MATH2060B Exercise 5

Deadline: Feb 25, 2015 (noon)

The questions are from Bartle and Sherbert, *Introduction to Real Analysis*, Wiley, 4th edition, unless otherwise stated. It is ordered in such a way that is intended to help your understanding.

Section 7.1 Q.1(a), 2(a)(b), 14.

Supplementary Exercise

1. Define $f : [-1, 1]$ by

$$f(x) = \begin{cases} -x & \text{if } x \in [-1, 0], \\ -x + 1 & \text{if } x \in (0, 1]. \end{cases}$$

   (a) Let P be the partition $\{-1, -\frac{1}{2}, 0, \frac{1}{3}, 1\}$ of $[-1, 1]$. Find the Darboux upper and lower sums of f with respect to the partition P.

   (b) By considering upper and lower sums, show that f is Riemann integrable on $[-1, 1]$.

   (c) Find the integral of f on $[-1, 1]$ by computing the limit of an appropriate sequence of upper sums of f.

Section 7.4 Q.12.
Section 7.2 Q.2, 12.

Supplementary Exercises

2. Show that if $f\colon [a, b] \to \mathbb{R}$ is Riemann integrable on $[a, b]$, and $[c, d] \subset [a, b]$, then f is also Riemann integrable on $[c, d]$.

3. Show that if $f\colon [a, b] \to \mathbb{R}$ is Riemann integrable on $[a, b]$, then $|f|$ is also Riemann integrable on $[a, b]$. Is the converse true?

4. (a) Show that if $f, g\colon [a, b] \to \mathbb{R}$ are Riemann integrable on $[a, b]$, with

$$f(x) \leq g(x) \quad \text{for all } x \in [a, b],$$

   then

$$\int_a^b f(x)dx \leq \int_a^b g(x)dx.$$

   (You may do this by either considering tagged partitions, or considering upper and lower sums.)

   (b) Hence, or otherwise, show that if $f\colon [a, b] \to \mathbb{R}$ is Riemann integrable on $[a, b]$, then

$$\left| \int_a^b f \right| \leq \int_a^b |f|;$$

   also, if in addition $|f(x)| \leq M$ for all $x \in [a, b]$, then

$$\int_a^b |f| \leq M(b - a).$$

5. Prove that if $f, g\colon [a, b] \to \mathbb{R}$ are Riemann integrable on $[a, b]$, and there exists $c > 0$ such that
$$|g(x)| > c$$
for all $x \in [a, b]$, then f/g is Riemann integrable on $[a, b]$. (Hint: we may assume $f = 1$ if we use Theorem 2.8(b) in Notes 2.)

6. Let f, $g\colon [a, b] \to \mathbb{R}$ be bounded functions. Show that the lower integrals of f and g over $[a, b]$ satisfies
$$\underline{\int_a^b} f + \underline{\int_a^b} g \leq \underline{\int_a^b} (f + g).$$
Can you give an example where the inequality here is strict?