2)' = \frac{2x}{1+x^6}.$$

   (b)
$$F'(x) = \sqrt{1+x^2} - \sqrt{1+x^4}(x^2)' = \sqrt{1+x^2} - 2x\sqrt{1+x^4}.$$

15. Note
$$g(x) = \int_0^{x+c} f(t)dt - \int_0^{x-c} f(t)dt.$$
Since f is continuous at both $x+c$ and $x-c$, by Fundamental theorem of calculus and the chain rule, g is differentiable at x, and
$$g'(x) = f(x+c)\frac{d}{dx}(x+c) - f(x-c)\frac{d}{dx}(x-c) = f(x+c) - f(x-c).$$

21. a. Since $(tf \pm g)^2 \geq 0$, thus $\int_a^b (tf \pm g)^2 \geq 0$ for any $t \in \mathbb{R}$.

   b. It follows from $\int_a^b (tf \pm g)^2 \geq 0$, we have $t\int_a^b f^2 + \frac{1}{t}g^2 \geq 2\int_a^b f(\mp g)$ for any $t > 0$.
   By G-M inequality, $2\int_a^b f(\mp g) \geq -\left(t\int_a^b f^2 + \int_a^b \frac{1}{t}g^2\right)$.
   Hence,
$$2|\int_a^b fg| \leq t\int_a^b f^2 + \frac{1}{t}g^2.$$

   c. By (b), $2|\int_a^b fg| \leq \int_a^b \frac{1}{t}g^2$ for any $t > 0$. Let $t \to +\infty$, we obtain $\int_a^b fg = 0$.

   d. Note that
$$\begin{aligned} & \left(t\int_a^b f^2 + \int_a^b \frac{1}{t}g^2\right)^2 \\ = & \left(t\int_a^b f^2 - \int_a^b \frac{1}{t}g^2\right)^2 + 4\int_a^b f^2 \int_a^b g^2 \\ \geq & \ 4\int_a^b f^2 \int_a^b g^2. \end{aligned}$$
   In fact, when $t^2 = \frac{\int_a^b f^2}{\int_a^b g^2}$, $\left(t\int_a^b f^2 + \int_a^b \frac{1}{t}g^2\right)^2 = 4\int_a^b f^2 \int_a^b g^2$. Therefore, $\int_a^b f^2 \int_a^b g^2$ is the minimum of $\left(t\int_a^b f^2 + \int_a^b \frac{1}{t}g^2\right)^2$ with respect to t.
   Using G-M inequality, we have $\left(t\int_a^b f^2 + \int_a^b \frac{1}{t}g^2\right)^2 \geq 4\left(\int_a^b |fg|\right)^2$.
   Hence,
$$|\int_a^b fg|^2 \leq \left(\int_a^b |fg|\right)^2 \leq \int_a^b f^2 \int_a^b g^2.$$